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An Interior Dispersion Relation Program for Extracting Low Energy πN and $\pi\pi$ Scattering Parameters from πN Phase Shifts

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Abstract

This memorandum contains the basic definitions and formalisms for interior dispersion relation (IDR) analyses of πN phase shifts for the purpose of determining low energy πN and $\pi - \pi$ scattering parameters by extrapolation of a discrepancy function to the threshold regions of the s and t–channels.

I. Basic Kinematics

The Mandelstam variables s, u, and t have the usual connotations. The pion mass is μ, and the nucleon mass is m. In πN scattering, the s-channel will be the direct channel and the t-channel will be the cross channel, representing the reaction $\pi\pi \to N\overline{N}$. In this channel, the $\pi\pi \to \pi\pi$ reaction threshold introduces a unitarity cut with branch point at $t = 4\mu^2$. We have

$$s + t + u = 2\mu^2 + 2m^2 \equiv \Sigma \tag{1}$$

In what follows, all center-of-mass momenta will be labeled as p_{ab}, where a and b are the particles occuring in the scattering state (initial *or* final) together; thus, $p_{\pi\pi}$, $p_{N\overline{N}}$, and $p_{\pi N}$. Finally, we use the convention $\nu = s - u$.

Interior dispersion relations (IDR) are "written" along a family of hyper bolae[1] in the (doubly) complex $\nu - t$ plane distinguished by a curve parameter a and written parametrically as

$$(s-a)(u-a) = b(a) \equiv (s_t - a)(u_t - a) \tag{2}$$

where $s_t = (m+\mu)^2$ and $u_t = (m-\mu)^2$. The curve parameter relates ν and t by the equation

$$\begin{aligned} \nu^2 - 4at &= (4p_{\pi\pi}p_{N\bar{N}})^2 && (3)\\ &= (t-4\mu^2)(t-4m^2), && (4)\\ \nu &= +\sqrt{(4p_{\pi\pi}p_{N\bar{N}})^2 + 4at}\,. && (5) \end{aligned}$$

One can also write ν in terms of its roots:

$$\nu^2 = (t - t_-(a))\,(t - t_+(a)) \tag{6}$$

where

$$t_\pm(a) = \Sigma - 2a \pm \sqrt{(\Sigma - 2a)^2 - 16m^2\mu^2}. \tag{7}$$

Furthermore,

$$\begin{aligned} s(t,a) &= \tfrac{1}{2}[\Sigma - t + \nu(t,a)], && (8)\\ u(t,a) &= \tfrac{1}{2}[\Sigma - t - \nu(t,a)]. && (9) \end{aligned}$$

[1] Interior dispersion relations are a form of "hyperbolic dispersion relations" for elastic reactions, such as $\pi - N$ scattering. Other hyperbolic trajectories can be formed by taking b in Eq. 2 to be other values. Contributions to the s-channel integral in IDR will always come from the physical region. For inelastic processes, such as pion photoproduction, the curves along which the dispersion relations are written differ significantly.

For fixed a, negative t and the positive root for ν, the curve described by Eqn. 2 is a curve passing through the s-channel physical region along which the s-channel *laboratory* scattering angle is constant:

$$\cos\theta_L = -\frac{a+m^2-\mu^2}{[a^2 - a\Sigma + (m^2-\mu^2)^2]^{\frac{1}{2}}}, \qquad a \leq 0. \tag{10}$$

The s-channel *center-of-mass* angle is given by

$$\cos\theta_{CM} = (a+s)/(a-s). \tag{11}$$

In particular, $a = 0$ corresponds to the backwards scattering boundary in the s-channel. Indeed, a is related directly to the *Kibble boundary function:*

$$\phi(s,t) \equiv \left[su - \left(m^2-\mu^2\right)^2\right] t \tag{12}$$

$$= -at^2. \tag{13}$$

The forward scattering boundary is, of course, described by $t = 0$.

Additional useful expressions are:

$$t(s,a) = -4sp_{\pi N}^2/(s-a) \tag{14}$$

$$= -\frac{[s-(m+\mu)^2]\,[s-(m-\mu)^2]}{s-a}, \tag{15}$$

$$(t'-t)(s'-a) = -(s'-s)(s'-u), \tag{16}$$

where, in the last, t' and $s'(t',a)$ are, say, integration variables with a fixed, and t, $s(t,a)$ and $u(t,a)$ are set values. That is, in the $t-s$ plane, (t,s) and (t',s') are two points on the same curve specified by a. Finally, we have

$$(z_t p_{\pi\pi} p_{N\bar{N}})^2 = (p_{\pi\pi} p_{N\bar{N}})^2 + at/4, \tag{17}$$

where $z_t = \cos\theta_t$, the cosine of the t-channel center-of-mass scattering angle, and

$$p_{\pi N}^2 = \frac{1}{4s}\left[s-(m+\mu)^2\right]\left[s-(m-\mu)^2\right]. \tag{18}$$

II. Interior Dispersion Relations

The IDR are written for functions (usually quantum mechanical scattering amplitudes) that are even functions of ν and are considered to be analytical functions in the Mandelstam variables with singularities given solely by the dynamics of the interaction and the kinematics of reaction thresholds, etc. That is to say, the functions are assumed not to have any spurious singularities. The IDR stem essentially from Cauchy's integral formula for these functions as expressed in the complex t-plane, and have the generic form

$$\begin{aligned}\mathrm{Re}\, A(t,a) &= A_P(t,a) + \frac{\wp}{\pi}\int_0^{-\infty} \frac{\mathrm{Im}\, A^s(t',a)}{t'-t}\, dt' + \\ &\quad + \frac{\wp}{\pi}\int_{t_0}^{\infty} \frac{\mathrm{Im}\, A^t(t',a)}{t'-t}\, dt' \qquad (19)\\ &\equiv A_P(t,a) + I_s(t,a) + D(t,a). \qquad (20)\end{aligned}$$

Here, $A_P(t,a)$ is related to the residue of a known pole, and is usually refered to as the *Born term*; $\wp$ indicates a principal-value integral (of course, the respective integral is treated as a principal-value integral only when the fixed value of t lies within the integration range); A^s and A^t indicate the value of the function in the s-channel and t-channel, resp.; t_0 is the *threshold* value of t in the t-channel of the reaction; $I_s(t,a)$ is the s-channel integral and $D(t,a)$ is called the *discrepancy function*. Throughout, we will assume that I_s and D refer to the amplitude at hand, and so will not use further identifiers on these symbols. The following form is a useful alternative for the first (s-channel) integral in Eqn. 19:

$$\frac{\wp}{\pi}\int_{(m+\mu)^2}^{\infty} \mathrm{Im}\, A^s\left(t'\left(s',a\right),a\right)\left(\frac{1}{s'-s}+\frac{1}{s'-u}-\frac{1}{s'-a}\right) ds'. \qquad (21)$$

It is frequently useful to use a subtracted form of the interior dispersion relation. These will be discussed as they appear in context.

The real $\nu - t$ plane with physical region boundaries and a typical fixed-a trajectory is shown in Figure a.

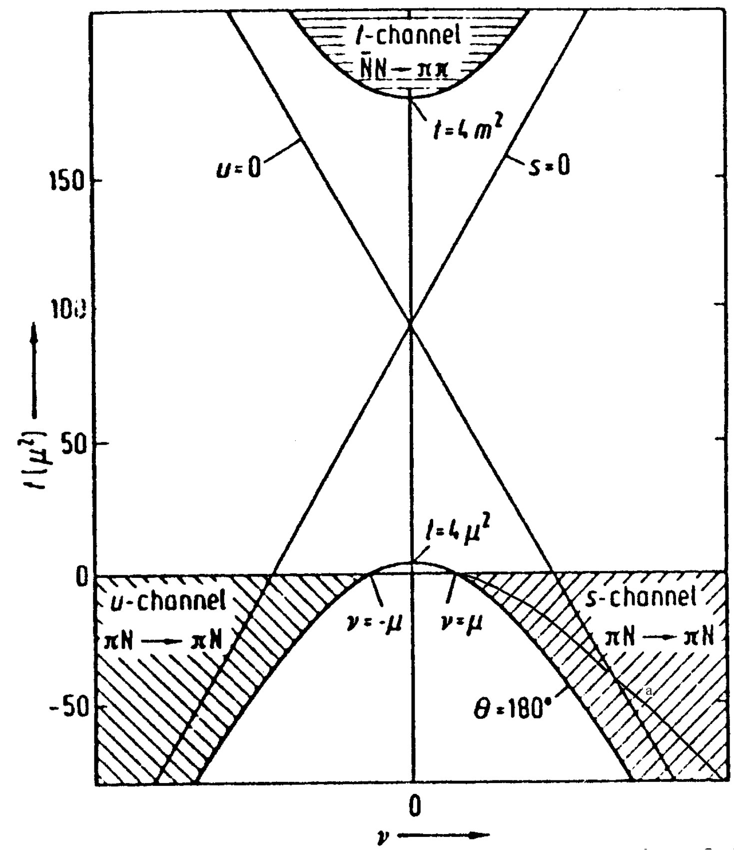

Fig. a The real $\nu - t$ plane with s, u and t-channel physical regions. The curve internal to the s-region represents a typical fixed-a trajectory. (Adopted from G. Hoehler, **Pion Nucleon Scattering**, Springer Publishing Co. In the figure, $\nu = (s-u)/4m$.)

III. Invariant Amplitudes

Interior dispersion relations can be written for functions that are even functions of ν. The πN *crossing-symmetric* invariant scattering amplitudes,

$$A^{(+)},\ A^{(-)}/\nu,\ B^{(+)}/\nu,\ B^{(-)},\ C^{(+)},\ C^{(-)}/\nu,\ D^{(+)},\ D^{(-)}/\nu, \qquad (22)$$

are such functions. The sign in the superscript indicates the symmetry $(+)$ or antisymmetry $(-)$ of the amplitude under isospin crossing. We will indicate such a crossing-symmetric amplitude with a *tilde:* $\tilde{A}^{(+)} \equiv A^{(+)}$, $\tilde{A}^{(-)} \equiv$

$A^{(-)}/\nu$, etc. The C and D amplitudes are not independent, but are defined in terms of the A and B amplitudes according to:

$$C^{(\pm)} \equiv A^{(\pm)} + \frac{m\nu}{4m^2 - t} B^{(\pm)}, \tag{23}$$

$$D^{(\pm)} \equiv A^{(\pm)} + \frac{\nu}{4m} B^{(\pm)}. \tag{24}$$

A. Born Terms

The s-channel Born terms are taken from the pseudoscalar *or* pseudovector πN Lagrangians. The so-called fixed-t Born terms for the invariant amplitudes are the obtained by applying fixed-t dispersion relations to the nucleon and pion pole contributions. These turn out to be the usual expressions obtained directly from the Feynman diagrams, and are (for $m_p = m_n$)

$$\tilde{A}_N^{(\pm)}(s,t) = 0; \tag{25}$$

$$\tilde{B}_N^{(+)}(s,t) = \frac{g^2}{(m^2 - s)(m^2 - u)} = \frac{4g^2}{\nu_N^2 - \nu^2}; \tag{26}$$

$$\tilde{B}_N^{(-)}(s,t) = g^2 \frac{t - 2\mu^2}{(m^2 - s)(m^2 - u)} = 4g^2 \frac{\nu_N}{\nu_N^2 - \nu^2}; \tag{27}$$

$$\tilde{C}_N^{(+)}(s,t) = \frac{mg^2}{4m^2 - t} \frac{\nu_N^2}{(m^2 - s)(m^2 - u)} \tag{28}$$

$$= \frac{4mg^2}{4m^2 - t} \left(\frac{\nu_N^2}{\nu_N^2 - \nu^2} \right); \tag{29}$$

$$\tilde{C}_N^{(-)}(s,t) = mg^2 \frac{t - 2\mu^2}{4m^2 - t} \frac{1}{(m^2 - s)(m^2 - u)} \tag{30}$$

$$= \frac{4mg^2}{4m^2 - t} \left(\frac{\nu_N}{\nu_N^2 - \nu^2} \right); \tag{31}$$

$$\tilde{D}_N^{(+)}(s,t) = \frac{g^2}{m} \left(\frac{\nu_N^2}{\nu_N^2 - \nu^2} \right); \tag{32}$$

$$\tilde{D}_N^{(-)}(s,t) = \frac{g^2}{m} \left(\frac{\nu_N}{\nu_N^2 - \nu^2} \right). \tag{33}$$

where $\nu_N = \nu(s = m^2, t) = t - 2\mu^2$ and g is the πN-coupling constant: $g^2/4\pi \simeq 14.5$. We have also used the identity

$$\nu_N^2 - \nu^2 = 4\left(m^2 - s\right)\left(m^2 - u\right) \tag{34}$$

$$= \; 4\left(m^2 - a\right)(t - t_N)\,. \tag{35}$$

Applying IDR to the same pole contributions results in minor changes only in $\tilde{A}^{(-)}$, $\tilde{B}^{(+)}$, and $\tilde{C}^{(-)}$,where in some factors, t has been replaced by $t_N = t(s = m^2, a) = \mu^2(4m^2 - \mu^2)/(m^2 - a)$. The IDR Born terms are

$$\tilde{A}^{(\pm)}_{IDR,N}(t,a) = 0 \tag{36}$$

$$\tilde{B}^{(+)}_{IDR,N}(t,a) = g^2 \frac{1}{(m^2-a)(t-t_N)} \tag{37}$$

$$\tilde{B}^{(-)}_{IDR,N}(t,a) = g^2 \frac{t_N - 2\mu^2}{(m^2-a)(t-t_N)} \tag{38}$$

$$\tilde{C}^{(+)}_{IDR,N}(t,a) = mg^2 \frac{(t_N - 2\mu^2)^2}{4m^2 - t_N} \frac{1}{(m^2-a)(t-t_N)} \tag{39}$$

$$\tilde{C}^{(-)}_{IDR,N}(t,a) = mg^2 \frac{t_N - 2\mu^2}{4m^2 - t_N} \frac{1}{(m^2-a)(t-t_N)}; \tag{40}$$

$$\tilde{D}^{(+)}_{IDR,N}(t,a) = \frac{g^2}{4m} \frac{(t_N - 2\mu^2)^2}{(m^2-a)(t-t_N)} \tag{41}$$

$$\tilde{D}^{(-)}_{IDR,N}(t,a) = \frac{g^2}{4m} \frac{t_N - 2\mu^2}{(m^2-a)(t-t_N)} \tag{42}$$

B. Partial Wave Expansions

The invariant amplitudes have the following s-channel partial-wave expansions (suppressing isospin indices):

$$A(s,t) = \frac{4\pi m}{p^2_{\pi N}} \sum_\ell \{ \; f_{\ell+} \left[(\omega + \omega_q)P'_\ell + (\omega - \omega_q)P'_{\ell+1}\right] - f_{\ell-} \left[(\omega + \omega_q)P'_\ell + (\omega - \omega_q)P'_{\ell-1}\right] \; \}; \tag{43}$$

$$B(s,t) = \frac{4\pi}{p^2_{\pi N}} \sum_\ell \{ \; - f_{\ell+} \left[(E+m)P'_\ell - (E-m)P'_{\ell+1}\right] + f_{\ell-} \left[(E+m)P'_\ell - (E-m)P'_{\ell-1}\right] \; \}; \tag{44}$$

$$C(s,t) = \frac{8\pi W}{4m^2 - t} \sum_\ell \{ \; (\ell+1) f_{\ell+} \left[(E+m)P_\ell - (E-m)P_{\ell+1}\right] + \ell f_{\ell-} \left[(E+m)P_\ell - (E-m)P_{\ell-1}\right] \; \}. \tag{45}$$

Here, $W = \sqrt{s}$, $\omega = (\nu - t)/4m$ is the pion lab energy, $\omega_q = \sqrt{p_{\pi N}^2 + \mu^2}$ and $E = \sqrt{p_{\pi N}^2 + m^2}$. In terms of the s-channel phase shifts and inelasticities,

$$f_{\ell\pm}(s) = \frac{1}{2ip_{\pi N}} \left[\eta_{\ell\pm} \exp(2i\delta_{\ell\pm}) - 1\right]. \tag{46}$$

The t-channel partial wave expansions are:

$$\begin{aligned} A &= -\frac{4\pi}{p_{N\bar{N}}^2} \sum_{j=0}^{\infty} (2j+1)\,(p_{N\bar{N}} p_{\pi\pi})^j \Big\{ \ f_+^{(j)}(t) P_j(z_t) - \\ &\qquad \left(m z_t \Big/ \sqrt{j(j+1)}\right) f_-^{(j)}(t) P_j'(z_t) \Big\}; \qquad (47) \\ B &= 4\pi \sum_{j=1}^{\infty} \frac{2j+1}{\sqrt{j(j+1)}} (p_{N\bar{N}} p_{\pi\pi})^{j-1} f_-^{(j)}(t) P_j'(z_t); \qquad (48) \\ C &= -\frac{4\pi}{p_{N\bar{N}}^2} \sum_{j=0}^{\infty} (2j+1)\,(p_{N\bar{N}} p_{\pi\pi})^j f_+^{(j)}(t) P_j(z_t). \qquad (49) \end{aligned}$$

In these expansions, the even j terms belong to the $(+)$ amplitudes and the odd j terms belong to the $(-)$ amplitudes. The expansions for the ν-even amplitudes $\tilde{A}^{(-)}$, $\tilde{B}^{(+)}$ and $\tilde{C}^{(-)}$ can easily be found by using the relationship $\nu = 4 p_{N\bar{N}} p_{\pi\pi} z_t$.

IV. πN Threshold Quantities

Among the πN scattering parameters defined at or near the s–channel threshold that possess theoretical importance, and therefore beg for reliable determinations, are:

1. the pion-nucleon coupling constant, g.
2. the s–wave scattering lengths, $\mathsf{a}_{0+}^{(\pm)}$, and effective ranges, $\mathsf{b}_{0+}^{(\pm)}$.
3. the pion-nucleon sigma term, $\sigma_{\pi N}$.

Each requires special attention to the selection of amplitude(s) used in its extraction as well as the method of parameterization of the discrepancy function in order to optimize confidence in extrapolation to the position in the $\nu - t$ plane at which it is defined. We present in the following the formalism and adopted methodology for each.

A. The Pion-Nucleon Coupling Constant

For fixed–a, the pion-nucleon coupling constant is defined in terms of the residues of $\tilde{B}^{(\pm)}_{IDR,N}(t,\,a)$, as given in Eqns. 37 and 38, at $t = t_N$. We note that $\tilde{B}^{(-)}_{IDR,N}$ is reduced by a factor of $(t_N - 2\mu^2)$ over $\tilde{B}^{(+)}_{IDR,N}$ at the nucleon pole. Furthermore, $\tilde{B}^{(-)}$ contains the odd-isospin ρ pole in its t–channel partial wave expansion. The extrapolation distance from the lowest s–channel data point to t_N varies with a. While it is not daunting, it is appreciable and warrants more realistic parameterizations of the discrepancy function.

In each of the strategies for obtaining g described here, consistency must be checked by testing the a–dependence of the result (which should, of course, be constant in a.)

1. By using the expression,

$$(t-t_N)(m^2-a)\left[\mathrm{Re}\,\tilde{B}^{(+)}(t,\,a) - I_s(t,\,a)\right] = g^2 + (t-t_N)(m^2-a)\,D(t,\,a), \tag{50}$$

   we obtain g^2 by extrapolating *the* left-hand-side to $t = t_N$.

2. We can subtract the IDR at $t = 0$, and extrapolate

$$t_N(t-t_N)(m^2-a)\left[\mathrm{Re}\left(\frac{\tilde{B}^{(+)}(t,\,a) - \tilde{B}^{(+)}(0,\,a)}{t}\right) - I_{s,SUB}(t,\,a)\right] \tag{51}$$

   to $t = t_N$, where

$$I_{s,SUB}(t,\,a) = \frac{I_s(t,\,a) - I_s(0,\,a)}{t} = \frac{\wp}{\pi}\int_0^{-\infty} \frac{\mathrm{Im}\,\tilde{B}^{(+)}(t',a)}{t'\,(t'-t)}\,dt'\,. \tag{52}$$

V. $\pi\pi$ Scattering Lengths

Between the two-pion threshold in $\pi\pi$ scattering and the four-pion threshold, unitarity provides the following relationship for

partial wave amplitudes:

$$\mathrm{Im}\, f^{I,J}_{\pi\pi\to N\bar{N}} = \frac{p^{2J+1}_{\pi\pi}}{\sqrt{t}}\, f^{I,J*}_{\pi\pi\to\pi\pi}\, f^{I,J}_{\pi\pi\to N\bar{N}}\,. \tag{53}$$

This implies that

$$\delta^{I,J}_{\pi\pi\to N\bar{N}}(t) = \delta^{I,J}_{\pi\pi\to\pi\pi}(t) \equiv \delta^{I,J}_{\pi\pi}, \;\; 4\mu^2 \leq t \lesssim 16\mu^2, \tag{54}$$

where $\delta^{I,J} = \arg f^{I,J}$.

The scattering length and effective range appear in the first two terms of the expansion of $\cot\delta^{I,J}$:

$$(p_{\pi\pi})^{2J+1}\cot\delta^{I,J} = \frac{1}{\mathsf{a}^I_J} + \tfrac{1}{2}\mathsf{r}^I_J(p_{\pi\pi})^2 + \bigcirc(p^4_{\pi\pi}). \tag{55}$$

In what follows, we will frequently suppress the isospin index, I, and set $p_{\pi\pi} = q$, $p_{N\bar{N}} = p$ and $J = j$. From the t-channel partial wave expansions of the invariant amplitudes (Eqns. $47-49$), we have for the first few terms:

$$\frac{p^2}{4\pi}\tilde{A}^{(+)} = -f^{(0)}_+ + \frac{15}{\sqrt{6}}m(qp)^2 z_t^2 f^{(2)}_- - \frac{5}{2}(qp)^2(3z_t^2-1)f^{(2)}_+ + \cdots \tag{56}$$

$$\frac{p^2}{3\pi}\tilde{A}^{(-)} = -f^{(1)}_+ + \frac{1}{\sqrt{2}}mf^{(1)}_- - \frac{7}{6}(qp)^2(5z_t^2-3)f^{(3)}_+ + \frac{7}{4\sqrt{3}}m(qp)^2(5z_t^2-1)f^{(3)}_- - \cdots \tag{57}$$

$$\frac{\sqrt{6}}{15\pi}\tilde{B}^{(+)} = f^{(2)}_- + \frac{9}{2\sqrt{30}}(qp)^2(7z_t^2-3)f^{(4)}_- + \frac{13\sqrt{7}}{40}(qp)^4(33z_t^4-30z_t^2+5)f^{(6)}_- + \cdots \tag{58}$$

$$\frac{1}{6\sqrt{2}\pi}\tilde{B}^{(-)} = f^{(1)}_- + \frac{7}{2\sqrt{6}}(qp)^2(5z_t^2-1)f^{(3)}_- + \frac{11\sqrt{15}}{24}(qp)^4(21z_t^4-14z_t^2+1)f^{(5)}_- + \cdots \tag{59}$$

$$-\frac{p^2}{4\pi}\tilde{C}^{(+)} = f^{(0)}_+ + \frac{5}{2}(qp)^2(3z_t^2-1)f^{(2)}_+ + \frac{9}{8}(qp)^4(35z_t^4-30z_t^2+3)f^{(4)}_+ + \cdots \tag{60}$$

$$-\frac{p^2}{\pi}\tilde{C}^{(-)} = 3f^{(1)}_+ + \frac{7}{2}(qp)^2(5z_t^2-3)f^{(3)}_+ + \frac{11}{8}(qp)^4(63z_t^4-70z_t^2+15)f^{(5)}_+ + \cdots \tag{61}$$

Because they do not mix helicity eigenstates, we will deal solely with $\tilde{B}$ and $\tilde{C}$ in the following. We have included sufficient terms in the above expressions to guarantee that we have achieved the proper expansions from which to obtain the scattering lengths and effective ranges. From Eqn. 17, we see, to these orders, anyway, that are all polynomials in $(z_t qp)^2$ and $(qp)^2$, in other words, in a and $(qp)^2$. This can be seen as true to all orders by considering the properties of $P_j'(z_t)$ and $P_j(z_t)/z_t$ for odd j and $P_j(z_t)$ and $P_j'(z_t)/z_t$ for even j. In the limit as $a \to 0$, $z_t^2 \to 1$, so that in this limit these functions are all finite and independent of q^2.

Furthermore, if P is a polynomial in z_t^2, then, since

$$\frac{\partial \mathsf{P}(z_t^2)}{\partial a} = \mathsf{P}' \frac{\partial z_t^2}{\partial a} = \mathsf{P}' \frac{t}{(2qp)^2},$$

any partial derivative of $\tilde{A}$ with respect to a will have the z_t^2-polynomials in $\tilde{A}$ replaced by q^{-2} times quantities of the order of 1 in the $a \to 0$, $q \to 0$ limits. In particular,

$$\left.\frac{2}{5\sqrt{6}\pi}\frac{\partial \tilde{B}^{(+)}}{\partial a}\right|_{a=0} = \frac{63}{8\sqrt{30}} t f_-^{(4)} + \frac{819}{40\sqrt{7}} (qp)^2 t f_-^{(6)} + \cdots \tag{62}$$

$$\left.\frac{1}{6\sqrt{2}\pi}\frac{\partial \tilde{B}^{(-)}}{\partial a}\right|_{a=0} = \frac{35}{8\sqrt{6}} t f_-^{(3)} + \frac{77\sqrt{15}}{24} (qp)^2 t f_-^{(5)} + \cdots \tag{63}$$

$$\left.-\frac{p^2}{4\pi}\frac{\partial \tilde{C}^{(+)}}{\partial a}\right|_{a=0} = \frac{15}{8} t f_+^{(2)} + \frac{45}{4} (qp)^2 t f_+^{(4)} + \cdots \tag{64}$$

$$\left.-\frac{p^2}{4\pi}\frac{\partial \tilde{C}^{(-)}}{\partial a}\right|_{a=0} = \frac{35}{8} t f_+^{(3)} + \frac{77}{4} (qp)^2 t f_+^{(5)} + \cdots\ . \tag{65}$$

If we now write schematically for a ν-even function,

$$\tilde{A} = \sum_{n=0} (qp)^{2n} f^{(j_0+2n)}(t)\, \mathsf{P}_{2n}(z_t^2), \tag{66}$$

where j_0 is the smallest J-value in the sum for the particular amplitude, P_{2n} absorbs overall numerical constants and, if necessary a factor of p^2, and where we recall $\mathrm{Im}\, f^{(j)} = \mathrm{Re}\, f^{(j)} \tan\delta_{\pi\pi}^{(j)}$, we have

$$\begin{aligned} \arg\tilde{A} &= \arctan \frac{\mathrm{Im}\sum_n (qp)^{2n} f^{(j_0+2n)}(t)\, \mathsf{P}_{2n}(z_t^2)}{\mathrm{Re}\sum_n (qp)^{2n} f^{(j_0+2n)}(t)\, \mathsf{P}_{2n}(z_t^2)} \\ &= \arctan \frac{\mathrm{Re}\sum_n (qp)^{2n} f^{(j_0+2n)}(t) \tan\delta_{\pi\pi}^{(j_0+2n)}\, \mathsf{P}_{2n}(z_t^2)}{\mathrm{Re}\sum_n (qp)^{2n} f^{(j_0+2n)}(t)\, \mathsf{P}_{2n}(z_t^2)}. \end{aligned} \tag{67}$$

A similar expansion for $\tilde{A}/q^{2j_0+1}$ would yield (recall that q is real in the range of t being considered here)

$$\frac{\arg\left(\tilde{A}\right)}{q^{2j_0+1}} = \arctan \frac{\gamma^{(j_0)}\,\mathrm{Re}\, f^{(j_0)}(\tan\delta_{\pi\pi}^{(j_0)}/q^{2j_0+1}) + \bigcirc(q^4)}{\gamma^{(j_0)}\,\mathrm{Re}\, f^{(j_0)} + (qp)^2\,\mathrm{Re}\, f^{(j_0+2)}\mathsf{P}_2(z^2) + \bigcirc(q^4)}, \tag{68}$$

since $\tan\delta_{\pi\pi}^{(j)} = \bigcirc(q^{2j+1})$. The coefficient $\gamma^{(j_0)}$ contains factors from the first term in the polynomial P_0 as well as, in the case of $\tilde{C}^{(\pm)}$, factors of p^2. Were it not for the second term in the denominator, factors would cancel and $\tan\delta_{\pi\pi}^{(j_0)}/q^{2j_0+1}$ would be available to second order in q. We could then use Eqn. 55 to determine r_J. The trick is to rewrite the denominator of Eqn. 68 as $\Gamma^{(j_0)}\,\mathrm{Re}\, f^{(j_0)}(1+\bigcirc(q^2))$, where the ration $\gamma^{(j_0)}/\ \Gamma^{(j_0)}$ is a known quantity. Examination of Eqns. 62 - 65 indicates that we can write generally

$$\left.\frac{\partial}{\partial a}\tilde{A}\right|_{q=0,\,a=0} = f^{(j_0+2)}\mathsf{P}'_{j_0+2}\cdot\mu^2 + \bigcirc(q^2).$$

Thus, an appropriate amplitude whose leading term will be proportional to $f^{(j_0)}$ to $\bigcirc(q^4)$ will be

$$\tilde{A}_{\mathsf{s}} \equiv \tilde{A} - \left.\frac{\partial\tilde{A}}{\partial a}\cdot\left(\frac{\mathsf{P}_{j_0+2}}{\mu^2\mathsf{P}'_{j_0+2}}\right)(qp)^2\right|_{a=0}. \tag{69}$$

Thus,

$$\frac{\left[\mathrm{Im}\,\tilde{A}/q^{2j_0+2}\right]_{q=0}}{\left.\mathrm{Re}\,\tilde{A}_{\mathsf{s}}\right|_{q=0,\,a=0}} = \tan\delta_{\pi\pi}^{(j_0)}/q^{2j_0+1} + \bigcirc(q^4),$$

or

$$\left(\frac{\mathrm{Im}\,\tilde{A}}{q^{2j_0+1}}\right)\left(\frac{1}{\mathsf{a}_J} + \tfrac{1}{2}\mathsf{r}_J\,q^2\right) = \mathrm{Re}\left[\tilde{A} + \eta q^2 \left.\frac{\partial\tilde{A}}{\partial a}\right|_{q=0,\,a=0}\right] + \bigcirc(q^4) \tag{70}$$

where

$$\eta = -(p^2/\mu^2)\frac{\mathsf{P}_{j_0+2}}{\mathsf{P}'_{j_0+2}}.$$

An expansion of both sides in q^2 can then yield the desired quantities.

As an example, consider finding a_0^0 from $\tilde{C}^{(+)}$. We have from Eqns. 60 and 64,

$$\begin{aligned}
-\frac{p^2}{4\pi}\tilde{C}^{(+)}(a,t) &= f_+^{(0)} + \frac{5}{2}\left[2(qp)^2 + \frac{3at}{4}\right] f_+^{(2)} \\
&\quad + \frac{9}{8}\left[8(qp)^4 + 40(qp)^2\left(\frac{at}{4}\right) + 35\left(\frac{at}{4}\right)^2\right] f_+^{(4)} + \cdots, \\
-\frac{p^2}{4\pi}\frac{\partial \tilde{C}^{(+)}(a,t)}{\partial a}\bigg|_{a=0} &= \frac{15t}{8} f_+^{(2)} + \frac{45t}{4}(qp)^2 f_+^{(4)} + \cdots .
\end{aligned}$$

where we have used Eqn. 17. From the second of these, we have

$$f_+^{(2)} = \left(-\frac{p^2}{4\pi}\right)\frac{8}{15t}\frac{\partial \tilde{C}^{(+)}(a,t)}{\partial a}\bigg|_{a=0} + \bigcirc(q^2),$$

so that,

$$\mathrm{Re}\, f_+^{(0)} = -\frac{p^2}{4\pi}\left\{\mathrm{Re}\,\tilde{C}^{(+)}\Big|_{a=0} - \frac{8}{3t}(qp)^2\frac{\partial \tilde{C}^{(+)}(a,t)}{\partial a}\bigg|_{a=0} + \bigcirc(q^4)\right\}.$$

Now, since

$$\begin{aligned}
-\frac{p^2}{4\pi}\,\mathrm{Im}\,\tilde{C}^{(+)} &= \mathrm{Re}\, f_+^{(0)} \tan\delta^{(0)} + (qp)^2 \mathsf{P}_2 \cdot \mathrm{Re}\, f_+^{(2)} \tan\delta^{(2)} + \cdots \\
&= \mathrm{Re}\, f_+^{(0)} \tan\delta^{(0)} + \bigcirc(q^5),
\end{aligned}$$

we have

$$\begin{aligned}
\frac{\mathrm{Im}\,\tilde{C}^{(+)}}{\left[\mathrm{Re}\,\tilde{C}^{(+)} - (8/3t)(qp)^2\frac{\partial \tilde{C}^{(+)}(a,t)}{\partial a}\right]_{a=0}} &= \frac{\mathrm{Re}\, f_+^{(0)} \tan\delta^{(0)} + \bigcirc(q^5)}{\mathrm{Re}\, f_+^{(0)} + \bigcirc(q^4)} \\
&= q\frac{\left[\frac{1}{\mathsf{a}_0^0} - \frac{1}{2}\mathsf{r}_0^0 q^2 + \bigcirc(q^4)\right]^{-1} + \bigcirc(q^4)}{1 + \bigcirc(q^4)} \\
&= \frac{q}{\left[\frac{1}{\mathsf{a}_0^0} - \frac{1}{2}\mathsf{r}_0^0 q^2 + \bigcirc(q^4)\right]},
\end{aligned}$$

or

$$\left(\frac{1}{\mathsf{a}_0^0} - \frac{1}{2}\mathsf{r}_0^0 q^2 + \bigcirc(q^4)\right)\frac{\operatorname{Im}\tilde{C}^{(+)}}{q} = \left[\operatorname{Re}\tilde{C}^{(+)} - (8/3t)(qp)^2\frac{\partial\tilde{C}^{(+)}(a,t)}{\partial a}\right]_{a=0}.$$

In applying the IDR for this amplitude, we will reconstruct $\tilde{C}^{(+)}$ at the $t = 4\mu^2$ point from the s-channel integral, $I(a,t)$, the IDR Born term, $\tilde{C}^{(+)}_{IDR,N}(a,t)$, and the discrepancy function, $D(a,t)$, according to

$$\tilde{C}^{(+)}(0,4\mu^2) = I(0,4\mu^2) + \tilde{C}^{(+)}_{IDR,N}(0,4\mu^2) + D(0,4\mu^2)\,.$$

We assume expansions:

$$\begin{aligned} I + \tilde{C}^{(+)}_{IDR,N} &= \sum_n s_n(a)(-iq/\mu)^n, \\ D &= \sum_n c_n(a)(-iq/\mu)^n\,, \\ \left.\frac{\partial\tilde{C}^{(+)}(a,t)}{\partial a}\right|_{a=0} &= \sum_n (s'_n + c'_n)(-iq/\mu)^n\,. \end{aligned}$$

Let $\kappa = -\left.\frac{2p^2}{3}\right|^{t=4\mu^2}$. Then,

$$\begin{aligned} \operatorname{Re}\tilde{C}^{(+)} - (8/3t)(qp)^2\frac{\partial\tilde{C}^{(+)}(a,t)}{\partial a} &= s_0 + c_0 \\ &\quad + \left[\kappa(s'_0 + c'_0) - (s_2 + c_2)\right](q/\mu)^2\Big|_{a=0} + \bigcirc(q^4), \\ \frac{\operatorname{Im} D}{q} &= \frac{\operatorname{Im}\tilde{C}^{(+)}}{q} = -\frac{c_1}{\mu} + \frac{c_3}{\mu^3}q^2 + \bigcirc(q^4), \end{aligned}$$

and

$$\left(\frac{1}{\mathsf{a}_0^0} - \frac{1}{2}\mathsf{r}_0^0 q^2\right)\left(-\frac{c_1}{\mu} + \frac{c_3}{\mu^3}q^2\right) = \left[(s_0 + c_0) + \left[\kappa(s'_0 + c'_0) - (s_2 + c_2)\right](q/\mu)^2\right]$$

to order q^4. Gathering terms, we finally obtain

$$q^0 \quad : \quad -\mu\mathsf{a}_0^0 = \frac{c_1}{s_0 + c_0|_{a=0}} = \frac{c_1}{\left[\operatorname{Re}\tilde{C}^{(+)}\right]_{q=0,\,a=0}}\,; \tag{71}$$

$$q^2 \quad : \quad \frac{\mu}{2}\mathsf{r}_0^0 = \frac{1}{c_1^2}\left[c_1\kappa(s'_0 + c'_0)|_{a=0} - c_1(s_2 + c_2)|_{a=0} - c_3(s_0 + c_0)|_{a=0}\right]. \tag{72}$$

We note that a_0^0 and r_0^0 are independent of a if c_1 and c_3 are.