1.1 Problem Solving

Objectives

1. Understand Polya's problem-solving method.
2. State and apply fundamental problem-solving strategies.
3. Apply basic mathematical principles to problem solving.
4. Use the Three-Way Principle to learn mathematical ideas.

My goal in writing this section is to introduce you to some practical techniques and principles that will help you to solve many personal and professional problems in your life—such as whether to buy or lease a car, borrow money for graduate school, or organize a large class project. You will find that although real-life problems are often more complex than those found in this text, by mastering the techniques presented here, you will increase your ability to solve problems throughout your life. It is important to remember that you cannot rush becoming a good problem solver. Like anything else in life, the more you practice problem solving, the better you become at it.

Much of the advice presented in this section is based on a problem-solving process developed by the eminent Hungarian mathematician George Polya (see the historical highlight at the end of this section). We will now outline Polya's method.

KEY POINT

George Polya developed a four-step problem-solving method.

George Polya's Problem-Solving Method

Step 1: Understand the problem. It would seem unnecessary to state this obvious advice, but yet in my years of teaching, I have seen many students try to solve a problem before they completely understand it. The techniques that we will explain shortly will help you to avoid this critical mistake.

Step 2: Devise a plan. Your plan may be to set up an algebraic equation, draw a geometric figure, or use some other area of mathematics that you will learn in this text. The plan you choose may involve a little creativity because not all problems succumb to the same approach.

Step 3: Carry out your plan. Here you do what many often think as "doing mathematics." However, realize that steps 1 and 2 are at least as important as the mechanical process of manipulating numbers and symbols to get an answer.

Step 4: Check your answer. Once you think that you have solved a problem, go back and determine if your answer fits the conditions originally stated in the problem. For example, if you are to find the number of snowboarders who participate

Math in Your Life

Problem Solving and Your Career

It may seem to you that some of the problems that we ask you to solve in this book are artificial and of no practical use to you. However, consider the following question that was asked of a prospective employee during a job interview. "How many quarters—placed one on top of the other—would it take to reach the top of the Empire State Building?" This question may strike you as strange and unrelated to your qualifications for getting a job. However, according to an article at monster.com, such puzzle-type questions can play a critical role in determining who is hired for an attractive, well-paying job and who is not. What is important about this question is not the answer, but rather the applicant's ability to display creative problem-solving techniques in a pressure situation.

If you are interested in learning more about how your ability to think creatively in solving puzzles can affect your future job prospects, see the book *How Would You Move Mount Fuji? Microsoft's Cult of the Puzzle—How the World's Smartest Company Selects the Most Creative Thinkers* by William Poundstone.

in the Winter X Games, $19\frac{1}{2}$ people is not an acceptable answer. Or, in an investment problem, it is highly unlikely that your deposit of \$1,000 would earn \$334 interest in a bank account. If your solution is not reasonable, then look for the source of your error. Maybe you have misunderstood one of the conditions of the problem, or perhaps you made a simple computational or algebraic mistake.

KEY POINT

Problem solving relies on several basic strategies.

Problem-Solving Strategies

Problem solving is more of an art than a science. We will now suggest some useful strategies; however, just as we cannot list a set of rules describing how to write a novel, we cannot specify a series of steps that will enable you to solve every problem. Artists, composers, and writers make creative decisions as to how to use their tools, and so you also must be creative in using your mathematical tools.

Mathematics is not as rigid as you may believe from your past experiences. It is important to use the strategies in this section to keep your focus on understanding concepts rather then memorizing formulas. If you do this, you may be surprised to find that a given problem can be solved in several different ways.

STRATEGY

Draw Pictures

Problems usually contain several conditions that must be satisfied. You will find it useful to draw pictures to understand these conditions before trying to solve the problem.

EXAMPLE 1 *Visualizing a Condition in a Word Problem*

Four architects are meeting for lunch to discuss preliminary plans for a new performing arts center on your campus. Each will shake hands with all of the others. Draw a picture to illustrate this condition, and determine the number of handshakes.

Quiz Yourself 1*

How might your diagram change in Example 1 if we were counting the ways the architects could send text messages to each other? Realize now that a message sent from A to B is not the same as a message sent from B to A.
Hint: Consider putting arrowheads on the edges.

SOLUTION: We will use points labeled A, B, C, and D, respectively, to represent the people, and join these points with lines representing the handshakes, as in Figure 1.1.

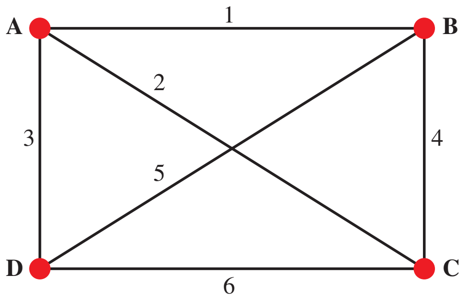


FIGURE 1.1 Visualizing handshakes.

If we represent the handshake between A and B by AB, then we see that there are six handshakes; namely, AB, AC, AD, BC, BD, and CD.

Now try Exercises 7 to 10. 1

In later chapters, you will often be interested in determining all the possibilities that can occur when a series of things are occurring. For example, in Chapter 14, you will be solving probability problems involving the flipping of coins and the rolling of dice. The next example illustrates one way to visualize such situations.

EXAMPLE 2 *Drawing a Tree Diagram*

Draw a diagram to illustrate the different ways that you can flip three coins.

SOLUTION: We can draw the following diagram (Figure 1.2), which is called a *tree diagram*, to show the different possibilities. To keep straight in our mind what each coin is

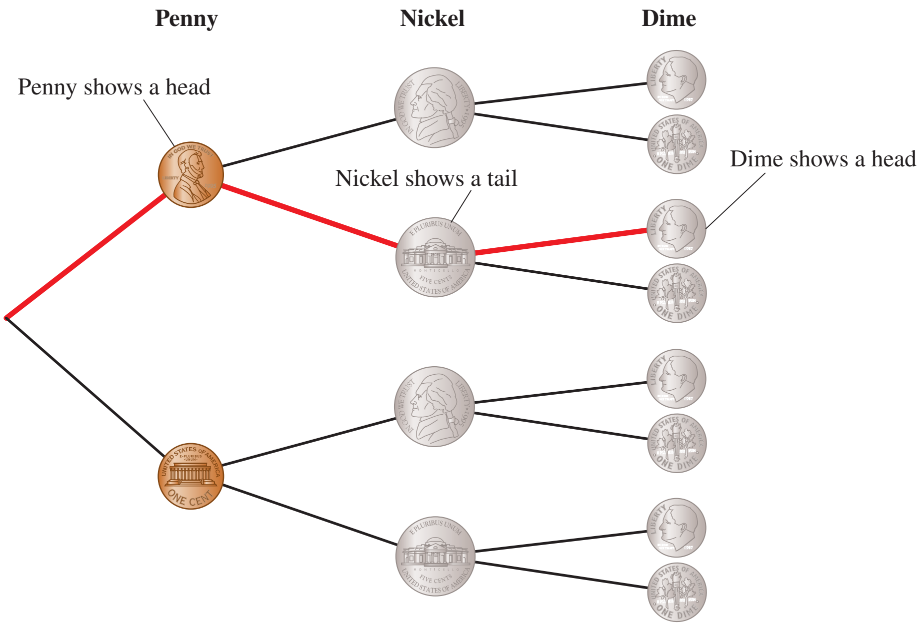


FIGURE 1.2 A tree diagram shows the eight ways to flip three coins.

*Quiz Yourself answers begin on page 778.

doing, we will assume that the three coins are a penny, a nickel, and a dime. The third branch of the tree (shown in red) illustrates that one possibility is that the penny shows a head, the nickel shows a tail, and the dime shows a head. By tracing through this diagram, you can see that there are eight different ways that the three coins can be flipped.

Now try Exercises 19 and 20.

STRATEGY
Choose Good Names for Unknowns

It is a good practice to name the objects in a problem so you can remember their meaning easily.

Example 3 combines good naming with the drawing strategy mentioned earlier.

EXAMPLE 3 Combining the Naming Strategy and the Drawing Strategy

Assume that one group of dance students is taking swing lessons and another is taking Latin dance lessons. Choose good names for these groups, and represent this situation with a diagram.

SOLUTION: In Figure 1.3, the region labeled S represents students taking swing dance lessons and region L represents the students taking Latin dance lessons.

Quiz Yourself 2

Describe the students who are represented by region r_3. What about r_4?

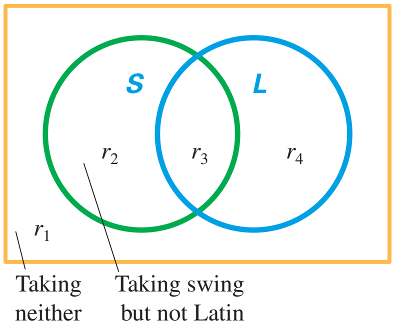


FIGURE 1.3 Representing several groups of people in a diagram.

Quiz Yourself 3

Choose meaningful names for the objects mentioned in the following situation.

Two amounts are invested—one at a high interest rate, the other at a low rate.

As you can see from Figure 1.3, the region marked r_2 indicates students who are taking swing lessons, but are not taking Latin dance lessons. Region r_1 represents students who are taking neither type of lesson.

Now try Exercises 25 and 26. 2 3

STRATEGY
Be Systematic

If you approach a situation in an organized, systematic way, frequently you will gain insight into the problem.

EXAMPLE 4 *Systematically Listing Options*

Javier is buying an iPhone and is considering which optional features to include with his purchase. He has narrowed it down to three choices: extended-life battery, deluxe ear buds, and 8 gigabytes of memory. Depending on price, he will decide how many of these options he can afford. In how many ways can he make his decision?

SOLUTION: In making his decision, we see that there are four cases.

Choose none *or* one *or* two *or* all three of the options.

We organize these possibilities in Table 1.1.

	Battery	Ear Buds	8 Gigabytes
choose none	No	No	No
choose one	Yes	No	No
	No	Yes	No
	?	?	?
choose two	Yes	Yes	No
	Yes	No	Yes
	?	?	?
choose three	Yes	Yes	Yes

TABLE 1.1 Systematic listing of choices of iPhone options.

Quiz Yourself 4

Complete Table 1.1.

We see that there are eight ways that Javier can make his decision.

Now try Exercises 17, 18, 21, and 22. 4

STRATEGY
Look for Patterns

If you can recognize a pattern in a situation you are studying, you can often use it to answer questions about that situation.

EXAMPLE 5 *Finding Patterns in Pascal's Triangle*

You will encounter the pattern that we show in Figure 1.4, called *Pascal's triangle*, in later chapters.

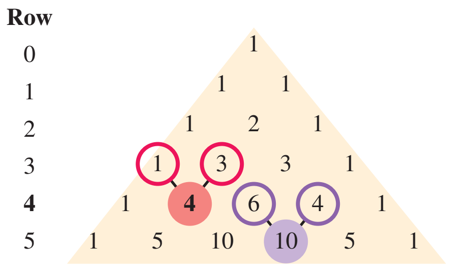


FIGURE 1.4 Pascal's triangle.

Notice how each number is the sum of the two numbers immediately above it, which are a little to the right and a little to the left. Suppose we want to find the total of all the numbers that will be in the ninth row of this diagram.

SOLUTION: (It will be convenient when we discuss this diagram in later chapters to begin numbering rows with 0 instead of 1.)

Notice that in the zeroth row, the total is 1; in the first row, the total is 2; in the second row, the total is 4; and in the third row, it is 8. We continue this pattern in Table 1.2.

Quiz Yourself 5

a) List the numbers in the sixth and seventh rows of Pascal's triangle.

b) What are the first two numbers in the 100th row of Pascal's triangle?

Row	0	1	2	3	4	5	6	7	8	9
Total	1	2	4	8	16	32	64	128	256	512

TABLE 1.2 The sum of the numbers in each row of Pascal's triangle.

We now easily see that the desired total is 512. 5

STRATEGY

Try a Simpler Version of the Problem

You can begin to understand a complex problem by solving some scaled-down versions of the problem. Once you recognize a pattern in the way you are solving the simpler problems, then you can carry over this insight to attack the full-blown problem.

In these days of identity theft, it is important when you send personal information, such as your Social Security number or bank account numbers, to another party that the information cannot be intercepted and your identity stolen. In Example 6, we consider a problem similar to the handshake problem you saw earlier.

Recall that in Example 1, by drawing a diagram of the possible handshakes among four people, you could see that there were six possibilities. If we had asked the same question for 12 people, it would be very cumbersome to draw a picture to count the handshakes, so we would have to have used another technique.

EXAMPLE 6 *Secure Communication Links*

Suppose that 12 branches of Bank of America need secure communication links among them so that financial transfers can be made safely. How many links are necessary?

SOLUTION: Instead of considering all 12 branches, we will look at much smaller numbers of bank branches, count the links, and see if we recognize a pattern. Let's call the branches A, B, C, D, E, F, G, H, I, J, K, and L. In Table 1.3, we will write AB for the link between A and B, AC for the link between A and C, etc.

From looking at these smaller examples, we see an emerging pattern. We notice that as we add new branches, the number of links increases first by 1, then by 2, then by 3, and then by 4, etc.

You can easily see why this is the case if you imagine establishing the branch offices of the banks one at a time. First A is established, so no links are required. Then when B is

Number of Branches	Branches	Links	Number of Links	
1	A	None	0	
2	A, B	AB	1	⟵ add 1 link
3	A, B, C	AB, AC, BC	3	⟵ add 2 links
4	A, B, C, D	AB, AC, AD, BC, BD, CD	6	⟵ add 3 links
5	A, B, C, D, E	AB, AC, AD, AE, BC, BD, BE, CD, CE, DE	10	⟵ add 4 links

TABLE 1.3 Looking for a pattern in the links between Bank of America branches.

built, one link is required between A and B. When C is built, two additional links are needed, namely, AC and BC. When branch D is added, we will need three additional links—AD, BD, and CD. If we continue this pattern as in Table 1.4, we can solve the original problem.

Quiz Yourself 6

Determine the number of ways e-mail can be sent between the various branches. Notice now that e-mail from A to B, written AB, is not the same as e-mail from B to A, written BA.

Number of Branches	1	2	3	4	5	6	7	8	9	10	11	12
Number of Links	0	1	3	6	10	15	21	28	36	45	55	66

TABLE 1.4 Counting the links.

We now see that for 12 branches, there will be 66 links.

Now try Exercises 33 to 38. 6

STRATEGY

Guessing Is OK

One of the difficulties in solving word problems is that you can be afraid to say something that may be wrong and consequently sit staring at a problem, writing nothing until you have the full-blown solution. Making guesses, even incorrect guesses, is not a bad way to begin. It may give you some understanding of the problem. Once you make a guess, evaluate it to see how close you are to meeting all the conditions of the problem.

Imagine that in a few years, when you have graduated, you decide to purchase a brand-new house. At first you are happy with your decision, but then one day when you open your mail, you are shocked to find a school real estate tax bill for $5,200. You are not alone, because in some areas of the country,* both young homeowners and senior citizens are struggling with excessive property taxes. As a result, taxpayer groups have urged politicians to consider other ways of funding public education. We consider a hypothetical case in our next example.

*As I was writing this fourth edition, a bitter controversy was going on in Pennsylvania, where property owners both young and old alike were losing their homes, in large part, due to rising property taxes.

EXAMPLE 7 *Solving a Word Problem by Guessing*

Suppose that you own a house in a school district that has a yearly budget of \$100 million. In order to reduce the portion of the budget borne by property owners, your local taxpayers' organization has negotiated the following political agreement:

1. The amount funded by the state income tax will be three times the amount funded by property taxes.
2. The amount funded by the state sales tax will be \$15 million more than the amount funded by property taxes.

How much of the budget will be funded by property taxes?

SOLUTION: In a later chapter, we will discuss how to solve problems algebraically, but for now, all we want to do is make several educated guesses and then keep adjusting them until we get an acceptable answer. Let us call the amount of the budget due to property taxes p, the amount due to income taxes i, and the amount due to sales taxes s. We will organize our guesses in the following chart.

Guesses for p, i, s (in millions of \$)	Evaluation of Guess	
	Good Points	Weak Points
20, 20, 60	Total is 100.	Amounts are not different.
20, 30, 60	Amounts are different.	Total is not 100. The amount i is not three times p. The amount s is not 15 more than p.
20, 60, 35	The amount i is three times p. The amount s is 15 more than p.	The total is greater than 100, so we have to reduce the amount p.
18, 54, 33	The amount i is three times p. The amount s is 15 more than p.	The total is still more than 100.
17, 51, 32	We have a solution. The amount from property taxes is \$17 million.	

Quiz Yourself 7

The Field Museum in Chicago, which houses Sue, the largest preserved *T. Rex*, has acquired three more mechanical dinosaurs for a new exhibit. The combined weight of the three new dinosaurs is 50 tons. If the weight of the Apatosaurus is seven times the weight of the Duckbill, and the Diplodocus is 14 tons more than the Duckbill, what are the weights of each dinosaur?

Now try Exercises 65 to 74. 7

You may not believe that the guessing approach to problem solving that we suggested in Example 7 is doing mathematics. However, if you are making intelligent guesses and systematically refining them, you probably have some solid, intuitive, underlying logical reasons for what you are doing. And, as a result of this thinking, you are doing legitimate mathematics. The problem with guessing is that it can be inefficient and, if the answer is complex, you probably will not be able to find it without doing some algebra. 7

STRATEGY
Convert a New Problem to an Older One

An effective technique in solving a new problem is to try to connect it with a problem you have solved earlier. It is often possible to rewrite a condition so that the problem becomes exactly like one you have seen before.

EXAMPLE 8 Converting a New Problem to an Older One

Assume that your dear Uncle Trump has died and bequeathed you a large sum of money and you are considering putting it in the following possible investments: stocks, bonds, mutual funds, real estate, certificates of deposit, collectibles, and precious metals. How many ways can you invest your money using these options if you can choose from none to all seven of these options?

SOLUTION: We could solve this problem by doing systematic listing or considering simpler examples as we have done earlier; however, we will solve this problem by recognizing that it is a problem that we have solved before. We are just using different language. Recall that in Example 2 when we were counting the ways to flip three different coins we used a tree diagram to list the possibilities. If instead of thinking "coins" we think "investment options," and instead of "heads or tails" we think "yes of no," then we see that we can draw a similar tree. So consider the tree diagram in Figure 1.5 illustrating the ways that you can select none to all three of the options: stocks, *s*, bonds, *b*, or mutual funds, *m*. A "yes" in the tree means that you are taking that option; a "no" means that you are not taking it. The branch highlighted in red indicates that one possibility is to invest in stocks and mutual funds, but not bonds.

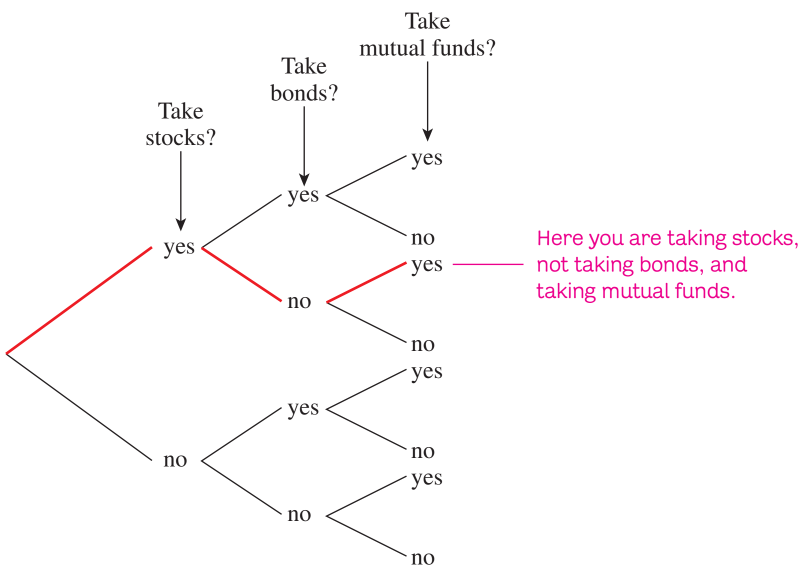


FIGURE 1.5 A tree diagram showing the different ways to choose none to all three investments.

Quiz Yourself 8

How many ways could you choose to invest your inheritance if you had 10 different investments to choose from?

From this diagram you can see that there are eight ways to choose among three investments. If we were to add a fourth investment, such as certificates of deposit, we would add two branches "yes or no" to the existing eight, to give us 16. So each time we add an investment, we double the number of possible ways to select investments. Table 1.5 shows us how to continue this pattern to solve the problem. 8

Number of Investments	1	2	3	4	5	6	7
Number of Selections	2	4	8	16	32	64	128

TABLE 1.5 Counting possibilities for investing.

KEY POINT

Remembering several fundamental principles helps us with problem solving.

Some Mathematical Principles

We will now discuss some basic mathematical principles that we will refer to frequently throughout this text.

The Always Principle

When we say a statement is true in mathematics, we are saying that the statement is true *100% of the time.* One of the great strengths of mathematics is that we do not deal with statements that are "sometimes true" or "usually true."

EXAMPLE 9 An Algebraic Statement That Is Always True

Find an example to illustrate that the statement

$$(x + y)^2 = x^2 + 2xy + y^2$$

is true.*

SOLUTION: In algebra, we prove that the statement $(x + y)^2 = x^2 + 2xy + y^2$ is true for all numbers x and y. This means that any number we substitute for x and y should make the statement true.

Suppose that $x = 3$ and $y = 4$. Then, substituting in the given statement we get

$$\overset{(x+y)^2}{(3 + 4)^2} = \underset{x^2}{3^2} + \overset{2\cdot x\cdot y}{2 \cdot 3 \cdot 4} + \underset{y^2}{4^2} \quad \text{or} \quad 49 = 9 + 24 + 16.$$ 9

Quiz Yourself 9

Find another example to show that

$$(x + y)^2 = x^2 + 2xy + y^2.$$

In a similar way, we accept a mathematical argument only if it holds up under every set of conditions. If we can think of even a single situation in which the argument fails, then the argument is not acceptable.

The Counterexample Principle

An example that shows that a mathematical statement fails to be true is called a *counterexample.* Keep in mind that if you want to use a mathematical property and someone can find a counterexample, then the property you are trying to use is not allowable. A hundred examples in which a statement is true do not prove it to be *always* true, yet a single example in which a statement fails makes it a false statement. Be careful to understand that when we say a statement is false, we are not saying that it is always false. We are only saying that the statement is *not always true.* That is, we can find at least one instance in which it is false.

*For a review of the order of operations in arithmetic and algebra, see Appendix A.

EXAMPLE 10 *Counterexamples to False Algebra Statements*

Although the following two statements are false, those who are not secure in working with algebra often believe them to be true.

a) $(x+y)^2 = x^2 + y^2$ b) $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$

Provide a counterexample for each statement.

SOLUTION:

a) Assume that $x = 3$ and $y = 4$. Then,

$$(x+y)^2 = (3+4)^2 = 7^2 = 49,$$

which is not equal to

$$x^2 + y^2 = 3^2 + 4^2 = 9 + 16 = 25.$$

b) Let $x = 16$ and $y = 9$. Then,

$$\sqrt{x+y} = \sqrt{16+9} = \sqrt{25} = 5.$$

However,

$$\sqrt{x} + \sqrt{y} = \sqrt{16} + \sqrt{9} = 4 + 3 = 7.$$

Now try Exercises 39 to 48. 10

Quiz Yourself 10

Find a counterexample to the following statement:

$$\frac{a+b}{a+c} = \frac{b}{c}.$$

The Order Principle

When you read mathematical notation, pay careful attention to the *order* in which the operations must be performed. The order in which we do things in mathematics is as important as it is in everyday life. When getting dressed in the morning, it makes a difference whether you first put on your socks and then your shoes, or first put on your shoes and then your socks. Although the difference may not seem as dramatic, reversing the order of mathematical operations can also give unacceptable results. Note that we are not saying that it is *always* wrong to reverse the order of mathematical operations; we are saying that if you reverse the order of operations, you *may* accidentally change the meaning of your calculations.

Quiz Yourself 11

Consider the equation

$$\sqrt{x+y} = \sqrt{x} + \sqrt{y}.$$

a) On the left side of the equation we are performing two operations. What are these operations and what is their order?

b) What is the order of the operations performed on the right side of the equation?

c) Summarize what is wrong with this equation, as we did in Example 11(a).

EXAMPLE 11 *Reversing the Order of Operations Can Change the Result of Mathematical Computations*

Explain how Example 10 illustrates the order principle.

SOLUTION: In Example 10, the problem with statements a) and b) is that in each case we carelessly changed the order in which we performed the operations.

a) In the equation $(x+y)^2 = x^2 + y^2$, the left side of the equation tells us to *first* add x and y and *then* square the resulting sum. The right side of the equation tells us to *first* square x and y separately and *then* add these two squares. To say this more succinctly, we can say that to "first add and then square" is not the same as to "first square and then add."

b) The problem here is similar. Test your understanding of the order principle by solving Quiz Yourself 11.

Now try Exercises 49 to 52. 11

Understanding the order principle will be important when you work with set theory in Chapter 2 and also when you study logic in Chapter 3.

The Splitting-Hairs Principle

You should "split hairs" when reading mathematical terminology. If two terms are similar but sound slightly different, they usually do not mean exactly the same thing. In everyday English, we may use the words *equal* and *equivalent* interchangeably; however, in mathematics they do not mean the same thing. The same is true for notation. When you encounter different-looking notation or terminology, work hard to get a clear idea of exactly what the difference is. Representing your ideas precisely is part of good problem solving.

Quiz Yourself 12

In your own words, explain the differences you see in each pair of symbols. Do not be concerned if you do not know what these symbols mean.

a) $\supset$ and $\supseteq$

b) $\{\emptyset\}$ and $\emptyset$

c) $\cup$ and $\vee$

d) $\{0\}$ and 0

EXAMPLE 12 Identifying Differences in Notation

Notice the difference in the following pairs of symbols. It is not important that you know what these symbols mean at this time; all we want you to do is recognize that there are slight differences in the notations.

a) $<$ and $\leq$. There is an extra line under the $<$ symbol on the right.

b) $\cap$ and $\wedge$. The symbol on the left is rounded; the symbol on the right is pointed.

c) $\in$ and $\subset$. The symbol on the left has an extra line.

d) $\emptyset$ and 0. The symbol on the right is the number 0; the symbol on the left is something else, not a number. 12

The Analogies Principle

Much of the formal terminology that we use in mathematics sounds like words that we use in everyday life. This is not a coincidence. Whenever you can associate ideas from real life with mathematical concepts, you will better understand the meaning behind the mathematics you are learning.

Quiz Yourself 13

Which English words are you reminded of by the following mathematical terminology?

a) intersection

b) simultaneous (as in simultaneous equations)

EXAMPLE 13 Relating Mathematical Terms to Everyday Language

In Table 1.6, the left column contains formal mathematical terminology. The right column contains ordinary English words that will help you remember the mathematical concepts.

Mathematical Concept	Related English Ideas
Union	Labor union, marriage union, united
Complement	Complete
Equivalent	Equivalent (in some way the same)
Slope	Ski slope, slope of a roof

TABLE 1.6 Mathematical terms related to words in everyday language.

Now try Exercises 61 to 64. 13

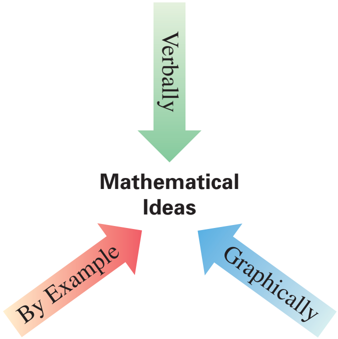


FIGURE 1.6 The Three-Way Principle.

The Three-Way Principle

We conclude this section with a method for approaching mathematical concepts that we illustrate in Figure 1.6.

Whether you are learning a new concept or trying to gain insight into a problem, it is helpful to use the ideas we have discussed in this chapter to approach mathematical situations in three ways.

- *Verbally*—Make analogies. State the problem in your own words. Compare it with situations you have seen in other areas of mathematics.
- *Graphically*—Draw a graph. Draw a diagram.
- *By example*—Make numerical or other kinds of examples to illustrate the situation.

Not every one of these three approaches fits every situation. However, if you get in the habit of using a verbal-graphical-example approach to doing mathematics, you will find that mathematics is more meaningful and less dependent on rote memorization. If you practice approaching mathematics using the strategies and principles that we have discussed, you will find eventually that you are more comfortable and more successful in your mathematical studies.

HISTORICAL HIGHLIGHT

George Polya

We could argue that George Polya (1887–1985) is the father of problem solving as we teach it in so many of today's mathematics textbooks (including this one). As a youth, Polya decided to study law but because of his dislike for memorization, he found it tedious and eventually chose a career in mathematics. While working as a mathematics tutor, he began to develop a problem-solving method that led him to write a book called *How to Solve It.* This book, which has sold more than 1 million copies, discusses many of the approaches to problem solving that you will learn in this section.

Polya loved to solve problems, and it seemed that he could find them everywhere. One day while he and his wife-to-be, Stella, were walking in a garden in Switzerland, they encountered another young couple six times. Polya wondered what the likelihood was that they would meet the same couple so many times on the same walk. His attempts to answer that question eventually led him to publish research papers on the topic of the random walk problem.

In the 1940s, due to their concern about the increasing influence of Nazism in Europe, George and Stella emigrated to the United States. Polya eventually accepted a position at Stanford University in California, where he conducted research and worked on problem solving until he was well into his nineties.

Exercises 1.1

Looking Back*

These exercises follow the general outline of the topics presented in this section and will give you a good overview of the material that you have just studied.

1. List the four steps in Polya's problem-solving method.
2. Name the seven problem-solving strategies that we have presented in this section.
3. Give an example of a situation in which you might use each problem-solving strategy that you mentioned in Exercise 2.
4. List the six principles that we discussed in this section. Give an example of how each principle is used.
5. What well-known book on problem solving did Polya write?

*Before doing these exercises, you may find it useful to review the note *How to Succeed at Mathematics* on page xix.

6. What is the point of the *Mathematics in Your Life* highlight on page 3?

Sharpening Your Skills

In Exercises 7–10, draw a picture to illustrate each situation. You are not being asked to solve any problems—just draw a picture.

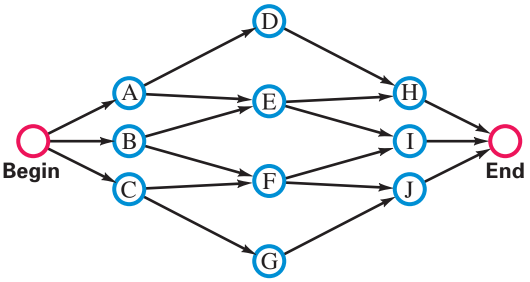
7. Five liters of a 10% sugar solution are mixed with pure water to get a 5% solution.
8. Six people clink each other's glasses in a toast.
9. Messages are being sent among three spies: (A)ustin Powers, James (B)ond, and (C)hloe O'Brien.
10. The Democratic, Republican, and Green political parties have no members in common.

In Exercises 11–16, choose names that would be meaningful for each item. Again, you are not being asked to solve the problem—just make a recommendation for some good names.

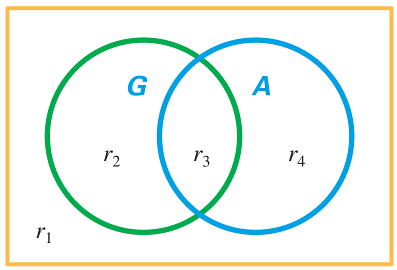
11. A solar panel is three times as long as it is wide.
12. In order to reduce dependence on foreign oil, a presidential commission is considering increasing funding for research in the development of hybrid automobiles, windmill turbines, and solar energy.
13. Meredith, Christina, Izzie, and Alex are planning a party for Derek.
14. Two pathways in the brain cross.
15. A person makes two investments—one in stocks, the other in bonds.
16. Lance Armstrong wants to include calcium and protein in his diet.

In Exercises 17 and 18, list the items mentioned. Try to organize your list in a systematic way.

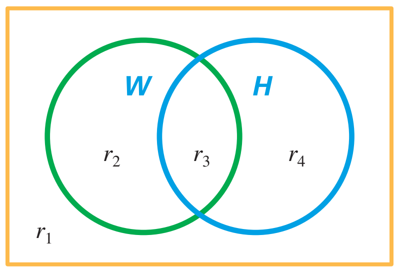
17. List the different combinations of heads and tails that can occur when a penny and a nickel are flipped.
18. Using the numbers 1, 2, and 3, form as many ordered pairs of numbers as you can. For example, (2, 3) is one such pair; (3, 2) is a different one. You are allowed to repeat numbers, so (1, 1) is an allowable pair.
19. If you draw a tree, as we did in Example 2, to show the number of ways to flip four coins, how many possibilities would there be?
20. If you draw a tree, as we did in Example 2, to show the number of ways to roll two dice (one red and one green), how many possibilities would there be? (*Hint:* Draw the branches for the roll of the first die and then attach more branches to correspond to the roll of the second die.)
21. Choose two recording artists to host the Grammy Awards. Your choices are Kanye West, Beyonce Knowles, Christina Aguilera, and Carrie Underwood. (Represent the artists by the first letter of their last name. One pair would be Kanye West and Beyonce Knowles, which you would represent by WK. Note that KW is the same as WK.)
22. Follow the arrows in the given diagram to travel from "Begin" to "End."



23. Some role playing games, such as *Dungeons and Dragons*, have dice with other than six sides. Assume that you are rolling two four-sided dice—with faces numbered 1, 2, 3, and 4. Draw a tree diagram and then list all of the possible ordered pairs of numbers that can be obtained when the two dice are rolled.
24. Repeat the previous problem, but now only list pairs of numbers that are different.
25. Consider the following diagram, which represents various groups of people. G is the group of people who are good singers, and A is the group of people who have appeared as contestants on *American Idol.* Describe the people who are in regions r_3 and r_4.



26. Redraw the diagram in Exercise 25, except now label the two groups of people as W, those who are working to reduce global warming, and H, those who are striving to reduce world hunger. Describe the people who are in regions r_2 and r_4.



In Exercises 27–32, continue the pattern for five more items in the list. (There may be more than one way to continue the pattern. We will provide only one solution.)

27. 7, 14, 21, 28, . . .
28. 1, 3, 9, 27, . . .
29. *ab*, *ac*, *ad*, *ae*, *bc*, *bd*, *be*, . . .
30. (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), . . .
31. 1, 1, 2, 3, 5, 8, 13, . . .
32. 2, 3, 5, 7, 11, 13, 17, . . .

In Exercises 33–38, we state a problem. Instead of trying to solve that problem, state a simpler problem and solve it instead.

33. Ten people are being honored for their work in reducing pollution. In how many ways can we line up these people for a picture?

34. If you guess at 10 true–false questions, how many different ways can you fill in the 10 answers?

35. Using all the letters of the alphabet, how many two-letter codes can we form if we are allowed to use the same letter twice? For example, *ah*, *yy*, *yo*, and *bg* are all allowable pairs. (*Note: bg* is different from *gb*.)

36. A family has seven children. If we list the genders of the children (for example, *bbggbgb*, where *b* is boy and *g* is girl), how many different lists are possible?

37. An electric-blue Ferrari comes with seven options: run flat tires, front heated seats, polished rims, front parking sensor, carbon interior trim, fender shields, and a custom tailored cover. You may buy the car with any combination of these options (including none). How many different choices do you have?

38. In printing your résumé, you have 10 different colors of paper to use and 20 different font styles. How many different ways can you print your résumé?

In Exercises 39–48, determine whether each statement is true or false. If a statement is true, give two examples to illustrate it. If it is false, give a single counterexample.

39. Months of the year have 31 days.

40. All past presidents of the United States are deceased.

41. $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$

42. If $a < b$, then $a + c < b + c$.

43. If A is the father of B and B is the father of C, then A is the father of C.

44. If X is acquainted with Y and Y is acquainted with Z, then X is acquainted with Z.

45. If $a < b$, then $a^2 < b^2$.

46. $x + y - z = x + z - y$

47. If the lengths of the sides of a square are doubled, then the area of the square is also doubled.

48. If the price on a Blu-ray player is increased by 10% and then later reduced by 10%, the price will be the same as the original price.

In Exercises 49–52, decide whether the two sequences of operations will give the same results.

49. Squaring a number, then adding 5 to it; adding 5 to a number and then squaring that sum.

50. Squaring two numbers x and y, then subtracting the results; subtracting x and y, then squaring the difference.

51. Adding two numbers x and y, then dividing the result by 3; dividing x and y by 3, then adding the results.

52. Multiplying two numbers x and y by 5, then adding the results; adding x and y, then multiplying the result by 5.

In Exercises 53–60, explain the differences you see in each pair of symbols.

53. 5 and $\{5\}$

54. A and A'

55. U and u

56. $(1, 2)$ and $[1, 2]$

57. $\{1, 2\}$ and $(1, 2)$

58. $(2, 3)$ and $(3, 2)$

59. $\varnothing$ and 0

60. $\{\ \}$ and $\varnothing$

In later chapters, we will use the following mathematical terminology. Of what common English usage do these terms remind you? Explain.

61. path, circuit, bridge, directed

62. dimension, reflection, translation, transform

63. intercept, best fit, compounding

64. median, deviation, central tendency, correlation

Applying What You've Learned

In Exercises 65–74, do not try to solve each problem algebraically. Instead, make a guess that satisfies one or more conditions of the problem and then evaluate your guess, as we did in Example 7. Keep adjusting your guesses until you have a solution that fits all the conditions of the problem.

65. The local historical society wants to preserve two buildings. The total age of the buildings is 321 years. If one building is twice as old as the other, what are the ages of the two buildings?

66. To celebrate Dunder Mifflin's 40th anniversary, Michael Scott has made a 40-foot-long sandwich. The sandwich is cut into three unequal pieces. The longest piece is three times as long as the middle piece, and the shortest piece is 5 feet shorter than the middle piece. What are the lengths of the three pieces?

67. Janine worked 15 hours last week. One job as a clerk in a sporting goods store paid her $7.25 per hour, while her job giving piano lessons paid $12 per hour. If she earned $137.25 between the two jobs, how many hours did she work at each job?

68. Vince worked 18 hours last week. Part of the time he worked in a fast-food restaurant and part of the time he worked tutoring a high school student in mathematics. He was paid $7.25 per hour in the restaurant and $15 per hour tutoring. If he earned $161.50 total, how many hours did he work at each job?

69. Last season, LaDainian Tomlinson carried the football two times more than Steven Jackson and 68 times less than Larry Johnson. If the three running backs carried the ball 1,110 times, how many times did each carry the ball?

70. In 2007, Alex Rodriguez of the New York Yankees and Magglio Ordoñez of the Detroit Tigers had a total of 295 runs batted in. If Rodriguez had 7 more than Ordoñez, how many did each have?

71. Heather has divided $8,000 between two investments—one paying 8%, the other 6%. If the return on her investment is $550, how much does she have in each investment?

72. Carlos has $9,000 in two mutual funds. One fund pays 11% interest and the other fund pays 8%. If his income from the two funds last year was $936, how much did he invest in each fund?

73. The administration at Center City Community College has formed a 26-person planning committee. There are five times as many administrators as there are students and five more faculty members than students. How many students are there on the committee?

74. Last week Minxia made 55 phone calls to gather support for the reelection of her local state representative. She contacted three times as many senior citizens as she did young adults. The number of middle-age adults was one-half the number of senior citizens. How many senior citizens did she contact?

Communicating Mathematics

75. What is the benefit of choosing good names for objects in a mathematical discussion?

76. Explain the advantage of listing items systematically in a discussion rather than in a random fashion.

77. How does the technique of considering simpler examples of a problem relate to the technique of looking for patterns in solving a problem?

78. Why is guessing a good way to gain insight into a problem?

79. Students often ignore the first and last step of Polya's problem-solving method. Why do you think that this is so? What are the dangers of ignoring these steps?

80. Explain in your own words the Three-Way Principle.

81. Does using the Three-Way Principle in learning mathematics seem different from the ways you have approached learning mathematics in the past? Explain.

82. The French mathematician Jacques Hadamard wrote a book, *An Essay on the Psychology of Invention in the Mathematical Field* (Dover 1954), that gives insights into how real mathematicians use the subconscious mind to solve problems. Write a brief report on this book.

Using Technology to Investigate Mathematics

83. Search the Internet for the term *problem solving*, and write a brief report on one of the sites that you find interesting.

84. Search the Internet for the terms *creativity* and *mathematics*. Write a brief report on an interesting site that you find.

85. Use a calculator to find the last digit in the number 2^{100}. If you try to do this directly, you will get an answer that looks something like 1.2676506 E 30, which is the number written in what is called *scientific notation* that we discuss later in Chapter 6. That is not what we are asking you to do. If you instead consider the sequence of computations $2, 2\times 2, 2\times 2\times 2, 2\times 2\times 2\times 2$, etc. until you see a pattern, that will allow you to predict the last digit in 2^{100}.

86. Proceed as in Exercise 85 and use a calculator to find the last digit in the number 7^{100}.

For Extra Credit

87. How many ordered triples are possible if we roll three 6-sided dice?

88. How many ordered triples are possible if we roll three 12-sided dice?

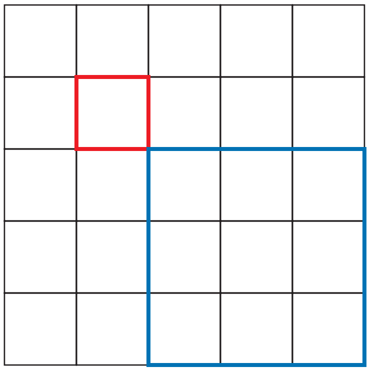
89. Draw a diagram to show all of the possible routes that a sales representative for a company could take by starting at Los Angeles, visiting Chicago, Houston, and Philadelphia in some order, and then returning home. (Describe each route by listing the first letter of each city visited. For example, LHPCL is one route.)

90. Reconsider Exercise 89, but now assume you have 10 different cities, including Los Angeles. By looking at simpler examples, find a pattern to determine the total number of routes that would be possible. Assume that you are always starting at Los Angeles.

91. Continue the following sequence of pairs of numbers by listing the next two pairs in the sequence: (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), ...

92. Continue the following sequence of pairs of numbers by listing the next two pairs in the sequence: (5, 11), (7, 13), (11, 17), (13, 19), (17, 23), (19, 29), ...

93. We will call the following figure a 5-by-5 square. In this figure, we have highlighted a 1-by-1 square in red and a 3-by-3 square in blue. Find the number of all possible squares in the figure. Try to be systematic in solving this problem by considering all 5×5 squares, then all 4×4 squares, etc.



94. Consider in the following map how many different ways there are to go directly from the Hard Rock Cafe to The Cheesecake Factory.