Global dynamics of a discrete two-species Lottery-Ricker competition model

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In this article, we study the global dynamics of a discrete two-dimensional competition model. We give sufficient conditions on the persistence of one species and the existence of local asymptotically stable interior period-2 orbit for this system. Moreover, we show that for a certain parameter range, there exists a compact interior attractor that attracts all interior points except Lebesgue measure zero set. This result gives a weaker form of coexistence which is referred to as *relative permanence*. This new concept of coexistence combined with numerical simulations strongly suggests that the basin of attraction of the locally asymptotically stable interior period-2 orbit is an infinite union of connected components. This idea may apply to many other ecological models. Finally, we discuss the generic dynamical structure that gives *relative permanence*.

1. A discrete two species competition model

Mathematical models can provide important insights into the general conditions that permit the coexistence of competing species and the situations that lead to competitive exclusion [8,9]. A model of resource-mediated competition between two competing species can be described as follows [1,10,11]:

$$x_{n+1} = \frac{r_1 x_n}{a + x_n + y_n},\tag{1}$$

$$y_{n+1} = y_n e^{r_2 - (x_n + y_n)},\tag{2}$$

where x_n and y_n denote the population sizes of two competing species x and y at generation n, respectively; all parameters r_1, r_2 and a are strictly positive. Franke and Yakubu [10] established the ecological principle of mutual exclusion as a mathematical theorem in a general discrete two-species competition system including (1)–(2). They [11] also gave an example that such

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exclusion principle fails where two species can coexist through a locally stable period-2 orbit. This phenomenon of coexistence has been observed in many other competition models (e.g., [8,9,18,19]) including system (1)–(2) with $a = 0$:

$$x_{n+1} = \frac{r_1 x_n}{x_n + y_n}, \tag{3}$$

$$y_{n+1} = y_n e^{r_2 - (x_n + y_n)}. \tag{4}$$

Note that Equation (3) is the non-overlapping lottery model (Chesson 1981) with singularity at the origin. Every initial condition $(x_0, 0)$ with $x_0 > 0$ maps to $(r_1, 0)$. The lottery model emphasizes the role of chance. It assumes that resources are captured at random by recruits from a larger pool of potential colonists [2,16, chapter 18]. When $y_n = 0$, Equation (3) can be a reasonably good approximation for plant species where a single individual can sometimes grow very big in the absence of competition from others or for a territorial marine species, such as coral reef fish, where a single individual puts out huge number of larvae (personal communications with P. Chesson; also see [16]). Chesson and Warner [3] used such non-overlapping lottery models to study competition of species in a temporally varying environment. In this paper, we focus on the dynamics of Equations (3)–(4). The system (3)–(4) may be an appropriate model for resource competition between a territorial species x and a non-territorial species y.

A recent study by Kang [13] shows that Equations (3)–(4) is persistent with respect to the total population of two species, i.e., all initial conditions in $\mathbb{R}_+^2 \setminus \{(0, 0)\}$ are attracted to a compact set which is bounded away from the origin. The results obtained in [13] allow us to explore the structure of the basin of attraction of the asymptotically stable period-2 orbit of the system (3)–(4) lying in the interior of the quadrant. In this article, we study the global dynamics of (3)–(4). The objectives of our study are two-fold:

(1) Mathematically, it is interesting to study the global dynamics of (3)–(4) since it has singularity at the origin. Thus, the first objective of our study is to give sufficient conditions for competitive exclusion and coexistence of Equations (3)–(4).
(2) Biologically, it is very important to classify and give sufficient conditions for the coexistence of species in ecological models. Among many forms of coexistence, permanence is the strongest concept since it requires all strictly positive initial conditions converge to the bounded interior attractor. Although permanence fails for Equations (3)–(4), we establish the weaker notion *relative permanence*: almost all (relative to Lebesgue measure) strictly positive initial conditions converge to the bounded interior attractor. Numerical simulations of other ecological models (e.g., [4,10,11,14,15]) suggest the possibility that relative permanence may apply where permanence fails. Our second objective of this article is to draw attentions on the concept of *relative permanence*. Our study could potentially provide insights into weaker forms of coexistence for general ecological models and open problems on the basins of attractions of stable cycles for a discrete competition model studied by Elaydi and Yakubu [8].

Simple analysis combined with numerical simulations suggest the following interesting dynamics of the system (3)–(4)

(1) There is no interior fixed point. The eigenvalue governing the local transverse stability of the boundary equilibrium on the x-axis (i.e., $y = 0$) is given by $e^{r_2 - r_1}$ (or r_1/r_2 on the y-axis). If this eigenvalue is less than 1, then we say that the equilibrium point on the x-axis (or y-axis) is transversally stable, otherwise, it is transversally unstable. Thus, if $r_1 > r_2$, then the boundary equilibrium $\xi^* = (r_1, 0)$ is transversally stable and $\eta^* = (0, r_2)$ is transversally unstable; while $r_1 < r_2$, $\xi^* = (r_1, 0)$ is transversally unstable and $\eta^* = (0, r_2)$ is transversally stable.

(2) For a certain range of r_1 and r_2 values, there exists an asymptotically stable periodic-2 orbit in the interior of the quadrant which attracts almost every interior point in $\mathbb{R}_+^2$. For example, when $r_1 = 2$ and $r_2 = 2.2$, the periodic-2 orbit is given by

$$(x_1^i, y_1^i) = (0.1536, 2.9629) \quad \text{and} \quad (x_2^i, y_2^i) = (0.0986, 1.1849)$$

and the eigenvalues of the product of the Jacobian matrices along the orbit are 0.91 and 0.26.

(3) There exists a heteroclinic orbit connecting ξ^* to η^* (see Figure 1).

(4) The basin of attraction of the interior periodic-2 orbit P_2^i consists of all interior points of $\mathbb{R}_+^2$ except all the pre-images of the heteroclinic curve C (where C is the closure of the union of all heteroclinic orbits, see Figure 2).
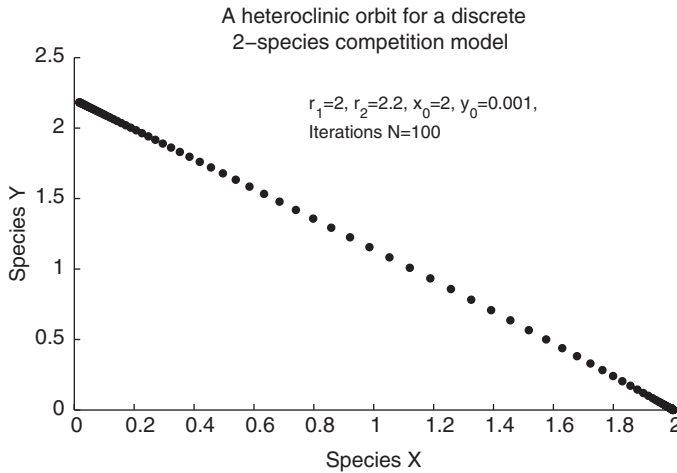
Figure 1.   A heteroclinic orbit of the system (3)–(4) when $r_1 = 2, r_2 = 2.2, x_0 = 2$ and $y_0 = 0.001$.
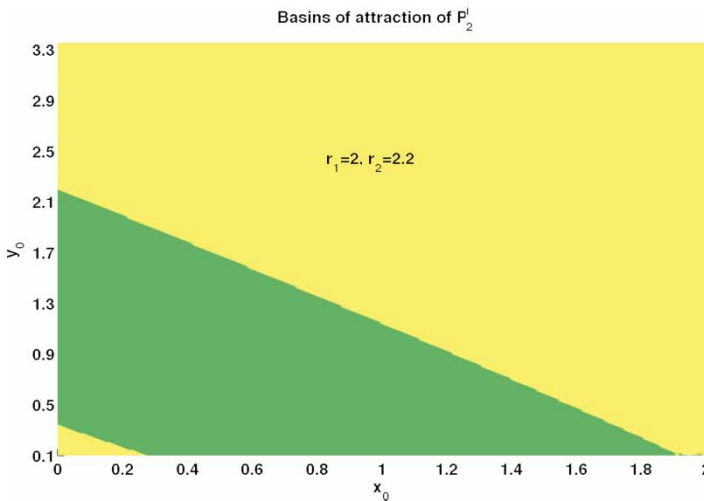
Figure 2.   The basin of attraction of the interior period-2 orbit is the open quadrant minus the pre-images of the heteroclinic curve C. The latter partitions the quadrant into components which are coloured according to which of the two periodic points attract points in the component under the second iterate of the map. Given a point in one of the regions, there is a large number N, such that the point will be very close to (x_1^i, y_1^i) at the iteration t and will be very close to (x_2^i, y_2^i) at the iteration $t + 1$ for all $t > N$.
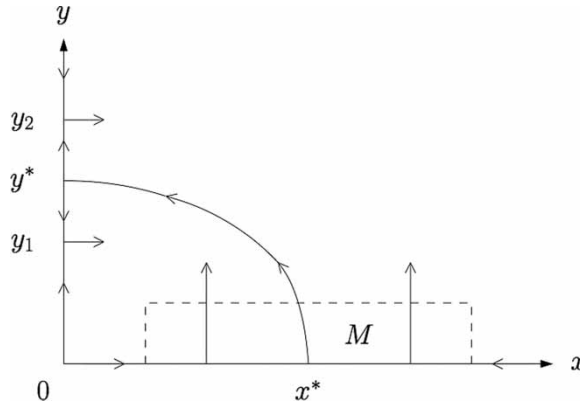
Figure 3.   Schematic features of the system (3)–(4) when $r_1 = 2$ and $r_2 = 2.2$.

Moreover, further analysis and numerical simulations suggest that if the system (3)–(4) satisfies the following conditions C1–C3, then it has the same global dynamics as the case $r_1 = 2$ and $r_2 = 2.2$:

- C1: The values of r_1, r_2 satisfy

$$2 < r_2 < 2.52, \ r_2 > r_1 > 1 \quad \text{and} \quad e^{2r_2 - 1 - e^{r_2 - 1}} > 1.$$

- C2: There is a boundary period-2 orbit $M_y = \{\eta_1, \eta_2\} = \{(0, y_1), (0, y_2)\}$ where $r_1^2 / y_1 y_2 > 1$.
- C3: There is a heteroclinic orbit connecting ξ^* to η^* (see Figure 1).

Condition C1 implies that the equilibria ξ^* and η^* of the system (3)–(4) are saddle nodes, where ξ^* is transversally unstable and η^* is transversally stable. Moreover, species y can invade species x. Condition $2 < r_2 < 2.52$ combined with Condition C2 indicates that species x can invade species $y$ on its periodic-2 orbit $\{(0, y_1), (0, y_2)\}$. Figure 3 describes the schematic scheme of the global dynamics of the system (3)–(4) when it satisfies Condition C1–C3.

The structure of the rest of the article is as follows: In Section 2, we give the basic notations and preliminary results that will be used in proving our main theorems. In Section 3, we obtain sufficient conditions on the persistence of one species and the extinction of the other species by using Lyapunov functions (Theorem 3.1). In Section 4, we first give a sufficient condition on the existence of locally asymptotically stable interior period-2 orbit for the system (3)–(4) (Theorem 4.1); then we show that for a certain parameter range, the system (3)–(4) is *relative permanent*, i.e., it has a compact interior attractor that attracts almost points in $\mathbb{R}_+^2$ (Theorem 4.3) by applying theorems from persistent theory. In Section 5, we discuss the fact that the global dynamics of the system (3)–(4) are generic rather than rare. Similar dynamic behaviours of Equations (3)–(4) have been observed in many biological models. Studying sufficient conditions for the relative permanence of the generalization of such biological models can be our future direction.

2.   Notation and preliminarily results

Note that the system (3)–(4) has singularity at the origin (0,0), thus its state space is defined as $X = \{(x, y) \in \mathbb{R}_+^2 : 0 < x + y < \infty\}$. Let H denote the map defined by Equations (3)–(4). Then $H : X \to X$ is a discrete semi-dynamical system where $H^0(\xi_0) = \xi_0 = (x_0, y_0)$ and $H^n(\xi_0) = \xi_n = (x_n, y_n), n \in \mathbb{Z}_+$. Here, we give some definitions that will be used in the rest of the article.

DEFINITION 2.1 (Pre-images of a point) *For a given point $\xi_0 \in X$, we say $\xi \in X$ is a rank-k pre-image of ξ_0 if $H^k(\xi) = \xi_0$. The collection of rank-k($k \geq 1$) pre-images of ξ_0 is defined as*

$$H^{-k}(\xi_0) = \{\xi \in X : H^k(\xi) = \xi_0\}$$

and the collection of all pre-images of ξ_0(including $k = 0$) is defined as

$$EF_{\xi_0} = \left(\bigcup_{k \geq 1} H^{-k}(\xi_0)\right)\bigcup\{\xi_0\}.$$

DEFINITION 2.2 (Invariant set) *We say $M \subset X$ is an invariant set of H if $H(M) = M$.*

DEFINITION 2.3 (Pre-images of an invariant set) *Let M be an invariant set for the system* (3)–(4), *then $H^0(M) = H(M) = M$. The collection of rank-k pre-images of M($k \geq 1$) is defined as*

$$H^{-k}(M) = \bigcup_{\xi_0 \in M} \{\xi \in X \setminus M : H^k(\xi) = \xi_0\};$$

and the collection of all pre-images of M(including $k = 0$) is defined as

$$EF_M = \left(\cup_{k \geq 1} H^{-k}(M)\right)\bigcup M = \left[\bigcup_{k \geq 1}\left(\bigcup_{\xi_0 \in M}\{\xi \in X \backslash M : H^k(\xi) = \xi_0\}\right)\right]\bigcup M.$$

Note: If *M is an invariant set of H, then $H^{-k}(M)$ should not contain points in M for all $k \geq 1$.*

DEFINITION 2.4 (Uniform weak repeller) *Let $\tilde{X}$ be a positively invariant subset of X. We call the compact invariant set C is a* uniformly weak repeller *with respect to $\tilde{X}$ if there exists some $\epsilon > 0$ such that*

$$\limsup_{n \to \infty} d(H^n(\xi), C) > \epsilon \quad \text{for any } \xi \in \tilde{X} \setminus C.$$

DEFINITION 2.5 (Uniform weak ρ-persistence) *Let $\tilde{X}$ be a positively invariant subset of X. The semi-flow H is called uniformly weakly ρ-persistent in $\tilde{X}$ if there exists some $\epsilon > 0$ such that*

$$\limsup_{n \to \infty} \rho(H^n(\xi)) > \epsilon \quad \text{for any } \xi \in \tilde{X},$$

where $\rho : X \to \mathbb{R}_+$ is a persistence function (e.g., $\rho(x, y) = x$ can be a persistence function if we want to study whether species x is uniformly weakly persistent *or not). We say species x is* uniformly weakly persistent in $\tilde{X}$ *if there exists a $\epsilon > 0$ such that*

$$\limsup_{n \to \infty} x_n > \epsilon \quad \text{for any } \xi \in \tilde{X}.$$

DEFINITION 2.6 (Uniform persistence) *Let $\tilde{X}$ be a positively invariant subset of X. We say species x is* uniformly persistent *in $\tilde{X}$ if there exists some $\epsilon > 0$ such that*

$$\liminf_{n \to \infty} x_n > \epsilon \quad \text{for any } \xi \in \tilde{X}.$$

DEFINITION 2.7 (Permanence) *Let $\tilde{X}$ be a positively invariant subset of X. We say the system H is* permanent *in $\tilde{X}$ if there exists some $\epsilon > 0$ such that*

$$\liminf_{n \to \infty} \min\{x_n, y_n\} > \epsilon \quad \text{for any } \xi \in \tilde{X}.$$

DEFINITION 2.8 (Relative permanence) *We say the system H is relative permanent in X if there exists some $\epsilon > 0$ such that $\liminf_{n \to \infty} \min\{x_n, y_n\} > \epsilon$ for almost all initial condition taken in X (i.e., all initial conditions in X except a Lebesgue measure zero set).*

LEMMA 2.9 (Compact positively invariant set) *Assume that $r_1 \neq r_2$, then for any*

$$0 < \epsilon \le \min\{r_1, r_2, e^{2r_2 - 1 - e^{r_2 - 1}}, r_1 e^{r_2 - r_1}\} = r^m,$$

the compact region defined by

$$D_\epsilon = \{(x, y) \in X : \epsilon \le x + y \le \max\{r_1, e^{r_2 - 1}\}\}$$

is positively invariant and attracts all points in X.

LEMMA 2.10 (Pre-images of invariant smooth curve) *Assume that*

$$r_2 > r_1 > 1 \quad \text{and} \quad e^{2r_2 - 1 - e^{r_2 - 1}} > 1.$$

Let C be an invariant smooth curve of the system (3)–(4) and M be any compact subset of X, then $m_2(EF_C \bigcap M) = 0$ where m_2 is a Lebesgue measure in $\mathbb{R}^2$.

Remark 1 Lemmas 2.9 and 2.10 are a direct corollary from Theorem 2.2 and 3.3 in Kang [13].

3. Sufficient conditions for persistence

In this section, we investigate sufficient conditions for the extinction of one species and the persistence of the other species in system (3)–(4). Let D_ϵ be the set defined in Lemma 2.9 and denote $\mathring{D}_\epsilon$ as the interior of D_ϵ. We can obtain sufficient conditions for the extinction of one species by using a Lyapunov function $V : \mathring{D}_\epsilon \to \mathbb{R}_+$ where $V(x, y) = x^c y^d$ and c and d are some constants. In addition, we give a sufficient condition on the persistence of species y by applying Theorem 2.2 and its corollary of Hutson [12] through defining an average Lyapunov function $P(x, y) = y$ in the compact positively invariant region D_ϵ. Now we are going to give detailed proof of the following theorem:

THEOREM 3.1 (Persistence of one species) (1) *If $r_1 > r_2 > 0$, then the system (3)–(4) has a global stability at $(r_1, 0)$, i.e., for any initial condition $\xi_0 = (x_0, y_0) \in \{(x, y) \in X : x_0 > 0\}$, we have*

$$\lim_{n \to \infty} H^n(\xi_0) = \lim_{n \to \infty} H^n(x, y) = (r_1, 0).$$

(2) *If $0 < r_1 < r_2$, then the species y is uniformly persistent in X, i.e., there exists a positive number $\delta > 0$ such that for any initial condition $\xi_0 = (x_0, y_0) \in \{(x, y) \in X : y > 0\}$, we have*

$$\liminf_{n \to \infty} y_n \ge \delta,$$

where $(x_n, y_n) = H^n(\xi_0)$. Moreover, if $e^{2r_2 - 1 - e^{r_2 - 1}}/r_1 > 1$, then the species x goes to extinct for any $\xi_0 = (x_0, y_0) \in \{(x, y) \in X : y > 0\}$, i.e.,

$$\lim_{n \to \infty} x_n = 0.$$

Proof According to Lemma 2.9, any point in X is attracted to the compact positively invariant set D_ϵ for any $\epsilon \in (0, r^m]$. Therefore, we can restrict the dynamics of Equations (3)–(4) to D_ϵ.

If $r_1 > r_2 > 0$, define $V(x, y) = x^{-r_1} y$, then

$$\frac{V(H(x, y))}{V(x, y)} = r_1^{-r_1}(x + y)^{r_1} e^{r_2 - x - y}.$$

Let $f(u) = r_1^{-r_1} u^{r_1} e^{r_2 - u}$. Since $f'(u) = r_1^{-r_1} u^{r_1 - 1}(r_1 - u) e^{r_2 - u}$, we can conclude that the maximum value of $f(u)$ achieves at $u = 1$, i.e.,

$$\max_{\epsilon \leq u \leq K} \{f(u)\} = f(r_1) = \left(\frac{r_1}{r_1}\right)^{r_1} e^{r_2 - r_1} < 1 \quad \text{where } K > \max\{r_1, r_2, e^{r_2 - 1}, 1\}.$$

According to Lemma 2.9, we know that D_ϵ is positively invariant and attracts all points in X. Therefore, any point in the region $\{(x_0, y_0) \in D_\epsilon : x_0 > 0\}$ has the following two situations:

(1) If $y_0 = 0$, then $V(H^n(x_0, y_0)) = V((x_n, y_n)) = 0$ or
(2) If $x_0, y_0 > 0$, then

$$\frac{V(H(x_0, y_0))}{V(x_0, y_0)} \leq \max_{(x, y) \in D_\epsilon} \{r_1^{-r_1}(x + y)^{r_1} e^{r_2 - x - y}\}$$

$$= \max_{\epsilon \leq u \leq K} \{r_1^{-r_1} u^{r_1} e^{r_2 - u}\} = \left(\frac{r_1}{r_1}\right)^{r_1} e^{r_2 - r_1} < 1.$$

Thus,

$$\frac{V(H^n(x_0, y_0))}{V(x_0, y_0)} = \frac{V(H^n(x_0, y_0))}{V^{n-1}(x_0, y_0)} \cdots \frac{V(H(x_0, y_0))}{V(x_0, y_0)} = \left[\left(\frac{r_1}{r_1}\right)^{r_1} e^{r_2 - r_1}\right]^n \to 0 \quad \text{as } n \to \infty.$$

Therefore, the positively invariant property of D_ϵ implies that

$$\lim_{n \to \infty} x_n^{-r_1} y_n = 0.$$

Therefore,

$$\lim_{n \to \infty} y_n = 0 \quad \text{and} \quad \liminf_{n \to \infty} x_n \geq \epsilon.$$

This indicates that

$$\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \frac{r_1 x_n}{x_n + y_n} = \lim_{n \to \infty} \frac{r_1}{1 + y_n/x_n} = r_1.$$

Therefore, if $r_1 > r_2 > 0$, then the system (3)–(4) has global stability at $\xi^* = (r_1, 0)$. The first part of Theorem 3.1 holds.

If $r_2 > r_1 > 0$, then the omega limit set of $S_x = \{(x, 0) : x > 0\}$ is ξ^*, i.e., $\omega(S_x) = \{\xi^*\}$. The external Lyapunov exponent of S_x is $e^{r_2 - r_1} > 1$, therefore, it is transversal unstable. According to Lemma 2.9, for any $0 < \epsilon \leq r^m$, D_ϵ attracts all points in X. Thus, the uniform persistence of species y follows from Theorem 2.2 and its corollary of Hutson [12] by defining a Lyapunov function $P(x, y) = y$ on the compact positively invariant region D_ϵ, i.e., there exists a positive

number $\delta > 0$ such that for any $y_0 > 0$, we have

$$\liminf_{n \to \infty} y_n > \delta.$$

If, in addition, $r_1 < e^{2r_2 - 1 - e^{r_2 - 1}}$, then we can define a Lyapunov function as $V(x, y) = xy^{-1}$, then we have

$$\frac{V(H(x, y))}{V(x, y)} = \frac{r_1}{(x + y)e^{r_2 - (x+y)}}.$$

Now choose $\epsilon = \min\{r_1, r_2, e^{2r_2 - 1 - e^{r_2 - 1}}, r_1 e^{r_2 - r_1}\}$, then $\epsilon = r_1$ since $r_2 < r_1 < e^{2r_2 - 1 - e^{r_2 - 1}}$. Therefore, any point $(x_0, y_0) \in D_{r_1}$ satisfies $r_1 \le x + y \le e^{r_2 - 1}$ and will stay in D_{r_1} for all future time. This implies that for any point (x_0, y_0) in D_{r_1} with $y_0 > 0$, we have

$$\frac{V(H(x_0, y_0))}{V(x_0, y_0)} \le \max_{(x,y) \in D_{r_1}} \left\{ \frac{r_1}{(x + y)e^{r_2 - (x+y)}} \right\} = \frac{r_1}{\min_{(x,y) \in D_{r_1}} \{(x + y)e^{r_2 - (x+y)}\}}$$

$$= \frac{r_1}{\min_{r_1 < u < e^{r_2 - 1}} \{u e^{r_2 - u}\}} = \frac{r_1}{e^{2r_2 - 1 - e^{r_2 - 1}}} < 1.$$

Hence, $\lim_{n \to \infty} x_n = 0$. Now if $(x_0, y_0) \in X \backslash D_{r_1}$, then according to Lemma 2.9, (x_0, y_0) will either enter D_{r_1} in some finite time or converge to $(r_1, 0)$. Now we consider the following two cases for any initial condition $(x_0, y_0) \in X \backslash D_{r_1}$ with $y_0 > 0$:

(1) If $x_0 = 0$, then $x_n = 0$ for all positive integer n.
(2) If $x_0 > 0$, then (x, y) will not converge to $(r_1, 0)$ since the equilibrium point $(r_1, 0)$ is a saddle and transversal unstable when $r_2 > r_1$, therefore, (x, y) will enter D_{r_1} in some finite time.

Thus, the condition $r_1 < r_2$ and $r_1 < e^{2r_2 - 1 - e^{r_2 - 1}}$ guarantees that

$$\lim_{n \to \infty} x_n = 0.$$

Therefore, the second part of Theorem 3.1 holds.   ∎

Remark 2 The first part of Theorem 3.1 can be considered as a special case of rational growth rate dominating exponential [10,11] which states that if species x with rational growth rate can invade species y with exponential growth rate at species y's fixed point, i.e., $(0, r_2)$ is transversal unstable, then the exponential species goes extinct irrespective of the initial population sizes. The second part of Theorem 3.1 shows that the exponential species can be persistent whenever $(r_1, 0)$ is transversal unstable (i.e., $r_2 > r_1$). However, the rational species may not go extinct unless $r_1 < e^{2r_2 - 1 - e^{r_2 - 1}}$. In fact, simulations (e.g., Figure 2) suggest that two species of the system (3)–(4) may coexist for almost every initial conditions in X under certain conditions. This point will be illustrated with greater details in the next section.

4. Coexistence of two species

In this section, we give sufficient conditions for the existence of the interior period-2 orbits and its local stability for the system (3)–(4) as the following theorem states:

THEOREM 4.1 (Sufficient conditions on the existence of interior period-2 orbits)   *If $r_2 > 2$, then the Ricker map $y_{n+1} = y_n e^{r_2 - y_n}$ has period two orbits $\{y_1, y_2\}$ where $0 < y_1 < r_2 < y_2$ and $y_1 + y_2 = 2r_2$. The system (3)–(4) has an interior period-2 orbit $P_2^i = \{(x_1^i, y_1^i), (x_2^i, y_2^i)\}$ where*

$$x_1^i = \frac{s_1(s_1 e^{r_2 - s_1} - s_2)}{s_1 e^{r_2 - s_1} - r_1}, \quad y_1^i = \frac{s_1(s_2 - r_1)}{s_1 e^{r_2 - s_1} - r_1},$$

$$x_2^i = \frac{r_1 x_1^i}{s_1}, \quad y_2^i = y_1^i e^{r_2 - s_1},$$

$$s_1 = x_1^i + y_1^i = r_2 - \sqrt{r_2^2 - r_1^2}, \quad s_2 = x_2^i + y_2^i = r_2 + \sqrt{r_2^2 - r_1^2} \tag{5}$$

if one of the following holds:

(1) $s_1 e^{r_2 - s1} > s_2$, *or*
(2) $r_2 - \sqrt{r_2^2 - r_1^2} > y_1$, *or*
(3) $2 \le r_1 < r_2 < 2.5$ *and* $r_1 > r_2 - (r_2 - 2/0.26)^2/2r_2$, *or*
(4) $2.085 \le r_1 \le r_2 \le 2.5$

   In particular, (4) implies (3); (3) implies (2) and (2) implies (1). Moreover, if $r_1 = 2$ and $\delta = r_2 - r_1 = r_2 - 2$ is small enough, then P_2^i is locally asymptotically stable.

Proof   If $r_2 > r_1 > 0$, then

$$r_2 - r_1 - \sqrt{r_2^2 - r_1^2} = \sqrt{r_2 - r_1}\left(\sqrt{r_2 - r_1} - \sqrt{r_2 + r_1}\right) < 0,$$

thus, we have the following inequalities:

$$s_1 = r_2 - \sqrt{r_2^2 - r_1^2} < r_1 < r_2 < s_2 = r_2 + \sqrt{r_2^2 - r_1^2}.$$

Therefore, from Equation (5), we find that $s_1 e^{r_2 - s_1} - s_2 > 0$ is a sufficient condition for the existence of P_2^i.

   Note that the Ricker map $y_{n+1} = y_n e^{r_2 - y_n}$ goes through period-doubling two bifurcation at $r = 2$, thus if $r_2 > 2$, the Ricker map has a period-2 orbit $\{y_1, y_2\}$, where

$$0 < y_1 < r_2 < y_2, \quad y_2 = y_1 e^{r_2 - y_1} \quad \text{and} \quad y_1 + y_2 = 2r_2.$$

Since $s_1 + s_2 = 2r_2$, then from the graphic representation (see Figure 4), we can see that

$$s_1 e^{r_2 - s_1} - s_2 > 0 \quad \text{whenever } y_1 < s_1 < r_2.$$

Therefore, the condition $s_1 = r_2 - \sqrt{r_2^2 - r_1^2} > y_1$ is a sufficient condition for $s_1 e^{r_2 - s_1} - s_2 > 0$, therefore, it is a sufficient condition for the existence of P_2^i.

   Let $a = \sqrt{r_2^2 - r_1^2}$, then we have the following equivalent relationships:

$$s_1 e^{r_2 - s_1} > s_2 \iff (r_2 - a)e^a > r_2 + a \iff r_2 > \frac{2a(e^a + 1)}{e^a - 1}$$

$$= a + \frac{2a}{e^a - 1} \iff r_2 - a > \frac{2a}{e^a - 1}. \tag{6}$$

Thus we find that $r_2 - a > 2a/e^a - 1$ implies $s_1 e^{r_2 - s_1} - s_2 > 0$.
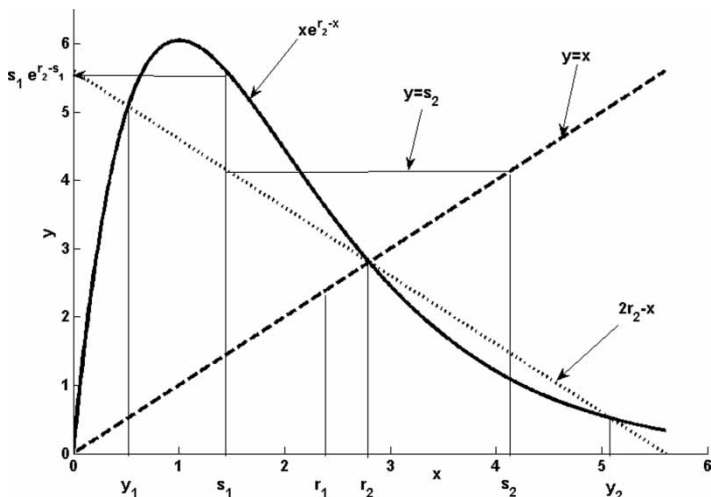
Figure 4. The location between $y_i, s_i, i = 1, 2$ and $s_1 e^{r_2-s_1}$. The solid line is $f(y) = y e^{r_2-y}$; the dashed line is $f(y) = y$; the dot line is $f(y) = 2r_2 - y$.

If $2 \leq r_1 < r_2 < 2.5$, then

$$0 < a = \sqrt{r_2^2 - r_1^2} = \sqrt{r_2 - r_1}\sqrt{r_2 + r_1} < \sqrt{0.5}\sqrt{(2.5 + 2.5)} = \frac{\sqrt{10}}{2}.$$

Notice that $h(a) = 2a/e^a - 1$ is a decreasing convex function with respect to a, thus

$$h(a) \leq k(a) = 2 - \frac{2 - 2\sqrt{10}/2/e^{\sqrt{10}/2} - 1}{\sqrt{10}/2}a,$$

where $k(a)$ is a straight line going through $(0,2)$ and $\left(\frac{\sqrt{10}}{2}, h(\frac{\sqrt{10}}{2})\right)$. Since $(2 - 2\sqrt{10}/2/e^{\sqrt{10}/2} - 1)/\sqrt{10}/2 > 0.74$, therefore,

$$2 - 0.74a \geq 2 - \frac{2 - 2\sqrt{10}/2/e^{\sqrt{10}/2} - 1}{\sqrt{10}/2}a \geq \frac{2a}{e^a - 1} \quad \text{for all } 0 < a < \frac{\sqrt{10}}{2}.$$

Hence, from Equation (6), we can conclude that $r_2 - a > 2 - 0.74a$ implies $r_2 - a > \frac{2a}{e^a-1}$, therefore it implies $s_1 e^{r_2-s_1} - s_2 > 0$. Note the following equivalent relationships,

$$r_2 - a > 2 - 0.74a \iff a < \frac{r_2 - 2}{0.26} \iff r_2^2 - r_1^2 < \left(\frac{r_2 - 2}{0.26}\right)^2$$

$$\iff r_1 > r_2 - \frac{((r_2 - 2)/0.26)^2}{r_2 + r_1}, \tag{7}$$

therefore, we can conclude that $r_1 > r_2 - ((r_2 - 2)/0.26)^2/(r_2 + r_1)$ implies $r_2 - a > 2 - 0.74a$, therefore, it implies $r_2 - a > (2a/e^a - 1)$, therefore, it implies $s_1 e^{r_2-s_1} - s_2 > 0$, therefore, it implies the existence of P_2^i.

Since $2 \leq r_1 < r_2 < 2.5$, then $r_2 + r_1 < 2r_2 \leq 5$, thus

$$r_1 > r_2 - \frac{((r_2 - 2)/0.26)^2}{5} \Rightarrow r_1 > r_2 - \frac{((r_2 - 2)/0.26)^2}{2r_2} \Rightarrow r_1 > r_2 - \frac{((r_2 - 2)/0.26)^2}{r_2 + r_1}.$$

Therefore, $r_1 > r_2 - ((r_2 - 2)/0.26)^2/2r_2$ implies $r_1 > r_2 - ((r_2 - 2)/0.26)^2/(r_2 + r_1)$, therefore, it implies the existence of P_2^i.

Note that

$$r_2 - \frac{((r_2 - 2)/0.26)^2}{5} = -2.958579881(r_2 - 2.169000001)^2 + 2.08450001 \leq 2.085,$$

hence we can conclude that

$$2.085 \leq r_1 < r_2 \leq 2.5$$

implies $r_1 > r_2 - ((r_2 - 2)/0.26)^2/(r_2 + r_1)$, therefore, it implies the existence of P_2^i.

So far, we have shown the first part of Theorem 4.1. Now we are going to see that local stability of P_2^i. Let $r_1 = 2$ and $\delta = r_2 - r_1 = r_2 - 2$, then we have

$$s_1 = r_2 - \sqrt{r_2^2 - r_1^2} = 2 + \delta - \sqrt{\delta(4 + \delta)} \quad \text{and} \quad s_2 = 2 + \delta + \sqrt{\delta(4 + \delta)}.$$

Thus, if δ is small enough, then

$$s_1 e^{r_2 - s_1} - s_2 = \left(\frac{10}{3}\right)\delta^{3/2} - \left(\frac{10}{3}\right)\delta^2 + \left(\frac{41}{12}\right)\delta^{5/2} + O(\delta^3) > 0.$$

Therefore, from the proof for the first part of Theorem 4.1, we can conclude that the system (3)–(4) has an interior period-2 orbit P_2^i when $r_1 = 2$ and $\delta = r_2 - r_1 = r_2 - 2$ is small enough. The local stability of P_2^i is determined by the eigenvalues of the product of the Jacobian matrices along the periodic-2 orbit which can be represented as follows:

$$J|_{P_2^i} = \begin{pmatrix} \dfrac{y_1^i y_2^i}{r_1^2} + \dfrac{r_1 x_1^i y_2^i e^{r_2 - s_2}}{s_1^2} & -\dfrac{y_1^i x_2^i}{r_1^2} + \dfrac{r_1 x_1^i (-1 + y_2^i) e^{r_2 - s_2}}{s_1^2} \\ -\dfrac{r_1 y_1^i y_2^i e^{r_2 - s_1}}{s_2^2} + y_2^i (y_1^i - 1) & -\dfrac{r_1 y_1^i x_2^i e^{r_2 - s_1}}{s_2^2} + (y_1^i - 1)(y_2^i - 1) \end{pmatrix}. \tag{8}$$

If δ is small enough, then the trace and determinant of (8) can be approximated by

$$\det(J) + 1 = 2 - \frac{8\delta}{3} + \frac{49\delta^2}{30} + O(\delta^3) \quad \text{and} \quad \text{trace}(J) = 2 - \frac{8\delta}{3} + \frac{3\delta^2}{10} + O(\delta^3).$$

By the Jury test in [5, p. 57], we see that P_2^i is locally asymptotically stable if

$$2 > 1 + \det(J) = 2 - \frac{8\delta}{3} + \frac{49\delta^2}{30} + O(\delta^3) > |\text{trace}(J)| = \left|2 - \frac{8\delta}{3} + \frac{3\delta^2}{10} + O(\delta^3)\right| \tag{9}$$

which is true when δ is small enough.

Therefore, the statement of Theorem 4.1 holds. ∎

Remark 3 Theorem 4.1 provides a sufficient condition on the existence of the interior period two orbit and their stability. Numerical simulations suggest that the system (3)–(4) has an interior period two orbit whenever $2 \leq r_1 < r_2 < 2.5$. In the case that $r_1 = 2$ and $r_2 = 2 + \delta$, the interior period two orbit P_2^i is locally asymptotically stable whenever $\delta < 0.95$ (see Figures 5 and 6).

LEMMA 4.2 *Assume that the system* (3)–(4) *satisfies Condition C1 and C3, then there exists a smooth invariant curve C that connecting ξ^* to η^*. Denote EF_C as the collection of all ranks preimages of C, then $m_2(EF_C \cap M) = 0$, where M is any compact subset of X and m_2 is Lebesgue measure in $\mathbb{R}_+^2$.*
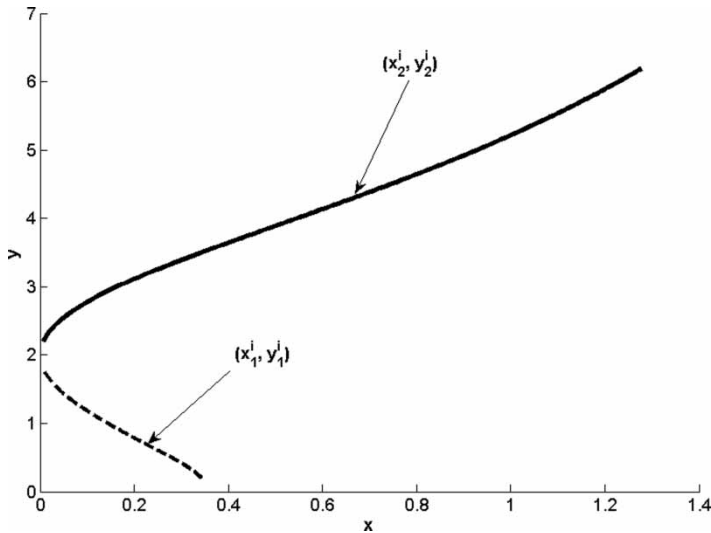
Figure 5. Interior period two orbit P_2^i of the system (3)–(4) when $r_1 = 2, r_2 = 2 + \delta$ and δ is varying from 0 to 2. The solid line is (x_2^i, y_2^i) and the dashed line is (x_1^i, y_1^i).
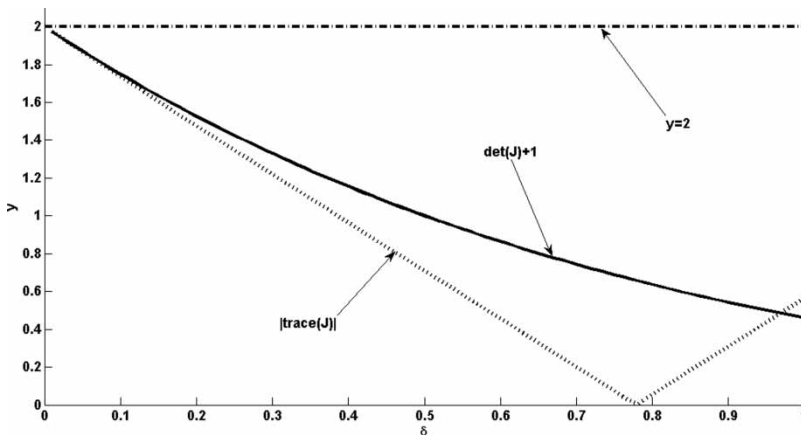

Figure 6. The stability of the interior period-2 orbit P_2^i of the system (3)–(4) when $r_1 = 2, r_2 = 2 + \delta$ and δ is varying from 0 to 1. The solid line is $\det(J) + 1$; the dashed-dot line is constant 2 and the dot line is $|\mathrm{trace}(J)|$. This figure indicates that P_2^i is locally asymptotically stable when $r_1 = 2$ and $2 < r_2 = 2 + \delta < 2.95$.

Proof First we show that C is a smooth curve connecting $\xi^* = (r_1, 0)$ to $\eta^* = (0, r_2)$. Let $\omega_l^u(\xi^*)$ be the local unstable manifold of ξ^* and $\omega_l^s(\eta^*)$ be the local stable manifold of η^*. Since the map H is smooth, then according to stable manifold theorem [6, Theorem D.1 in Appendix], we can conclude that both $\omega_l^u(\xi^*)$ and $\omega_l^s(\eta^*)$ are smooth curves. Since Equations (3)–(4) satisfies Condition C3, then there exists some positive integer k such that $H^k(\omega_l^u(\xi^*))$ is smoothly connected with $\omega_l^s(\eta^*)$. Thus, C is a smooth invariant curve with ξ^*, η^* as its two end points. Then according to Lemma 2.10, the statement holds. ∎

4.1. *Persistence of species x in new space $\tilde{X}$*

Let

$$S_x = \{(x, 0) \in D_\epsilon\}, \quad S_y = \{(0, y) \in D_\epsilon\}, \quad S = S_x \bigcup S_y = \{(x, y) \in D_\epsilon : xy = 0\}.$$

Let C be the closure of all heteroclinic orbits connecting from ξ^* to η^* and denote EF_C as the collection of all rank pre-images of C. Then we can conclude that $E = \overline{EF_C \cap D_\epsilon}$ is compact and forward invariant. Define the following new spaces:

$$\tilde{S} = S \bigcup E, \quad \tilde{X} = X \setminus \tilde{S}.$$

Then both $\tilde{S}$ and $\tilde{X}$ are positively invariant. In addition, $\tilde{S}$ is compact since both S and E are compact.

The rest of this section, we assume that the system (3)–(4) satisfies Condition C1–C3. We will prove the following theorem:

THEOREM 4.3 (Relative permanence) *Assume that the system* (3)–(4) *satisfies C1–C3. Denote EF_C as the collection of all pre-images of the heteroclinic curve C. Then there exists a compact interior attractor in R^2_+ that attracts all points in the interior of X except points in EF_C. In particular, the interior attractor attracts almost every point with respect to Lebesgue measure in $\mathbb{R}^2$ of any compact subset M in the interior of X, i.e., $m_2(EF_C \bigcap M) = 0$ where m_2 is a Lebesgue measure in $\mathbb{R}^2$.*

Proof We use the following three main steps to prove the statement. We provide the detailed proof of the first two steps in the Appendix and the remaining proof here.

(1) C is a uniform weak repeller with respect to $\tilde{X}$, i.e., there exists some $b > 0$, for any $\xi \in \tilde{X}$, we have

$$\limsup_{n \to \infty} \mathrm{d}(H^n(\xi), C) > b.$$

The detailed proof of this part has been shown in the Appendix (Lemma A.1). This implies that any point in $\tilde{X}$ is going to be away from C in some distance in some future time even if the point is very close to C.

(2) Species x is uniformly weakly persistent in $\tilde{X}$, i.e., there exists some $\delta > 0$, such that for any initial condition $\xi_0 = (x_0, y_0) \in \tilde{X}$, the system has

$$\limsup_{n \to \infty} x_n > \delta.$$

The detailed proof of this part has been shown in the Appendix (Lemma A.2). This implies that any point in $\tilde{X}$ is going to be away from $\tilde{S}$ in some distance in some future time even if the point is very close to $\tilde{S}$.

(3) Species x is uniformly persistent in $\tilde{X}$, i.e., there exists some $\epsilon > 0$, such that for any initial condition $\xi = (x, y) \in \tilde{X}$, the system has

$$\liminf_{n \to \infty} x_n > \epsilon.$$

Now we will show the last step. Define a continuous and not identically zero persistent function $\rho(\xi) = d(\xi, \tilde{S})$ where $\xi \in X$. Then by the definition of the persistent function ρ, we have

$$\tilde{S} = \{\xi \in X : \rho(H^t(\xi)) = 0, \ \forall t \geq 0\}$$

is nonempty, closed and positively invariant. In addition, the system H satisfies the following two conditions:

(1) There exists no bounded total trajectory ϕ such that $\rho(\phi(0)) = 0$, $\rho(\phi(-r)) = 0$ and $\rho(\phi(t)) > 0$ for some positive integers r and t.
(2) H has a compact attractor D_ϵ which attracts all points in X.

From Lemma A.2, we know that species x is uniformly weakly persistent in $\tilde{X}$. Thus by applying Theorem 5.2 [17], we can conclude that species x is uniformly persistent in $\tilde{X}$, i.e., there exists some $\epsilon > 0$, such that for any initial condition $\xi = (x, y) \in \tilde{X}$, the system has

$$\liminf_{n \to \infty} x_n > \epsilon.$$

Note that species y is uniformly persistent in X whenever $r_2 > r_1$ according to Theorem 3.1. Thus, species y is also uniformly persistent in $\tilde{X}$ since $\tilde{X}$ is a positively invariant subset of X. Therefore, based on the argument above, we can conclude that there exists some positive constant $\epsilon > 0$, such that for any initial condition taken in $\tilde{X}$, we have

$$\liminf_{n \to \infty} \min\{x_n, y_n\} > \epsilon.$$

Hence there exists a compact interior attractor that attracts all points in the interior of X except points in EF_C.

Let M be any compact subset of the interior of X. Then any initial condition ξ_0 taken in M will enter D_ϵ in some future time through the following two cases:

(1) $\xi_0 \in EF_C$ which will enter C in some finite time.
(2) $\xi_0 \in \tilde{X}$ which is attracted to the interior compact attractor.

Since there are only Lebesgue measure zero of points in M that belong to EF_C, therefore, according to Lemma 2.10, we can conclude that $m_2(EF_C \bigcap M) = 0$ for any compact subset M in the interior of X. ■

5. Discussion and future work

In this article, we study the global dynamics of the system (3)–(4). We give sufficient conditions for the uniform persistence of one species and the existence of locally asymptotically stable interior period-2 orbits for this system. We also show that for a certain parameter range, the system (3)–(4) is *relative permanent*, i.e., there exists a compact interior attractor that attracts almost all points in X. Numerical simulations strongly suggest that this compact interior attractor is the locally asymptotically stable interior period-2 orbit P_2^i and its basin of attractions consists of a infinite union of connected regions that are separated by all pre-images of the heteroclinic curve C (see Figure 2).

The results that we obtained in Theorem 3.1 are a special case for model (1)–(2) when $a = 0$. Our Theorem 4.1 can be extended to the general model (1)–(2) when $a > 0$. If $r_2 > 2$ and $r_1^2 > 2ar_2$, the explicit expressions of the interior periodic-2 orbit $\{(x_1^i, y_1^i), (x_2^i, y_2^i\}$ of the system (1)–(2) can be found as

$$x_1^i = \frac{(a + s_1)(s_2 - s_1 e^{r_2 - s_1})}{r_1 - (a + s_1)e^{r_2 - s_1}}, \quad y_1^i = \frac{r_1 s_1 - s_2(a + s_1)}{r_1 - (a + s_1)e^{r_2 - s_1}},$$

$$x_2^i = \frac{(a + s_2)(s_1 - s_2 e^{r_2 - s_2})}{r_1 - (a + s_2)e^{r_2 - s_2}}, \quad y_2^i = \frac{r_1 s_2 - s_1(a + s_2)}{r_1 - (a + s_2)e^{r_2 - s_2}},$$

where

$$s_1 = x_1^i + y_1^i = r_2 + \sqrt{(r_2 + a)^2 - r_1^2}, \quad s_2 = x_2^i + y_2^i = r_2 - \sqrt{(r_2 + a)^2 - r_1^2}.$$

This interior periodic-2 orbit can have local stability for a certain range of parameters' values. For instance, if

$$r_1 = 2.1, \ a = 0.1 \quad \text{and} \quad r_2 = 2.5,$$

then the system (1)–(2) has locally stable interior periodic-2 orbit

$$(x_1^i, y_1^i) = (0.17, 0.80) \quad \text{and} \quad (x_2^i, y_2^i) = (0.33, 3.70)$$

along which the eigenvalues of the product of the Jacobian matrices are 0.11 and -0.24. Moreover, numerical simulation suggests follows:

(1) There exists a heteroclinic orbit C connecting ξ^* to η^* (see Figure 7);
(2) The basin of attraction of the interior periodic-2 orbit P_2^i is all points in the interior of $\mathbb{R}_+^2$ except a Lebesgue measure zero set in $\mathbb{R}^2$ which is a collection of all pre-images of the heteroclinic curve C (see Figure 8).

However, more mathematical techniques need to be developed in order to obtain results similar to those in Lemma 2.10 for the system (1)–(2) when $a > 0$. This is an area for future study.

Our results may apply to a two species discrete-time Lotka–Volterra competition model with stocking where both species are governed by Ricker's model and one species is being stocked at the constant per capita stocking rate s_1 per generation (10)–(11) [8,9]. We may infer from simulations (see Figure 9) that the basin of attraction of the 2-cycle is the infinite union of connected regions that are separated by all pre-images of the heteroclinic curve C when $s_1 = .5, q_1 = 1.5, q_2 = 2.2, p_1 = p_2 = 1$.

$$x_{n+1} = x_n[s_1 + e^{q_1 - p_1(x_n + y_n)}], \tag{10}$$

$$y_{n+1} = y_n e^{q_2 - p_2(x_n + y_n)}. \tag{11}$$

The system (1)–(2) is not the only competition model that has a local stable interior period-2 orbit that attracts all points of $\mathbb{R}_+^2$ except all pre-images of the heteroclinic curve C that is connecting two nontrivial boundary equilibria. In general, if a discrete two-species competition model satisfies the following conditions (see Figure 10 for a schematic presentation), numerical simulations suggest
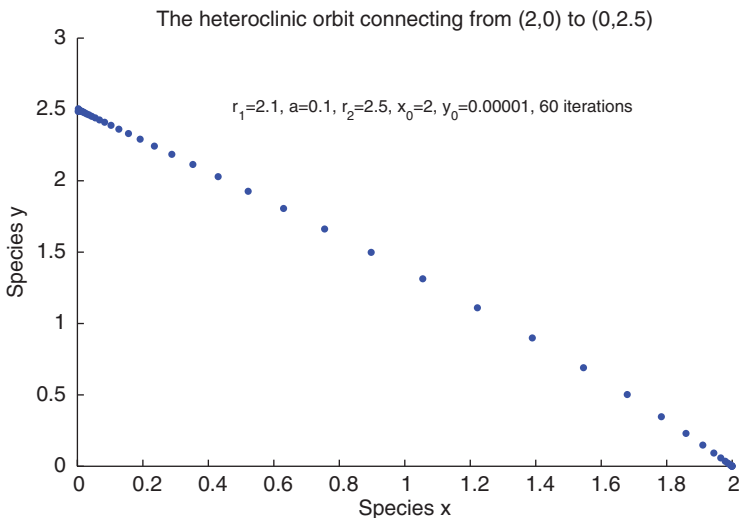

Figure 7. The heteroclinic orbit of the system (1)–(2) when $r_1 = 2.1, a = 0.1$ and $r_2 = 2.5$.
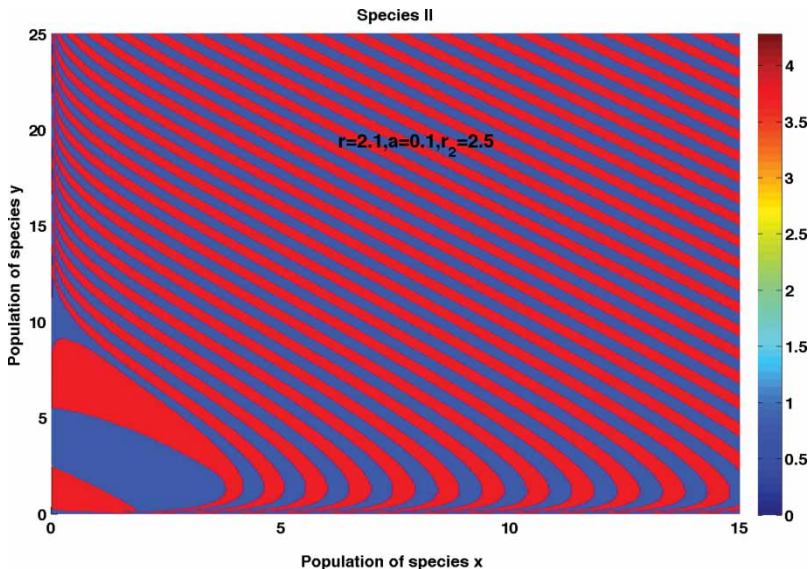
Figure 8. The basin of attraction of the interior period-2 orbit of the system (1)–(2) when $r_1 = 2.1$, $a = 0.1$ and $r_2 = 2.5$ is the open quadrant minus all pre-images of the heteroclinic curve C. The latter partitions the quadrant into components which are coloured according to which of the two periodic points attract points in the component under the second iterate of the map. Given a point in one of the regions, there is a large number N, such that the point will be very close to (x_1^i, y_1^i) at the iteration t and will be very close to (x_2^i, y_2^i) at the iteration $t + 1$ for all $t > N$.
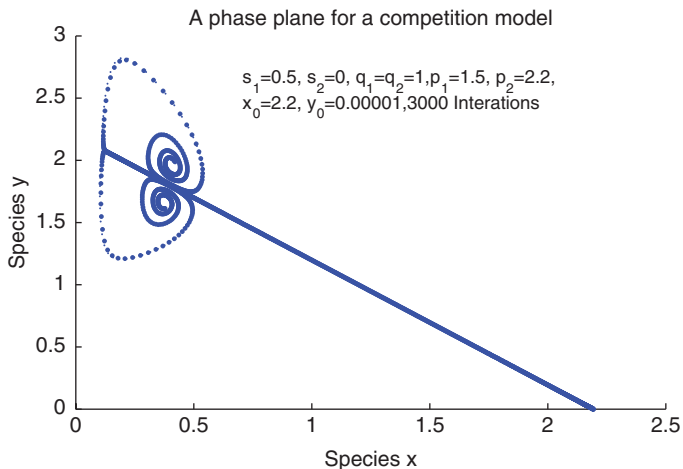

Figure 9. A single forward orbit of the system (10)–(11) starting near the fixed point on x-axis.

that it can have an interior attractor that attracts all points of $\mathbb{R}_+^2$ except all pre-images of the heteroclinic orbit that is connecting two nontrivial boundary equilibria. It will be our future work to develop more powerful analytic tools to rigorously prove this.

- G1: The system has only two nontrivial boundary equilibria $(x^*, 0)$ and $(0, y^*)$. Moreover, species y is persistent.
- G2: The omega limit set of y-axis is a unique attracting period-2 orbit $M_y = \{\eta_1, \eta_2\} = \{(0, y_1), (0, y_2)\}$ on y-axis, which attracts all points in y-axis except Lebesgue measure zero set.
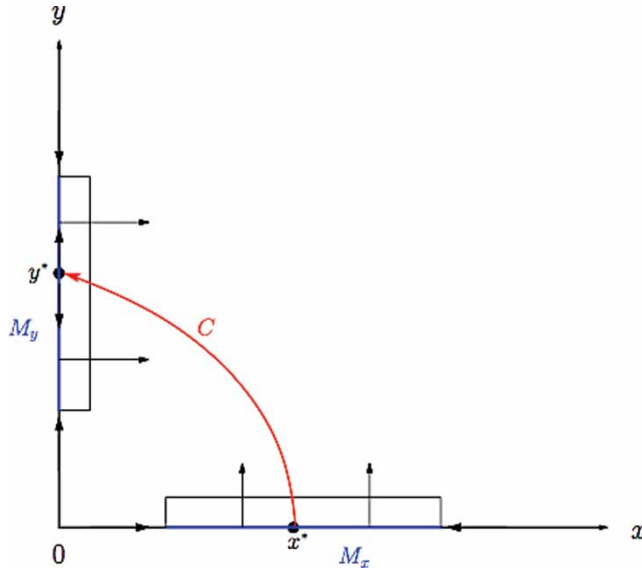
Figure 10. The general structure of dynamics that the basin of attraction of interior attractors (e.g., the interior periodic-2 orbit) is all points in $\mathbb{R}_+^2$ except the collection of all pre-images of the heteroclinic curve C.

In addition, the external Lyapunov exponent of M_y is greater than 1, i.e., species x can invade species y on M_y.

- G3: There is a heteroclinic orbit connecting the boundary equilibrium $(x^*, 0)$ to $(0, y^*)$.

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Appendix 1. Important lemmas

LEMMA A.1 (Uniform weak repeller)   *If the system* (3)–(4) *satisfies Condition C1–C3, then there exists some $b > 0$, such that*

$$\limsup_{n \to \infty} d(H^n(\xi), C) > b \quad \text{for any } \xi \in \tilde{X}.$$

Proof   The condition $2 < r_2 < 2.52$ indicates that species y has a unique attracting period-two orbit

$$\{\eta_1, \eta_2\} = \{(0, y_1), (0, y_2)\}$$

in its single state and the condition $r_1 < r_2$ implies that the boundary equilibrium $\eta^* = (0, r_2)$ is a saddle. By Hartman–Grobman–Cushing theorem [6], there exists some neighbourhood $U_{\epsilon_1}(\eta^*)$ of η^*, such that any point $\eta \in U_{\epsilon_1}(\eta^*) \cap \tilde{X}$ will exit from this neighbourhood in some finite time. If we choose ϵ small enough, then η is attracted to a compact neighborhood

$$B = \overline{U_\delta(M_y)} = \{\xi \in \tilde{X} : d(\xi, M_y) \leq \delta\}$$

where $M_y = \{\eta_1, \eta_2\}$ in some finite time. Similarly, the condition $0 < r_1 < r_2$ implies that the boundary equilibrium ξ^* is also a saddle, by Hartman–Grobman–Cushing theorem, there exists some neighbourhood $U_{\epsilon_2}(\xi^*)$ of ξ^*, such that any point $\xi \in U_{\epsilon_2}(\xi^*) \cap \tilde{X}$ will exit from this neighbourhood in some finite time.

Choose $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Let $K = \overline{C \setminus (U_\epsilon(\xi^*) \bigcup U_\epsilon(\eta^*))}$, then K is a compact subset of C. Since C is the closure of the family of heteroclinic orbits connecting ξ^* to η^*, then any point $\xi \in K$ will reach $U_\epsilon(\eta^*)$ in some finite time $m_\epsilon(\xi)$. Moreover, there exists a neighborhood of ξ, denoted by $U_{\delta_\xi}(\xi)$ will contain in $U_\epsilon(\eta^*)$ in time $m_\epsilon(\xi)$, i.e.,

$$H^{m_\epsilon(\xi)}(U_{\delta_\xi}(\xi)) \subset U_\epsilon(\eta^*).$$

Then we can see that

$$K \subset \bigcup_{\xi \in K} U_{\delta_\xi}(\xi).$$

Since K is compact, it has a finite open cover, i.e.,

$$K \subset \bigcup_{i=1}^{\bar{m}} U_{\delta_{\xi_i}}(\xi_i).$$

Choose $\delta = \min\{\epsilon, \min_{1 \leq i \leq \bar{m}}\{\delta_{\xi_i}\}\}$. Then any point $\xi \in \tilde{X}$ with $d(\xi, K) < \delta$, then there exists some $m_\xi = m_\epsilon(\xi_i)$, $1 \leq i \leq \bar{m}$, such that $H^{m_\xi}(\xi) \in U_\epsilon(\eta^*)$.

Now assume that the statement of Lemma A.1 is not true. Then for any k large enough, there exists some $\xi_k \in \tilde{X}$ and a positive integer n_k such that

$$d(H^n(\xi_k), C) < \frac{1}{k} \quad \text{for any } n \geq n_k. \tag{A1}$$

Choose $k > 1/\delta$. Then $d(H^{n_k}(\xi_k), C) < \delta$. We show the contradiction in the following three situations:

(1) If $H^{n_k}(\xi_k) \in U_\epsilon(\eta^*)$, then by Hartman–Grobman–Cushing theorem, $H^{n_k}(\xi_k)$ will exit from $U_\epsilon(\eta^*)$ in some finite time $n_\epsilon(\xi_k, \eta^*)$ and be attracted to a compact neighbourhood B in some finite time $l_\epsilon(\xi_k)$. Let $b = d(C, B)$, then we have

$$d(H^{n_k + n_\epsilon(\xi_k, \eta^*) + l_\epsilon(\xi_k)}(\xi_k), C) > b$$

which is a contradiction to Equation (A1).

(2) If $d(H^{n_k}(\xi_k), K) < \delta$, then there exists some $m_{\xi_k} = m_\epsilon(\xi_i)$, $1 \leq i \leq \bar{m}$, such that

$$H^{m_{\xi_k}}(H^{n_k}(\xi_k)) = H^{n_k + m_{\xi_k}}(\xi_k) \in U_\epsilon(\eta^*),$$

which we go back to the first case, therefore, there is a contradiction to (A1).

(3) If $H^{n_k}(\xi_k) \in U_\epsilon(\xi^*)$, then by Hartman–Grobman–Cushing theorem, $H^{n_k}(\xi_k)$ will exit from $U_\epsilon(\xi^*)$ in some finite time $n_\epsilon(\xi_k, \xi^*)$, i.e.,

$$d(H^{n_\epsilon(\xi_k, \xi^*)}(H^{n_k}(\xi_k)), \xi^*) = d(H^{n_k + n_\epsilon(\xi_k, \xi^*)}(\xi_k), \xi^*) \geq \epsilon.$$

From Equation (A1), we have

$$d(H^{n_k + n_\epsilon(\xi_k, \xi^*)}(\xi_k), C) < \delta$$

which we go back to either the first case or the second case, therefore, there is a contradiction to Equation (A1).

Based on the arguments above, we can conclude that the statement of Lemma A.1 is true.   ∎

LEMMA A.2 (Uniform weak persistence) *If the system* (3)–(4) *satisfies Condition C1–C3. Then there exists some* $\delta > 0$, *such that for any initial condition* $\xi_0 = (x_0, y_0) \in \tilde{X}$, *the system has*

$$\limsup_{n \to \infty} x_n > \delta.$$

Proof Since the system satisfies Condition C1–C3, then there exists a compact neighbourhood $W \subset S_y \cap \tilde{S}$ of the stable periodic-2 orbit $M_y = \{\eta_1, \eta_2\}$ attracting all points $(0, y) \in \tilde{S}$ from Theorem 4.3 [7]. Condition C3 implies that M_y is transversal unstable, i.e., its external Lyapunov exponent is greater than 1.

Define $P(\xi) = x$ where $\xi = (x, y) \in \tilde{X}$ and

$$r(t, \xi) = \begin{cases} cc\, \dfrac{P(H^t(\xi))}{P(\xi)} & (\xi \in \tilde{X}) \\ \lim_{\eta \in \tilde{X} \to \xi \in S_y \cap \tilde{S}} \inf \dfrac{P(H^t(\eta))}{P(\eta)} & (\xi \in S_y \cap \tilde{S}) \end{cases}$$

Then $r(t, \cdot)$ is lower semicontinuous. For $h > 0, t \geq 0$ set

$$\alpha(h, t) = \{\xi \in \tilde{X} : r(t, \xi) > 1 + h\}.$$

Then $\alpha(h, t)$ is an open set from the semicontinuity and it has property that $\alpha(h_1, t) \subset \alpha(h_2, t)$ if $h_1 > h_2$. Since W is compact, then there exists a $\bar{h} > 0$, and a finite increasing positive integers $k_i, i = 1 \ldots N$, such that

$$W \subset B \subset \bigcup_{i=1}^{N} \alpha(\bar{h}, k_i),$$

where

$$B = \overline{U_\epsilon(M_y)} = \{\xi \in \tilde{X} : \mathrm{d}(\xi, M_y) \leq \epsilon\}$$

is a compact neighborhood of W in $\tilde{X}$. We want to show that for any point $\xi = (x, y) \in B \backslash W$, its semi-orbit $\gamma^+(\xi)$ eventually exits from B with $x_n > \epsilon$ for some positive integer n. If this is not true, then there exists some point $\xi = (x, y) \in B \backslash W$ such that $H^n(\xi) \in B \backslash W$ for all $n \geq 0$. Since any point in B belongs to some $\alpha(\bar{h}, k_i), 1 \leq i \leq \bar{h}$. This implies that there is a sequence of integer $n_i \to \infty$ with $n_i - n_{i-1} \in \{k_1, \ldots, k_N\}$ for each i such that

$$P(H^{n_i}(\xi)) > P(H^{n_{i-1}}(\xi))[1 + \bar{h}] > x[1 + \bar{h}]^i \to \infty \quad \text{since } x > 0 \quad \text{and } n_i \to \infty$$

which is a contradiction to the fact that all points are attracted to the compact set D_ϵ. Thus, for any point $\xi = (x, y) \in B \backslash W$, its semi-orbit $\gamma^+(\xi)$ eventually exits from B with $x_n > \epsilon$ for some positive integer n. Combined with Lemma A.1, we can conclude that for any point $\xi \in \tilde{X}$ that is close enough to C, it will enter the compact neighbourhood B of M_y and exit from B in some finite time. Therefore, there exists some $\epsilon > 0$, such that for any initial condition $\xi \in \tilde{X}$, the system has

$$\limsup_{n \to \infty} x_n \geq \epsilon.$$

∎