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Pre-images of invariant sets of a discrete-time two-species competition model

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Pre-images of invariant sets of a discrete-time two-species competition model

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In this paper, we explore the structure of pre-images of invariant sets of a discrete-time two-species competition model with singularity at the origin. We first show that this competition model is persistent with respect to the total population of two species, i.e. all initial conditions in $\mathbb{R}^2_+ \setminus \{(0,0)\}$ are attracted to a compact set which is bounded away from the origin. Then we study the properties of pre-images of this system and give the explicit structure of all pre-images of invariant sets for this system under certain parameter range. These results are analogous to the one-dimensional discrete system. Our study is the first step to explore the structure of the basins of attractions of interior attractors of a general discrete-time two-species model (e.g. the locally asymptotically stable interior period-2 orbit). Finally, we discuss how our results give useful insights on the future study for coexistence of the species and list some open problems related to our system.

Keywords: basins of attraction; competition models; critical curves; invariant sets; persistent; rank-*k* pre-images

AMS 2000 Subject Classifications: Primary 37B25; 39A11; 54H20; Secondary 92D25

1. Introduction

It is common that ecosystems have multiple attractors, in which the final densities of species, or even the persistence of species, depend on their initial densities [8,12]. The basins of attractions of attractors can measure the resilience of ecosystems and the possible initial conditions allowing the coexistence of species. Thus, understanding the structure of the basins of attractions of their attractors can provide important information on the strategies for the sustainable management of such ecosystems [12]. Discrete-time ecological models are known to exhibit multiple attractors including periodic and chaotic attractors. However, except for a few exceptions (e.g. Elaydi and Sacker [4]), the qualitative studies of the structure of the basins of attraction of these periodic and chaotic attractors are rare. Since pre-images of invariant sets can define the boundaries of the basins of attractions of attractors, we explore the structure of pre-images of invariant sets of a discrete-time two-species competition model in this paper by using the properties of critical curves of non-invertible maps. This novel approach can also be extended to general discrete-time two-species interaction models.

Let x_n and y_n denote the population sizes of two competing species x and y at generation n, respectively. Suppose that species x suffers from the extremes of contest intra-specific

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competitive interaction and species y suffers from the extremes of scramble intra-specific competitive interactions [11], then a model of resource-mediated competition between species x and y can be described as follows in the model (1), (2) [1,5,6]

$$x_{n+1} = \frac{r_1 x_n}{a + x_n + y_n}, \tag{1}$$

$$y_{n+1} = y_n e^{r_2 - (x_n + y_n)}, \tag{2}$$

where all parameters r_1, r_2 and a are strictly positive. Franke and Yakubu [5,6] studied global dynamics of general two-species competition models including system (1), (2). In particular, they (1991b) established an exclusion principle for discrete competition models where the density-dependent growth functions are either all exponential or all rational, and gave an example that such exclusion principle fails where two species can coexist through a locally stable period-2 orbit. This phenomenon of coexistence has been observed in other competition models [2,3,13,14] as well as in system (3), (4) which is the case when $a = 0$ for system (1), (2)

$$x_{n+1} = \frac{r_1 x_n}{x_n + y_n}, \tag{3}$$

$$y_{n+1} = y_n e^{r_2 - (x_n + y_n)}. \tag{4}$$

It is easy to check that for any $a \geq 0$ and $r_1 \neq r_2$, system (1), (2) has only two equilibrium points $(r_1, 0)$ and $(0, r_2)$ (i.e. no interior equilibrium point) with the following property: one of them is transversally stable (i.e. the external Lyapunov exponent evaluated at this equilibrium point is less than 1) and the other one is transversally unstable (i.e. the external Lyapunov exponent evaluated at this equilibrium point is greater than 1). The simulation suggests that under certain parameter range, system (1), (2) has locally asymptotically stable interior periodic-2 orbits with heteroclinic orbits connecting two boundary equilibria points. In addition, the basins of attractions of these interior periodic-2 orbits attract all interior points except all the pre-images of heteroclinic orbits. In order to show this phenomenon observed from simulations, we need to understand the structure of all pre-images of heteroclinic orbits first. However, it is very challenging to show this for the general case rigorously. Thus, we focus on system (3), (4) which is the special case of system (1), (2). We will extend the work to more general case $a > 0$ and other more general models in the future study.

The structure of the rest of this paper is as follows. In Section 2, we give the basic terminologies used in the paper and introduce the concept of critical curves. In addition, we show that there exists a compact positively invariant set that attracts all points in R_+^2 (Theorem 6). In Section 3, we study the properties of the rank-1 pre-image of system (3), (4) and give the explicit structure of all rank pre-images of any invariant set (Theorem 9). This result is analogous to Theorem 4.2 by [4]. In Section 4, we summarize the results of our study and discuss how to apply these results to obtain the structure of the basins of attraction for the interior attractors of system (3), (4). In the Appendix, we derive a major technical lemma which will be used in proving the main results.

2. Definitions and compact positively invariant set

Let X be a metric space and H be a two-dimensional discrete system on X that is described by (3), (4). Then $H : X \to X$ is a discrete semi-dynamical system, where $H^0(\xi_0) = \xi_0 =$

(x_0, y_0) and $H^n(\xi_0) = \xi_n = (x_n, y_n), n \in \mathbb{Z}_+$. Note that system (3), (4) has singularity at the origin (0, 0); thus, its state space X can be defined as $X = \{(x, y) \in \mathbb{R}^2_+ : 0 < x + y < \infty\}$. The local dynamics of (3), (4) can be summarized as follows:

Lemma 1 [Local Dynamics of (3), (4)].

1. *If $r_1 \neq r_2$, then system* (3), (4) *has only two equilibria $\xi^* = (r_1, 0)$ and $\eta^* = (0, r_2)$. If $r_1 > r_2$, then ξ^* is locally asymptotically stable and η^* is transversally unstable; if $r_1 < r_2$, then ξ^* is transversally unstable and η^* is transversally stable.*
2. *If $r_1 = r_2 = r$, then all points in the line $\{(x, y) \in \mathbb{R}^2_+ : x + y = r\}$ are equilibria of system* (3), (4).
3. *If system* (3), (4) *has an interior periodic-*2 *orbit $P_2^i = \{(x_1^i, y_1^i), (x_2^i, y_2^i)\}$, then it can be explicitly found as*

$$x_1^i = \frac{s_1(s_1 e^{r_2 - s_1} - s_2)}{s_1 e^{r_2 - s_1} - r_1}, \quad y_1^i = \frac{s_1(s_2 - r_1)}{s_1 e^{r_2 - s_1} - r_1},$$
$$x_2^i = \frac{r_1 x_1^i}{s_1}, \quad y_2^i = y_1^i e^{r_2 - s_1}$$

where

$$s_1 = x_1^i + y_1^i = r_2 - \sqrt{r_2^2 - r_1^2} \quad \text{and} \quad s_2 = x_2^i + y_2^i = r_2 + \sqrt{r_2^2 - r_1^2}.$$

The proof of Lemma 1 can be obtained from straight algebraic calculations; thus, we omit the details. Lemma 1 indicates that system (3)–(4) is not permanent since it has no interior steady state (Theorem 6.3, Hutson and Schmitt [7]). In the case that $r_1 \neq r_2$, the system always has one boundary steady state that is transversally stable. However, it can have a locally asymptotically stable interior period-2 orbit P_2^i for certain r_1, r_2 values as observed in other models [2,3,6,13,14]. For instance, when $r_1 = 2$, $r_2 = 2.2$, system (3), (4) has a locally asymptotically stable interior periodic-2 orbit

$$P_2^i = \{(x_1^i, y_1^i), (x_2^i, y_2^i)\} = \{(0.1536, 2.9629), (0.0986, 1.1849)\}$$

at which the eigenvalues of the product of the Jacobian matrix along these orbits are 0.91 and 0.26.

Denote $B(P_2^i)$ as the basins of attractions of P_2^i. The interesting question is: What is the structure of $B(P_2^i)$? This question has been posed as an open problem for a discrete competition model studied by [2]. As we know for planar maps, the basins of attraction of omega limit sets can have very complicated structures. Instead of studying the structure of $B(P_2^i)$ directly, we study the pre-images of invariant sets instead. Now, we give some important definitions.

Definition 2 [Pre-images of a point]. *For a given point $\xi_0 \in X$, we say $\xi \in X$ is a rank-k pre-image of ξ_0 if $H^k(\xi) = \xi_0$. The collection of rank-k ($k \geq 1$) pre-images of ξ_0 is defined as*

$$H^{-k}(\xi_0) = \{\xi \in X : H^k(\xi) = \xi_0\}$$

and the collection of all pre-images of ξ_0 (including $k = 0$) is defined as

$$EF_{\xi_0} = \left(\bigcup_{k \geq 1} H^{-k}(\xi_0)\right) \bigcup \{\xi_0\}.$$

DEFINITION 3 [INVARIANT/POSITIVELY INVARIANT SET]. *We say $M \subset X$ is an invariant set of H if $H(M) = M$. And $M \subset X$ is a positively invariant set of H if $H(M) \subset M$.*

Note. If M is an invariant set of H, then M includes both positively invariant orbits and negatively invariant orbits but not all negatively invariant orbits.

DEFINITION 4 [PRE-IMAGES OF AN INVARIANT SET]. *Let M be an invariant set for system* (3), (4), *then $H^0(M) = H(M) = M$. The collection of rank-k pre-images of M ($k \geq 1$) is defined as*

$$H^{-k}(M) = \bigcup_{\xi_0 \in M} \{\xi \in X \backslash M : H^k(\xi) = \xi_0\};$$

and the collection of all pre-images of M (including $k = 0$) is defined as

$$EF_M = \bigcup_{k \geq 0} H^{-k}(M) = \left[\bigcup_{k \geq 1} \left(\bigcup_{\xi_0 \in M} \{\xi \in X \backslash M : H^k(\xi) = \xi_0\} \right)\right] \bigcup M.$$

Note. If M is an invariant set of H, then $H^{-k}(M)$ should not contain points in M for all $k \geq 1$.

If $r_2 > r_1$, then from Lemma 1 we know that η^* is transversally stable. Let $W^s(\eta^*)$ be the stable manifold associated with this transversal direction. Then $EF_{W^s(\eta^*)}$ contains all pre-images of $W^s(\eta^*)$, which is invariant and does not belong to $B(P_2^i)$. The simulation (Figure 4) also suggests that all interior points of $\mathbb{R}^2_+$ are attracted to P_2^i except $EF_{W^s(\eta^*)}$. In order to show this rigorously, we investigate the structure of pre-images of invariant sets for system (3), (4), which includes $EF_{W^s(\eta^*)}$ in this paper.

Note that while species x is absent, system (3), (4) reduces to the following well known Ricker's map

$$y_{t+1} = y_n e^{r_2 - y_n}. \tag{5}$$

According to Theorem 4.2 (Elaydi and Sacker [4]), single species system (5) has a stable 2-cycle $\{y_1, y_2\}$ when $2 < r_2 < 2.52$ with its basins of attraction being $\{y \in \mathbb{R}_+ : y > 0\}\backslash EF_y$ where EF_y is a set of all pre-images of $y = r_2$ (EF_y is called the set of *eventually fixed points* by [4]). Moreover, they gave the explicit structure of EF_y as follows:

$$EF_y = \Omega^-(y) \bigcup \{q_{-n} : n \geq 2\}$$

where

1. $\Omega^-(y) = \{y_{-n} : n \geq 1\}$ is a full negative orbit with $y_n \to 0$ monotonically.
2. For each $n \geq 2$, there exists a unique monotonically increasing sequence of rank-n pre-image q_{-n} such that

$$\lim_{n\to\infty} q_{-n} = \infty \text{ and } q_{-n} \text{ is rank-1 pre-image of } y_{-n+1}.$$

Our study in this paper can answer the interesting question such as whether pre-images of the invariant sets of (3), (4), e.g. $EF_{W^s(\eta^*)}$, have the similar structure as E_y. To continue our study, we define the critical curves of (3), (4) in the next subsection.

2.1 Critical curves

Notice that the system H, i.e. (3), (4), is a two-dimensional non-invertible map. As the point $\xi_0 = (x_0, y_0)$ varies in its state space X, the number of rank-1 pre-images of ξ_0 changes. Real pre-images appear or disappear as the point ξ_0 crosses the boundary separating regions for which the points have a different number of rank-1 pre-images. Such boundaries, called *rank*-1 *critical curve* and denoted by LC, are generally characterized by the presence of multiple coincident (merging) pre-images. The locus of these *coincident first rank pre-images* is called *rank*-1 *curve of merging pre-images* and denoted by LC_{-1}.

The critical curve of rank-1 LC is the two-dimensional generalization of the notion of critical value (local minimum or maximum value) of a one-dimensional map (for the Ricker's map $h(y) = y\mathrm{e}^{r_2-y}$, it has $LC = \mathrm{e}^{r_2-1}$), LC_{-1} is the generalization of the notion of critical point (the image of local extremum point, e.g. the Ricker's map h has its extreme point as $LC_{-1} = 1$). Arcs of LC separate the plane into regions characterized by a different number of real pre-images. For convenience, $H^k(LC_{-1})$, $k \geq 1$ are called *critical curves*.

For a two-dimensional continuous map H, the set LC_{-1} is included in the set of points, denoted by J_C, through which $\det(J)(x, y)$ (i.e. the determinant of Jacobian matrix) changes sign. From the geometric action of the foliation of the Riemann plane, we can also say that the critical set LC_{-1} must belong to J_C, i.e. $LC_{-1} \subset J_C$. In fact, a plane region U which intersects LC_{-1} is 'folded' along LC to the side with more pre-images, and the two folded images have opposite orientation [10]. This implies that the map has the Jacobian matrix with a different sign in the two portions of U separated by LC_{-1}. Our map H is smooth in X; thus, the sign of $\det(J)$ occurs when it vanishes and $LC = H(LC_{-1})$ constitutes the boundary lines which separate regions Z_k characterized by a k number of rank-1 pre-images (for system (3), (4), $k = 0, 1, 2, \infty$, which we demonstrate in Lemma 7).

The Jacobian matrix of system (3), (4) is represented as

$$J = \begin{bmatrix} \frac{r_1 y}{(x+y)^2} & -\frac{r_1 x}{(x+y)^2} \\ -y\mathrm{e}^{r_2-x-y} & -\mathrm{e}^{r_2-x-y}(y-1) \end{bmatrix} \tag{6}$$

with its determinant

$$\det(J) = -\frac{r_1 y\mathrm{e}^{r_2-x-y}(x+y-1)}{(x+y)^2}.$$

This gives two critical curves $LC_{-1} = LC^1_{-1} \cup LC^2_{-1}$ of system (3), (4), where the determinant of Jacobian $\det(J)$ vanishes in X, i.e.

$$LC^1_{-1} : y = 1 - x, \quad 0 \leq x < 1, \tag{7}$$

$$LC^2_{-1} : y = 0, \quad x > 0. \tag{8}$$

The curve LC^2_{-1} is an invariant manifold of (3),(4) and is mapped to the rank-1 critical curve $LC^2 = (r_1, 0)$ through H, i.e. $H\big(LC^2_{-1}\big) = (r_1, 0)$. The other critical curve $LC^1 = H\big(LC^1_{-1}\big)$ is given by

$$LC^1 : \big(r_1 x, (1-x)\mathrm{e}^{r_2-1}\big), \quad 0 \leq x < 1$$

which can be simplified as follows:

$$LC^1 : g(x) = \left(1 - \frac{x}{r_1}\right)e^{r_2-1}, \ \ 0 \le x < r_1 \tag{9}$$

Both $LC^1_{-1} : y = 1 - x, 0 \le x < 1$ and $LC^1 : g(x) = (1 - x/r_1)e^{r_2-1}, \quad 0 \le x < r_1$ are linear decreasing functions with respect to x. Moreover, the curves $LC^1_{-1}, LC^1 = H\left(LC^1_{-1}\right)$ do not have intersections if $r_2 > r_1 > 1$ or $\max\{r_1, e^{r_2-1}/r_1\} < 1$, and $LC^1_{-1} = H^{-1}(LC^1)$ is a one-to-one mapping. In the next subsection, we show that H has a compact positively invariant set attracting all points in X.

2.2 Compact positively invariant set

Recall that

$$X = \left\{(x,y) \in \mathbb{R}^2_+ : 0 < x + y < \infty\right\},$$

we denote

$$\begin{aligned}
&Z_\infty = \{(r_1, 0)\} = LC^2,\\
&Z_0 = \{(x,y) \in X : y > g(x)\} \bigcup \{(x,0) : x > 0, x \neq r_1\},\\
&Z_1 = \{(x,y) \in X : y = g(x), 0 \le x < r_1\} = LC^1,\\
&Z_2 = \{(x,y) \in X : 0 < y < g(x), 0 \le x < r_1\},\\
&Z_{1,2,\infty} = Z_1 \bigcup Z_2 \bigcup Z_\infty,\\
&Z_{\text{all}} = \{(x,y) \in X : 0 \le x \le r_1, 0 \le y \le g(x)\},\\
&Z_2^1 = \left\{(x,y) \in Z_2 : 0 < y < f(f(y_c)) = \left(1 - \frac{x}{r_1}\right)e^{2r_2-1-e^{r_2-1}}\right\},\\
&Z_2^2 = \left\{(x,y) \in Z_2 \backslash Z_2^1\right\},\\
&B = \{(x,y) \in Z_2 : x + y < 1\}
\end{aligned} \tag{10}$$

where $g(x)$ is given by (9), $y_c = 1 - (x/r_1)$ and $f(y) = ye^{r_2-y/(r_1-x)}$. These defined notations are used throughout the rest of this paper.

LEMMA 5 [Z_2 POSITIVELY INVARIANT]. *System H defined by* (3), (4) *maps X to $Z_{1,2,\infty}$, where $Z_{1,2,\infty}$ is positively invariant. Moreover, Z_{all} is also positively invariant and attracts all points in X.*

Proof. Given any point $\xi = (x,y) \in X$, we have the following three cases:

1. If $y = 0$, then $\xi_1 = H(\xi) = (r_1, 0) = Z_\infty$;
2. If $x = 0$, then $\xi_1 = H(\xi) = (0, ye^{r_2-y}) \in Z_1 \cup Z_2$;
3. If $xy > 0$, then y can be represented as kx for some positive number k; thus, we have

$$x_1 = \frac{r_1 x}{x + y} = \frac{r_1}{k + 1}, \tag{11}$$

$$y_1 = kxe^{r_2-(k+1)x} \le \frac{k}{k+1}e^{r_2-1} = \left(1 - \frac{x_1}{r_1}\right)e^{r_2-1} = g(x_1). \tag{12}$$

The last two cases imply that for any point ξ in the region $\{(x,y) \in X : y > 0\}$, we have

$$\xi_1 = H(\xi) = (x_1, y_1) \in \{(x,y) \in X : 0 \le x < r_1, 0 < y \le g(x)\} = Z_1 \cup Z_2.$$

Hence, H maps X to $Z_{1,2,\infty}$. Since $Z_{1,2,\infty}$ is a subset of X; therefore, $Z_{1,2,\infty}$ is positively invariant and attracts all points in X.

Note that $Z_{1,2,\infty}$ is a subset of Z_{all} and Z_{all} is a subset of X; therefore,

$$H(Z_{\text{all}}) \subset H(X) \subset Z_{1,2,\infty} \subset Z_{\text{all}}$$

which implies that Z_{all} is also positively invariant and attracts all points in X. □

Lemma 5 indicates that the dynamics of (3), (4) can be restricted to a bounded positively invariant region Z_{all}. However, the non-compactness of Z_{all} is inconvenient for us to study the global dynamics of (3), (4). The following theorem shows that there exists a compact positive invariant set that attracts all points in X.

Theorem 6 [Compact positively invariant sets]. *Assume that $r_1 \neq r_2$, then for any ϵ such that the following inequality holds*

$$0 < \epsilon \le \min\left\{r_1, r_2, \mathrm{e}^{2r_2-1-\mathrm{e}^{r_2-1}}, r_1\mathrm{e}^{r_2-r_1}\right\} = r^m,$$

the compact region defined by

$$D_\epsilon = \{(x_0, y_0) \in Z_{\text{all}} : x_0 + y_0 \ge \epsilon\}$$

is positively invariant and attracts all points in X.

Proof. From Lemma 5, we can restrict system (3), (4) to the bounded positively invariant region Z_{all}. Assume that a point $\xi_0 = (x_0, y_0)$ is in the region defined by $\{(x,y) \in Z_{\text{all}} : y > 0\}$, then it can be characterized by two variables m and k where

$$0 < m = x_0 + y_0 \le \max\{r_1, \mathrm{e}^{r_2-1}\} \text{ and } x_0 = ky_0, \ \ 0 \le k < \infty.$$

Thus, we have

$$(x_0, y_0) = \left(\frac{km}{k+1}, \frac{m}{k+1}\right).$$

According to (3), (4), the first iteration of ξ_0 can be represented as

$$\xi_1 = (x_1, y_1) = H(\xi_0) = \left(\frac{r_1 k}{k+1}, \frac{m\mathrm{e}^{r_2-m}}{k+1}\right).$$

Denote the sum of x_1 and y_1 as the following new function $F(m,k)$

$$F(m,k) = x_1 + y_1 = \frac{r_1 k}{k+1} + \frac{m\mathrm{e}^{r_2-m}}{k+1}. \tag{13}$$

Then we have

$$\frac{\partial F}{\partial k} = \frac{r_1 - me^{r_2 - m}}{(k+1)^2}, \tag{14}$$

$$\frac{\partial F}{\partial m} = \frac{(1-m)e^{r_2 - m}}{k+1}. \tag{15}$$

Equation (14) indicates that for a given m, if $r_1 < me^{r_2-m}$, then $\partial F/\partial k < 0$, hence,

$$\inf_{x_0+y_0=m}\{x_1+y_1\} = \inf_{k\geq 0}\{F(k,m)\} = \lim_{k\to\infty} F(k,m) = r_1.$$

And if $r_1 \geq me^{r_2-m}$, then $\partial F/\partial k \geq 0$, hence,

$$\inf_{x_0+y_0=m}\{x_1+y_1\} = \inf_{k\geq 0}\{F(k,m)\} = F(0,m) = me^{r_2-m}.$$

Now let m vary from

$$r^m = \min\left\{r_1, r_2, e^{2r_2-1-e^{r_2-1}}, r_1 e^{r_2-r_1}\right\} \text{ to } r^M = \max\left\{r_1, e^{r_2-1}\right\}.$$

Note that

$$\left\{(x_0,y_0)\in Z_{\text{all}} : r^m \leq x_0+y_0 \leq r^M\right\} \subseteq \left\{(x_0,y_0)\in X : r^m \leq x_0+y_0 \leq r^M\right\}.$$

Thus, if $(x_0, y_0) \in Z_{\text{all}}$, then from part 3 of Lemma 11, we have

$$\inf_{r^m\leq m\leq r^M}\inf_{x_0+y_0=m}\{x_1+y_1\} \geq \inf_{r^m\leq m\leq r^M}\inf_{k\geq 0}\{F(k,m)\} = \min_{r^m\leq m\leq r^M}\{r_1, me^{r_2-m}\} \geq r^m. \tag{16}$$

Now define

$$D_\epsilon = \{(x_0,y_0)\in Z_{\text{all}} : \epsilon \leq x_0+y_0\} \text{ and } D_\epsilon^c = Z_{\text{all}} \backslash D_\epsilon,$$

then

$$D_\epsilon \subset \left\{(x_0,y_0)\in X : \epsilon \leq x_0+y_0 \leq r^M\right\}.$$

If $\epsilon = r^m$, then according to (16) and Lemma 5, we can see that if a point (x_0, y_0) is in D_{r^m}, then its first iteration (x_1, y_1) is still in D_{r^m} since $(x_1, y_1) \in Z_{\text{all}}$ and $x_1 + y_1 \geq r^m$. Therefore, D_{r^m} is positively invariant.

Let (x_0, y_0) be a point in the region $D^c_{r^m}$, then

$$x_0 + y_0 < r^m = \min\left\{r_1, r_2, e^{2r_2-1-e^{r_2-1}}, r_1 e^{r_2-r_1}\right\}.$$

Thus, we have

$$x_1 = \frac{r_1 x_0}{x_0+y_0} > x_0 \text{ and } y_1 = y_0 e^{r_2-(x_0+y_0)} > y_0$$

which indicates that we have the following two situations:

1. $\xi_0 = (x_0, y_0)$ stays in $D^c_{r^m}$ for all future time. In this case, we have a strictly increasing sequence $\{H^i(\xi_0)\}_{i=1}^{\infty}$ where

$$H^i(\xi_0) \in D^c_{r^m} \text{ and } H^i(\xi_0) \ll H^{i+1}(\xi_0) \text{ for all } i \in \mathbb{Z}_+$$

where $\ll$ is the strong usual component-wise order relation. Therefore,

$$\lim_{i\to\infty} H^i(\xi_0) = \xi^*$$

where $\xi^* = (r_1, 0)$ or $(0, r_2)$ is a boundary equilibrium of system H and is contained in D_{r^m}.

2. There exists some positive integer k, such that $\xi_k = H^k(\xi_0) \in D_{r^m}$, then $\xi_n \in D_{r^m} \subset D_{r^m}$ for all $n \geq k$.

The above argument indicates that D_{r^m} attracts all points in Z_{all}. Since Z_{all} attracts all points in X; therefore, D_{r^m} attracts all points in X.

If $\epsilon < r^m$, then D_ϵ is a compact neighbourhood of D_{r^m}. Since D_{r^m} attracts all points in X, any point in X will enter D_ϵ in some finite time. By applying the similar arguments (the above two steps for showing that D_{r^m} attracts all points in Z_{all}), we can show that any point in D_ϵ will either stay in D_ϵ for all future time and converge to one of the two boundary equilibria or will enter D_{r^m} in some finite time and stay in D_{r^m} for all future time. Therefore, D_ϵ is positively invariant and attracts all points in X. □

Theorem 6 indicates that system (3), (4) is persistent with respect to the total population of species x and y, i.e. if $r_1 \neq r_2$, then for any initial condition $(x_0, y_0) \in X$ we have

$$\liminf_{n\to\infty} (x_n + y_n) \geq r^m.$$

This result allows us to restrict system (3), (4) to the compact positively invariant region D_{r^m}.

3. Pre-images of invariant sets

In this section, we explore the properties of all rank pre-images of invariant sets of system (3), (4). The following two lemmas give us useful information on the properties of pre-images of points in X, which can help us to find the explicit structure of all rank pre-images of invariant sets.

Lemma 7 [$Z_0 - Z_2$ map]. *Each point in $Z_i, i = 0, 1, 2, \infty$ has i rank-1 pre-images. If $\xi \in Z_2$, then its rank-1 pre-images can be represented as $\xi^i_{-1}, i = a, b$ with the following properties*

$$H^{-1}(\xi) = \left\{\xi^a_{-1}, \xi^b_{-1}\right\} \text{ and } \xi^b_{-1} \ll \eta \ll \xi^a_{-1}$$

where $\ll$ is the strong usual component-wise order relation and η is a point in LC^1_{-1}. Moreover, the maps

$$G^b_{-1} : Z_2 \to B \text{ and } G^a_{-1} : Z_2 \to \mathbb{X}$$

defined by $G^i_{-1}(\xi) = \xi^i_{-1}, i = a, b$ are smooth.

Proof. Note that $H\left(LC_{-1}^2\right) = (r_1, 0)$; thus, $LC_{-1}^2 \backslash \{(r_1, 0)\}$ has no pre-images and $(r_1, 0)$ has infinite many pre-images, i.e. $\left(LC_{-1}^2 \backslash \{(r_1, 0)\}\right) \subset Z_0$ and $Z_\infty = \{(r_1, 0)\}$.

Given any point $(x_0, y_0) \in X \backslash LC_{-1}^2$, if it has pre-images (x, y), then the pre-images should satisfy the following two equations

$$x_0 = \frac{r_1 x}{x + y} \Rightarrow x = \frac{x_0 y}{r_1 - x_0}, \tag{17}$$

$$y_0 = y e^{r_2 - x - y}. \tag{18}$$

Now substituting (17) into (18) we get

$$y_0 = y e^{r_2 - \frac{r_1 y}{r_1 - x_0}} \leq \max_{y>0} \left\{ y e^{r_2 - \frac{r_1 y}{r_1 - x_0}} \right\} = f(y_c) = g(x_0) = \left(1 - \frac{x_0}{r_1}\right) e^{r_2 - 1} \tag{19}$$

where $y_c = 1 - x_0/r_1$ and $f(y) = y e^{r_2 - (r_1 y / r_1 - x_0)}$. Therefore, according to the properties of the Ricker's map 11 (also see results in [4]), we have the following three situations:

1. If $y_0 > g(x_0) > 0$, then y_0 has no pre-image for the Ricker's map (19); thus, (x_0, y_0) has no pre-image for (3), (4). Hence, we have shown that the region Z_0 defined in (10) has no pre-images.
2. If $y_0 = g(x_0) > 0$, then y_0 has exactly one rank-1 pre-image for the Ricker's map Lemma (19); thus, (x_0, y_0) has exactly one rank-1 pre-image.
3. If $0 < y_0 < g(x_0)$, then y_0 has two rank-1 pre-images for the Ricker's map (19); thus, (x_0, y_0) has two rank-1 pre-images.

Therefore, the first part of Lemma 7 holds.

Choose a point $\xi = (x_0, y_0) \in Z_2$, its two rank-1 pre-images can be denoted as

$$H^{-1}(\xi) = \left\{\xi_{-1}^a, \xi_{-1}^b\right\} = \left\{\left(x_{-1}^a, y_{-1}^a\right), \left(x_{-1}^b, y_{-1}^b\right)\right\}$$

where $\xi_{-1}^i, i = a, b$ satisfy (17) and (19).

Denote $\eta = (x_0/r_1, r_1 - x_0/r_1)$. Since

$$\eta = (1 - y, y) \text{ if let } y = \frac{r_1 - x_0}{r_1},$$

we can check that the point $\eta \in LC_{-1}^1$ by using (7). Note that $y = r_1 - x_0/r_1$ is a critical point of $y e^{r_2 - (r_1 y / r_1 - x_0)}$, then from (19), we can see that the pre-image

$$\xi_{-1}^b = \left(\frac{x_0 y_{-1}^b}{r_1 - x_0}, y_{-1}^b\right) \ll \eta \Rightarrow x_{-1}^b + y_{-1}^b < 1 \Rightarrow \xi_{-1}^b \in B$$

and the pre-image

$$\xi_{-1}^a = \left(\frac{x_0 y_{-1}^a}{r_1 - x_0}, y_{-1}^a\right) \gg \eta \Rightarrow x_{-1}^b + y_{-1}^b > 1.$$

Therefore, for each interior point $\xi \in Z_2$, there are two rank-1 pre-images $\xi_{-1}^i, i = a, b$ and a point $\eta \in LC_{-1}^1$, such that

$$\xi_{-1}^b \ll \eta \ll \xi_{-1}^a$$

where $\xi_{-1}^b \in B$.

From the inverse function theorem, we can conclude that the maps

$$G^a_{-1} : Z_2 \to B, G^b_{-1} : Z_2 \to \mathbb{X}$$

defined by $G^i_{-1}(\xi) = \xi^i_{-1}, i = a, b$ are smooth. □

In the rest of the paper, we use subscript $-n$ to denote a rank-n pre-image and superscript $i = a, b$ to denote the location of the pre-image, i.e. if $i = a$, then the pre-image is above LC^1_{-1}, otherwise it is below LC^1_{-1} and, therefore, in the set B.

LEMMA 8 [LOCATION OF PRE-IMAGES]. *Let* $\left(x^i_{-k}, y^i_{-k}\right), i = a, b$ *be rank-k pre-images of a point* $(x_0, y_0) \in Z_2$ *and* $\left(u^i_{-k}, u^i_{-k}\right), i = a, b$ *be rank-k pre-images of a point* $(u_0, v_0) \in Z_2$ *where* $k = 1,2$. *Then*

1. $\left(x^b_{-1}, y^b_{-1}\right) \ll (x_0, y_0)$ *if* $0 < x_0 + y_0 \leq \min\{r_1, r_2\}$.
2. $\left(x^a_{-1}, y^a_{-1}\right) \in Z_0$ *if* $r_2 > r_1 > 1$ *and* $(x_0, y_0) \in Z^1_2$. *Moreover, we have*

$$\left(x^b_{-2}, y^b_{-2}\right) \ll \left(x^b_{-1}, y^b_{-1}\right) \text{ and } y^a_{-1} < y^a_{-2}.$$

3. $\left(x^b_{-1}, y^b_{-1}\right) \ll \left(u^b_{-1}, v^b_{-1}\right)$ *if* $r_2 > 1$ *and* $(x_0, y_0) \ll (u_0, v_0)$.

Proof. If $(x_0, y_0) \in Z_2$, then $0 < x_0 + y_0 \leq \min\{r_1, r_2\}$ implies that $0 < y_0 \leq r_1 - x_0$. Thus, according to (17), we have

$$x^b_{-1} = \frac{x_0 y^b_{-1}}{r_1 - x_0} \leq \frac{x_0 y^b_{-1}}{y_0}. \tag{20}$$

On the other hand, according to (19), the following equality holds

$$y_0 = y^b_{-1} \mathrm{e}^{r_2 - \frac{r_1 y^b_{-1}}{r_1 - x_0}}. \tag{21}$$

The condition $0 < x_0 + y_0 \leq \min\{r_1, r_2\}$ implies that if $r_2 \leq r_1$, then

$$0 < x_0 + y_0 \leq r_2 \Rightarrow 0 < \frac{x_0}{r_2} + \frac{y_0}{r_2} \leq 1 \Rightarrow 0 < \frac{x_0}{r_1} + \frac{y_0}{r_2} \leq 1 \Rightarrow y_0 < y^* = r_2\left(1 - \frac{x_0}{r_1}\right).$$

Similarly, if $r_2 \geq r_1$, then

$$0 < x_0 + y_0 \leq r_1 \Rightarrow 0 < \frac{x_0}{r_1} + \frac{y_0}{r_1} \leq 1 \Rightarrow 0 < \frac{x_0}{r_1} + \frac{y_0}{r_2} \leq 1 \Rightarrow y_0 < y^* = r_2\left(1 - \frac{x_0}{r_1}\right).$$

Thus, according to Lemma 11 (also see results in [4]), we have $0 < y^b_{-1} < y_0$. Hence, according to (20), we can conclude that

$$x^b_{-1} \leq \frac{x_0 y^b_{-1}}{y_0} < x_0.$$

Therefore, the first part of Lemma 8 holds, i.e.

$$\left(x^b_{-1}, y^b_{-1}\right) \ll (x_0, y_0) \ \text{ if } \ 0 < x_0 + y_0 \leq \min\{r_1, r_2\}.$$

The condition $r_2 > r_1 > 1$ indicates that $(e^{r_2-1}/r_1) > 1$ and $B \subset Z^1_2$. If $(x_0, y_0) \in Z^1_2$, then $0 < y_0 < (1 - x_0/r_1)e^{2r_2-1-e^{r_2-1}}$. Therefore, according to Lemma 11 (also see results in [4]) and the inequality $(e^{r_2-1}/r_1) > 1$, we have

$$y^a_{-1} > \left(1 - \frac{x_0}{r_1}\right)e^{r_2-1}. \tag{22}$$

From (17) and (22), we have

$$x^a_{-1} = \frac{x_0 y^a_{-1}}{r_1 - x_0} > \frac{x_0 e^{r_2-1}}{r_1} > x_0. \tag{23}$$

This indicates that

$$y^a_{-1} > \left(1 - \frac{x_0}{r_1}\right)e^{r_2-1} > \left(1 - \frac{x^a_{-1}}{r_1}\right)e^{r_2-1}.$$

Therefore, $\left(x^a_{-1}, y^a_{-1}\right) \in Z_0$.

From the proof of the first part of Lemma 8, we can see that

$$\left(x^b_{-1}, y^b_{-1}\right) \ll (x_0, y_0) \Rightarrow \left(x^b_{-1}, y^b_{-1}\right) \in B \subset Z^1_2.$$

Thus, by applying the first part of Lemma 8, we have $\left(x^b_{-2}, y^b_{-2}\right) \ll \left(x^b_{-1}, y^b_{-1}\right)$ directly.

Note that $ye^{r_2-(r_1 y/r_1 - x_0)}$ is a decreasing function with respect to x_0. Thus,

$$y_0 = y^a_{-1}e^{r_2 - \frac{r_1 y^a_{-1}}{r_1 - x_0}} > y^b_{-1} = y^a_{-2}e^{r_2 - \frac{r_1 y^a_{-2}}{r_1 - x^b_{-1}}} > y^a_{-2}e^{r_2 - \frac{r_1 y^a_{-2}}{r_1 - x_0}}$$

Since $r_2 > r_1 > 1$, the fixed point $y^* = r_2 - (r_2 x/r_1)$ is strictly greater than the critical point $y_c = 1 - (x/r_1)$ for the Ricker's map $ye^{r_2-(r_1 y/r_1 - x)}$ where $x = x_0$ or $x = x^b_{-1}$. Let $u^i_{-1}, i = a, b$ be rank-1 pre-images of y^b_{-1} for the Ricker's map $ye^{r_2-(r_1 y/r_1 - x_0)}$. Since $ye^{r_2-(r_1 y/r_1 - x_0)} > ye^{r_2-(r_1 y/r_1 - x^b_{-1})}$, we have $u^a_{-1} < y^a_{-2}$. Recall that

$$u^a_{-1}e^{r_2-(r_1 u^a_{-1}/r_1 - x_0)} = y^b_{-1} < y_0 = y^a_{-1}e^{r_2-(r_1 y^a_{-1}/r_1 - x_0)};$$

therefore, we have $y^a_{-1} < u^a_{-1}$. Hence, $y^a_{-1} < y^a_{-2}$ (see Figure 1(a) for graphic representations). Therefore, the second part of Lemma 8 holds.

Since $(x_0, y_0) \ll (u_0, v_0)$, we have $x^b_{-1} = x_0 y^b_{-1}/(r_1 - x_0) < u_0 y^b_{-1}/(r_1 - u_0)$. The condition $r_2 > 1$ implies that

$$\begin{aligned} y^b_{-1}e^{r_2-(r_1 y^b_{-1}/r_1 - u_0)} &< y_0 = y^b_{-1}e^{r_2-(r_1 y^b_{-1}/r_1 - x_0)} < v_0 = v^b_{-1}e^{r_2-(r_1 v^b_{-1}/r_1 - u_0)} \\ &< v^b_{-1}e^{r_2-(r_1 v^b_{-1}/r_1 - x_0)}. \end{aligned}$$

Then from Figure 1(b), we can obtain that $y^b_{-1} < v^b_{-1}$. Therefore,

$$x^b_{-1} = \frac{x_0 y^b_{-1}}{r_1 - x_0} < \frac{u_0 y^b_{-1}}{r_1 - u_0} < \frac{u_0 v^b_{-1}}{r_1 - u_0} = u^b_{-1}.$$

Therefore, the third part of Lemma 8 holds. □

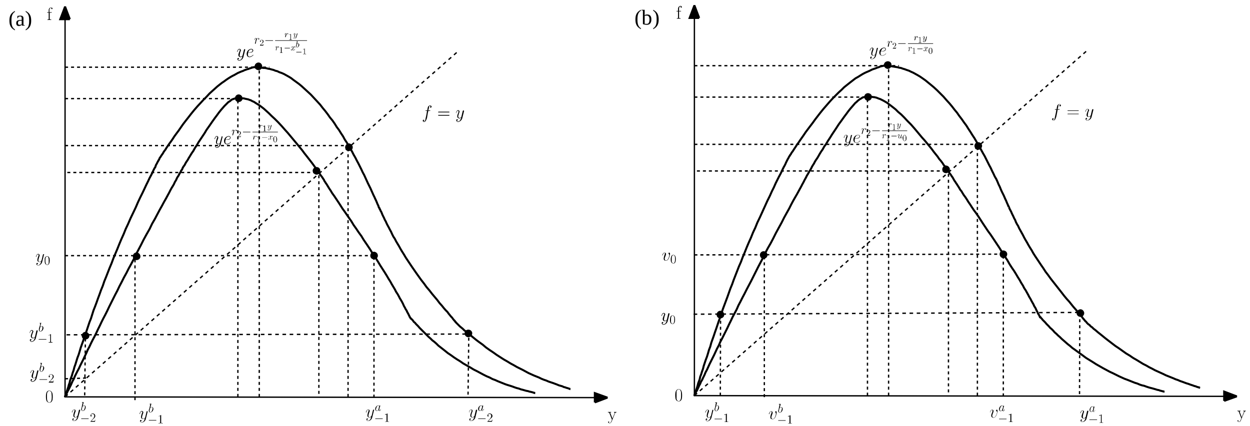

Figure 1. Schematic graphs of the pre-images of Ricker's maps $f(x,y) = ye^{r_2-(r_1 y/r_1-x)}$ where $f(x_0, y^i_{-1}) = y_0, f(x^b_{-1}, y^i_{-2}) = y^b_{-1}, f(u_0, v^i_{-1}) = v_0, i = a, b$. (a) Two Ricker's maps $f(x,y) = ye^{r_2-(r_1 y/r_1-x)}$ where $x = x_0, x^b_{-1}$ and $x_0 > x^b_{-1}$. In addition, $f(x_0, y^i_{-1}) = y_0, i = a, b$ and $f(x^b_{-1}, y^i_{-2}) = y^b_{-1}, i = a, b$ and (b) Two Ricker's maps $f(x,y) = ye^{r_2-(r_1 y/r_1-x)}$ where $x = x_0, u_0$ and $x_0 < u_0$. In addition, $f(x_0, y^i_{-1}) = y_0, i = a, b$ and $f(u_0, v^i_{-1}) = v_0, i = a, b$.

Part 3 of Lemma 8 indicates that the rank-1 pre-images have monotonic properties.

THEOREM 9 [STRUCTURE OF PRE-IMAGES OF INVARIANT SETS]. *Assume that*

$$r_2 > r_1 > 1, 2r_2 - 1 - e^{r_2 - 1} > 0$$

and M is an invariant set of system (3), (4) *such that* $(r_1, 0) \notin M$. *Then*

1. *M is contained in Z_2^2. Moreover, the rank-1 pre-images of M can be represented as $H^{-1}(M) = M_{-1}^b$, where $M_{-1}^b = \{\xi_{-1}^b = (x_{-1}^b, y_{-1}^b) \in B : H(\xi_{-1}^b) = \xi \in M\}$.*
2. *The collection of all rank pre-images of $\xi_{-1}^b \in M_{-1}^b$ can be represented as*

$$E_{\xi_{-1}^b} = \bigcup_{n\geq 0} H^{-n}(\xi_{-1}^b) = \bigcup_{n\geq 0, i=a,b} \xi_{-n-1}^i = \bigcup_{n\geq 1, i=a,b} (x_{-n}^i, y_{-n}^i)$$

where

$$H^{-1}(\xi_{-n}^b) = \{\xi_{-n-1}^a, \xi_{-n-1}^b\}, \xi_{-n}^b \in B, \xi_{-n-1}^b \ll \xi_{-n}^b \ \textit{for all} \ n \geq 1 \textit{ with}$$

$$\lim_{n\to\infty} \xi_{-n}^b = (0,0)$$

and

$$\xi_{-n}^a \in Z_0, y_{-n}^a < y_{-n-1}^a \ \textit{for all} \ n \geq 1 \ \textit{with} \ \lim_{n\to\infty} y_{-n}^a = \infty.$$

Moreover, $H^{-1}(\xi_{-n}^b) = H^{-k}(\xi_{-n-1+k}^b)$ for any $1 \leq k \leq n$.
3. *The collection of all rank pre-images of M can be represented as*

$$EF_M = M \bigcup \left(\bigcup_{n\geq 1} H^{-n}(M) \right) = M \bigcup M_{-1}^b \bigcup \left(\cup_{k\leq 2\leq n, i=a,b} M_{-k}^i \right)$$

where

$$M_{-n-1}^b = \bigcup_{\xi_{-1}^b \in M_{-1}^b} \{\xi \in B : H^n(\xi) = \xi_{-1}^b \in M_{-1}^b\},$$

$$M_{-n-1}^a = \bigcup_{\xi_{-1}^b \in M_{-1}^b} \{\xi \in Z_0 : H^n(\xi) = \xi_{-1}^b \in M_{-1}^b\},$$

$$H^{-1-n}(M) = \{M_{-n-1}^a, M_{-n-1}^b\}, M_{-n-1}^a \subset Z_0, M_{-n-1}^b \subset B, \ \textit{for all} \ 1 \leq n$$

and

$$\lim_{n\to\infty} M_{-n}^b = (0,0).$$

Moreover, $M_{-n}^b \cap M_{-k}^b = \emptyset$ for any two distinct positive integers and $M_{-n}^a \cap M_{-k}^a = \emptyset$ for any two distinct positive integers $n, k \geq 2$.

Proof. The condition $r_2 > r_1 > 1$ and $2r_2 - 1 - e^{r_2-1} > 0$ indicates that

$$r^m = \min\left\{r_1, r_2, e^{2r_2-1-e^{r_2-1}}, r_1 e^{r_2-r_1}\right\} = \min\left\{r_1, e^{2r_2-1-e^{r_2-1}}\right\} > 1.$$

Recall that $B = \{(x, y) \in Z_2 : x + y < 1\}$ and $LC^1_{-1} = \{(x, y) \in Z_2 : x + y = 1\}$, then

$$D_{r^m} \cap B = \emptyset \quad \text{and} \quad D_{r^m} \cap LC^1_{-1} = \emptyset.$$

According to Theorem 6, all points in X are attracted to the compact positively invariant set D_{r^m}; therefore,

$$M \subset D_{r^m} \quad \text{and} \quad \left(M \cap (B \cup LC^1_{-1})\right) = \emptyset.$$

The fact that

$$H(M) = M \quad \text{and} \quad H^{-1}(Z_1) = LC^1_{-1}$$

indicates that $(M \cap Z_1) = \emptyset$.

Since all points in X are also attracted to $Z_{1,2,\infty}$ by Lemma 5, $M \subset Z_{1,2,\infty}$. Therefore,

$$M \subset \left(D_{r^m} \cap Z_{1,2,\infty}\right).$$

Note that

$$Z_\infty = (r_1, 0) \notin M \quad \text{and} \quad (M \cap Z_1) = \emptyset;$$

therefore, $M \subset (D_{r^m} \cap Z_2)$.

Now suppose that there exists a point $\xi \in M \cap Z_2^1$. Then according to Lemmas 7 and 8, ξ has two rank-1 pre-images $\xi^i_{-1}, i = a, b$ where $\xi^a_{-1} \in Z_0$ and $\xi^b_{-1} \in B$. Since

$$(M \cap B) = \emptyset \text{ and } M \subset Z_2,$$

the point ξ has no pre-images in M. This contradicts to the fact that M is invariant. Therefore, $M \cap Z_2^1 = \emptyset$. This indicates that M should be contained in $Z_2 \backslash Z_2^1 = Z_2^2$.

Let $\xi = (x, y)$ be any point in M, then according to Lemma 7, ξ has two pre-images $\xi^i_{-1} = \left(x^i_{-1}, y^i_{-1}\right), i = a, b$. Since M is invariant and strictly above LC^1_{-1}; therefore, $\xi^a_{-1} \in M$ is strictly above LC^1_{-1} and $\xi^b_{-1} \in B$ is strictly below LC^1_{-1}.

Denote

$$M^a_{-1} = \bigcup_{\xi \in M} \left\{\xi^a_{-1} \in Z_0 : H\left(\xi^a_{-1}\right) = \xi\right\} \text{ and } M^b_{-1} = \bigcup_{\xi \in M} \left\{\xi^b_{-1} \in B : H\left(\xi^b_{-1}\right) = \xi\right\}.$$

Then the invariant property of M and $M \cap B = \emptyset$ implies that $M = M^a_{-1}$. Therefore, according to Definition 4, we have $H^{-1}(M) = M^b_{-1}$. Hence, the first part of Lemma holds.

Recall that

$$Z_2^1 = \left\{(x, y) \in Z_2 : 0 < y < \left(1 - \frac{x}{r_1}\right) e^{2r_2-1-e^{r_2-1}}\right\},$$

then we have $B \subset Z_2^1$. Let $\xi^i_{-k}, i = a, b$ be the rank k pre-image of ξ, where $\xi = (x, y) \in M$ and $k \in \mathbb{Z}_+$. Note that $\xi^b_{-1} = \left(x^b_{-1}, y^b_{-1}\right)$ is in B; therefore,

$$x^b_{-1} + y^b_{-1} < 1 < \min\{r_1, r_2\} \text{ and } y^b_{-1} < \left(1 - \frac{x^b_{-1}}{r_1}\right) e^{2r_2-1-e^{r_2-1}}.$$

Then according to Lemma 8, we have

$$H^{-1}(\xi^b_{-1}) = \{\xi^b_{-2}, \xi^a_{-2}\} = \{(x^b_{-2}, y^b_{-2}), (x^a_{-2}, y^a_{-2})\}$$

where

$$\xi^b_{-2} \in B, \xi^b_{-2} \ll \xi^b_{-1}, \xi^a_{-2} \in Z_0, H(\xi^a_{-2}) = \xi^b_{-1} \text{ and } y^a_{-1} < y^a_{-2}.$$

Since

$$\xi^b_{-2} \in B \subset Z^1_2 \text{ and } \xi^a_{-2} \in Z_0,$$

we can repeat the above argument to get

$$H^{-2}(\xi^b_{-1}) = H^{-1}(\xi^b_{-2}) = \{\xi^b_{-3}, \xi^a_{-3}\} = \{(x^b_{-3}, y^b_{-3}), (x^a_{-3}, y^a_{-3})\}$$

where

$$\xi^b_{-3} \in B, \xi^b_{-3} \ll \xi^b_{-2}, \xi^a_{-3} \in Z_0, H(\xi^a_{-3}) = \xi^b_{-2} \text{ and } y^a_{-2} < y^a_{-3}.$$

Apply the same above argument repeatedly, then for any $n \geq 1$ and $1 \leq k \leq n$, we have

$$\begin{aligned} H^{-1}(\xi^b_{-n}) &= H^{-k}(H^{-n+k-1}(\xi^b_{-1})) = \{\xi^b_{-n-1}, \xi^a_{-n-1}\} \\ &= \{(x^b_{-n-1}, y^b_{-n-1}), (x^a_{-n-1}, y^a_{-n-1})\} \end{aligned}$$

where $\{\xi^b_{-n}\}_{n=1}^{\infty}$ is a strictly monotone decreasing sequence converging to (0, 0) and $\{y^a_{-n}\}_{n=1}^{\infty}$ is a strictly monotone increasing sequence converging to infinity. Moreover, $H(\xi^a_{-n}) = \xi^b_{-n+1}$ for all $n \geq 2$. Thus, the collection of all rank pre-images of $\xi^b_{-1} \in M^b_{-1}$ is

$$\bigcup_{n\geq 1} H^{-n}(\xi) = \{\xi^b_{-n}\}_{n=1}^{\infty} \bigcup \{\xi^a_{-n}\}_{n=1}^{\infty}.$$

Therefore, the second part of Theorem 9 holds.

Denote

$$M^a_{-n-1} = \bigcup_{\xi^b_{-1} \in M^b_{-1}} \{\xi^a_{-n-1} \in Z_0 : H^n(\xi^a_{-n-1}) = \xi^b_{-1}\}$$

and

$$M^b_{-n-1} = \bigcup_{\xi^b_{-1} \in M^b_{-1}} \{\xi^b_{-n-1} \in B : H^n(\xi^b_{-n-1}) = \xi^b_{-1}\}.$$

Since $H^{-1}(M) = M^b_{-1} \subset B$; therefore,

$$\begin{aligned} M^a_{-n-1} &= \bigcup_{\xi^b_{-1} \in M^b_{-1}} \{\xi^a_{-n-1} \in Z_0 : H^n(\xi^a_{-n-1}) = \xi^b_{-1}\} \\ &= \bigcup_{\xi \in M} \{\xi^a_{-n-1} \in Z_0 : H^{n+1}(\xi^a_{-n-1}) = \xi\} \end{aligned}$$

and

$$M^b_{-n-1} = \bigcup_{\xi^b_{-1}\in M^b_{-1}} \left\{\xi^b_{-n-1} \in B : H^n\left(\xi^b_{-n-1}\right) = \xi^b_{-1}\right\}$$
$$= \bigcup_{\xi\in M} \left\{\xi^b_{-n-1} \in B : H^{n+1}\left(\xi^b_{-n-1}\right) = \xi\right\}.$$

Then according to the proof of the second part of Theorem 9 and the definition of rank-k pre-images of an invariant set 4, we have

$$H^{-n-1}(M) = M^a_{-n-1} \cup M^b_{-n-1}, n \geq 1.$$

Since $\lim_{n\to\infty} \xi^b_{-n} = (0,0)$, thus

$$\lim_{n\to\infty} M^b_{-n} = (0,0).$$

Note that $M^a_{-n-1} \in Z_0$ and $M^b_{-n-1} \in B$ for any $n \geq 1$, thus

$$H^{-n-1}(M) = \left\{M^a_{-1-n}, M^b_{-n-1}\right\} \text{ for any } n \geq 1.$$

Therefore, the collection of all rank pre-images of M is

$$EF_M = M \bigcup M^b_{-1} \bigcup \left(\bigcup_{n\geq 2} H^{-n}(M)\right) = M \bigcup M^b_{-1} \bigcup \left(\bigcup_{n\geq 2, i=a,b} M^i_{-n}\right).$$

Now assume that there is $\eta \in B$ and two distinct positive integers k, n such that $\eta \in M^b_{-n} \cap M^b_{-k}$. Without loss of generality, let $k < n$, then $H^{n-1}(\eta) = \xi \in M^b_{-1}$ and $H^{k-1}(\eta) = \zeta \in M^b_{-1}$. Since $\zeta \in M^b_{-1}$, then $H^i(\zeta) \in M$ for any $i \geq 1$. Therefore, $H^{n-k}(\zeta) \in M$. However, this contradicts to the fact that $H^{n-1}(\eta) = H^{n-k}\left(H^{k-1}(\eta)\right) = H^{n-k}(\zeta) = \xi \in M^b_{-1}$ and $M \cap M^b_{-1} = \emptyset$. Similarly, we can show that $M^a_{-n} \cap M^a_{-k} = \emptyset$ for any distinct positive integers $n, k \geq 2$. Therefore, the third part of Theorem 9 holds. □

Theorem 9 is an analogous result to Theorem 4.2 by [4]. Moreover, it indicates that $\{M^b_{-n}\}^{\infty}_{n=1}$ is a monotone decreasing sequence that converges to (0, 0). See Figure 2 for the structure of $E_{\xi^b_{-1}}$ and EF_M. In the case that M is a smooth curve, we have the following corollary:

Corollary 10 [Smooth invariant curve]. *Assume that all the conditions in Theorem* 9 *hold and notations are the same. Let M be an invariant smooth curve of system* (3), (4) *that does not contain* $(r_1, 0)$, *then*

$$EF_M = M \bigcup \left\{M^b_{-n}\right\}^{\infty}_{n=1} \bigcup \left\{M^a_{-n}\right\}^{\infty}_{n=2}$$

where $M^a_{-n-1} \subset Z_0, M^b_{-n} \subset B$ *are smooth curves. Let C be any compact subset of X, then* $m_2(EF_M \cap C) = 0$ *where* m_2 *is nature Lebesgue measure in* $\mathbb{R}^2$.

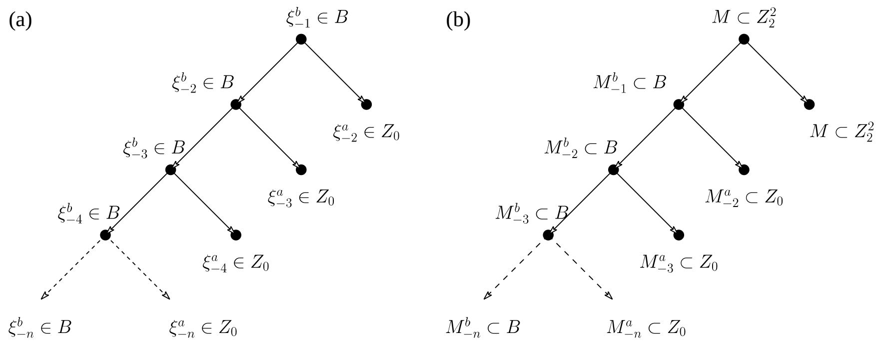

Figure 2. The structure of all rank pre-images of a point in M^b_{-1} and an invariant set M. (a) The structure of all rank preimages of a point ξ^b_{-1} in M^b_{-1}. (b) The structure of all rank pre-images of an invariant set M.

Proof. According to Theorem 9, we have

$$H^{-1}(M) = M^b_{-1} \text{ and } H^{-n-1}(M) = \{M^a_{-1-n}, M^b_{-1-n}\} \quad \text{for any} \quad n \geq 1.$$

Since M is a smooth curve in D_{r^m}, then from Lemma 7, we know that its rank-1 pre-image M^b_{-1} is also a smooth curve in B. Then apply Lemmas 7 and 8 repeatedly, we know that $M^b_{-n} \in B$ and $M^a_{-n-1} \in Z_0$ are smooth curves for any $n \geq 1$.

For any compact set $C \in X$, there are only finite number of $M^i_{-n}, i = a, b$ which intercept with C or are contained in C. Since $m_2\left(M^a_{-n-1}\right) = 0$ and $m_2\left(M^b_{-n}\right) = 0$ for any positive integer $n \geq 1$; therefore, $m_2\left(\cup_{n\geq 1} H^{-n}(M) \bigcap C\right) = 0$. □

Remark. If $(r_1, 0) \in M$, let $N = M \backslash \{(r_1, 0)\}$, then N is still an invariant smooth curve. From Corollary 10, we have

$$\begin{aligned} EF_M &= M \bigcup \left(\bigcup_{n\geq 1} H^{-n}(M) \right) = M \bigcup \left(\bigcup_{n\geq 1} H^{-n}(N) \right) \bigcup H^{-1}(Z_\infty) \\ &= M \bigcup \left\{N^b_{-n}\right\}_{n=1}^{\infty} \bigcup \left\{N^a_{-n}\right\}_{n=2}^{\infty} \bigcup \{(x, 0) \in X\}. \end{aligned}$$

Therefore, the result in Corollary 10 still holds, i.e. for any compact set $C \in X$, we have

$$m_2\left(EF_M \bigcap C\right) = 0$$

where m_2 is a nature Lebesgue measure in $\mathbb{R}^2$.

The simulations (see Figure 3) suggest that there are heteroclinic orbits connecting two boundary equilibria points. We are able to show the existence of such heteroclinic orbits when one species goes to extinction for (3), (4). However, it is still an open problem to show the existence of heteroclinic orbits when there are locally asymptotically stable interior periodic-2 orbits. Theorem 9 and its Corollary 10 give an explicit structure of all pre-images of invariant sets including heteroclinic orbits.

4. Discussion

The basins of attractions of attractors of an ecological system can provide important information on its resilience and initial conditions allowing the coexistence of all species.

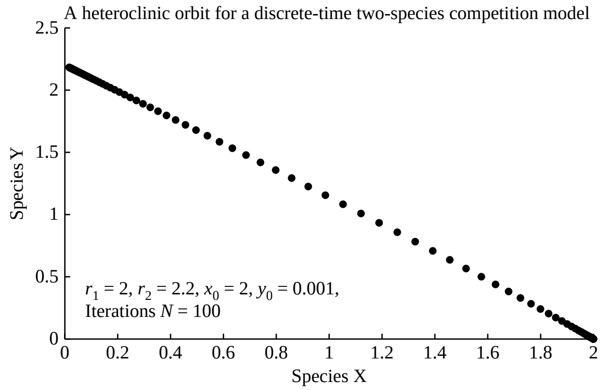

Figure 3. A heteroclinic orbit of system (3), (4) when $r_1 = 2, r_2 = 2.2, x_0 = 2, y_0 = 0.001$.

Since pre-images of invariant sets of a system can define the boundaries of the basins of attractions of attractors, we study the properties of all pre-images of invariant sets for a discrete-time two-species competition model (3), (4) and give the explicit structure of all pre-images of invariant sets for this system when

$$r_2 > r_1 > 1 \quad \text{and} \quad 2r_2 - 1 - e^{r_2 - 1} > 0. \tag{24}$$

Our results combined with simulations suggest that if inequalities (24) hold, then the basins of attractions of the interior periodic-2 orbits are all interior points in X except the measure zero sets. These measure zero sets are the pre-images of heteroclinic orbits (see Figure 4).

If we say that an ecosystem is *relatively permanent*, then the population of any species in this system is strictly bounded away from zero for almost all strictly positive initial

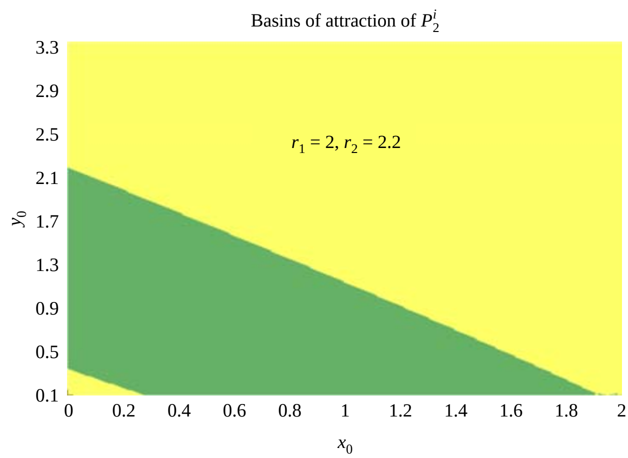

Figure 4. The basin of attraction of the interior period-2 orbit is the open quadrant minus the pre-images of the heteroclinic curve C. The latter partition the quadrant into components which are coloured according to which of the two periodic points attract points in the component under the second iterate of the map. Given a point in one of the regions, there is a large number N, such that the point will be very close to (x_1^i, y_1^i) at the iteration t and will be very close to (x_2^i, y_2^i) at the iteration $t + 1$ for all $t > N$.

conditions. Our work in this paper plays an important role in proving this relative permanence concept for (3), (4). The numerical simulations suggest the following dynamics of system (3), (4) when

$$r_2 > r_1 > 1 \quad \text{and} \quad 2r_2 - 1 - e^{r_2 - 1} > 0 :$$

1. There exists a heteroclinic orbit connecting ξ^* to η^* (see Figure 3);
2. The basins of attraction of the interior periodic-2 orbit P_2^i are all interior points of $\mathbb{R}_+^2$ except all the pre-images of heteroclinic orbits (see Figure 4).

Since our system is smooth in X, the closure of all heteroclinic orbits should be a smooth invariant curve. Then according to Theorem 9 and its Corollary 10, we can show that for any compact set $C \in X$, there is only Lebesgue measure zero set in C that will converge to heteroclinic orbits. If both the closure of heteroclinic orbits and the boundary of the system are repelling, then we are able to show that for any compact subset C of X, all interior points of C are attracted to the interior attractor except Lebesgue measure zero set, which is the collection of all pre-images of the closure of the heteroclinic orbits. All the details of proof are presented in a separate paper by [9].

In addition, our result is analogous to the result for the one-dimensional discrete system studied by [4] (Theorem 4.2). Partial results obtained in this paper can be applied to the general competition model (1), (2) while $a > 0$. For instance, Lemmas 7 and 8 are still valid for the general system (1), (2). However, we are not able to extend the analysis techniques used to prove Theorem 9 to prove the similar results for (1), (2). The reason is that while $a > 0$, the critical curve LC_{-1} is unbounded which always intersects with LC. We will seek additional analysis methods to prove the similar results for (1), (2).

4.1 Open problems

There are many interesting unsolved questions regarding system (1), (2). A partial list of these open problems is as follows:

1. How can we rigorously prove the existence of heteroclinic orbits in the planar Kolmogorov-type competition models with parallel isoclines (such as systems (1), (2) and (3), (4))?
2. If systems (1), (2) and (3), (4) have heteroclinic orbits connecting two boundary equilibria points, then how do the axis dynamics impact these heteroclinic orbits? Are there any other heteroclinic orbits?
3. How can we rigorously prove or disprove that the basins of attractions of the interior periodic-2 orbit P_2^i are all interior points of $\mathbb{R}_+^2$ except all the pre-images of heteroclinic orbits as suggested in Figure 4?

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Appendix A: Pre-images of Ricker's maps

Similar results (Parts 1 and 2 of Lemma 11) can also be found in the paper [4]. For convenience, we will re-derive their results when the Ricker map has the form of

$$y_{n+1} = y_n \mathrm{e}^{r_2-(r_1 y_n/r_1 - x_0)}.$$

And in addition, we derive the results of Part 3 in Lemma 11 that has been used in proving Theorem 6.

LEMMA 11 [PROPERTIES OF RICKER'S MAPS]. *Let* $f(y) = y\mathrm{e}^{r_2-(r_1 y/r_1 - x_0)}$ *where* $r_i > 0, 0 \le x_0 < r_1, i = 1, 2$. *Then* f *maps* $\mathbb{R}^+$ *to* $\left[0, (1 - x_0/r_1)\mathrm{e}^{r_2-1}\right]$. *The critical point of* f *is* $y_c = 1 - x_0/r_1$ *which is mapped to* e^{r_2-1}; *the fixed point is* $y^* = r_2(1 - x_0/r_1)$. *For any given point* $y_0 \in \left(0, (1 - x_0/r_1)\mathrm{e}^{r_2-1}\right)$, *it has two rank*-1 *pre-images* $y^i_{-1} \in \mathbb{R}^+, i = a, b$ *where* $y^b_{-1} < y_0 < y^a_{-1}$ *and* $f\left(y^i_{-1}\right) = y_0, i = a, b$. *Moreover, we have the following two situations depending on the values of* r_2

1. *When* $0 < r_2 \le 1$, *then one of the following inequalities holds*

$$0 < y^b_{-1} \le y_0 < y^* < y_c < y^a_{-1} \tag{25}$$

or

$$0 < y^* < y_0 < y^b_{-1} < y_c < y^a_{-1}. \tag{26}$$

In addition, y^a_{-1} *is always greater than* $(1 - x_0/r_1)\mathrm{e}^{r_2-1}$.

2. *If $r_2 > 1$, then one of the following inequalities holds*

$$0 < y_{-1}^b < y_c < y^* < y_{-1}^a < y_0 \tag{27}$$

or

$$0 < y_{-1}^b < y_c < y_0 < y^* < y_{-1}^a \tag{28}$$

or

$$0 < y_{-1}^b < y_0 < y_c < y^* < y_{-1}^a \tag{29}$$

In addition, if

$$y_0 < \left(1 - \frac{x_0}{r_1}\right) e^{2r_2 - 1 - e^{r_2 - 1}} \quad \textit{then} \quad y_{-1}^a > \left(1 - \frac{x_0}{r_1}\right) e^{r_2 - 1},$$

otherwise if

$$y_0 \geq \left(1 - \frac{x_0}{r_1}\right) e^{2r_2 - 1 - e^{r_2 - 1}} \quad \textit{then} \quad y_{-1}^a \leq \left(1 - \frac{x_0}{r_1}\right) e^{r_2 - 1}.$$

3. *Let* $r^m = \min\left\{r_1, r_2, e^{2r_2 - 1 - e^{r_2 - 1}}, r_1 e^{r_2 - r_1}\right\}$ *and* $r^M = \max\left\{r_1, e^{r_2 - 1}\right\}$*, then*

$$\min_{r^m \leq m \leq r^M} \min_{y = m} \{r_1, y e^{r_2 - y}\} \geq r^m.$$

Proof. It is easy to check that if $r_2 \leq 1$, then

$$y_c = r_2 \left(1 - \frac{x_0}{r_1}\right) e^{r_2 - 1} \leq y^* = \left(1 - \frac{x_0}{r_1}\right) e^{r_2 - 1}$$

where equality holds when $r_2 = 1$. And if $r_2 > 1$, then $y^* < y_c$. The detailed proof for (25)–(29) can be illustrated by the schematic diagrams (Figures 5 and 6).

Now we will show Part 3 of the lemma. First note that the map defined by $g(m) = m e^{r_2 - m}$ has the following properties

- $g(m) \geq m$ for $m \in [0, r_2)$ and $g(m) \leq m$ for $m \in [r_2, \infty)$.
- If $r_2 > 1$, map g is positively invariant in $[e^{2r_2 - 1 - e^{r_2 - 1}}, e^{r_2 - 1}]$; while $r_2 \leq 1$, map g is positively invariant in $[0, r_2]$ and $[r_2, e^{r_2 - 1}]$.
- $\min_{0 < r^m \leq m \leq r^M} \{m e^{r_2 - m}\} = \min\{r^m e^{r_2 - r^m}, r^M e^{r_2 - r^M}\}$.

Since $r^m = \min\left\{r_1, r_2, e^{2r_2 - 1 - e^{r_2 - 1}}, r_1 e^{r_2 - r_1}\right\}$ and $r^M = \max\left\{r_1, e^{r_2 - 1}\right\}$, thus we have

$$\begin{aligned}
\min_{r^m \leq m \leq r^M} \min_{y = m} \{r_1, y e^{r_2 - y}\} &= \min\left\{r_1, \min_{r^m \leq m \leq r^M} \{m e^{r_2 - m}\}\right\} = \min\{r_1, r^m e^{r_2 - r^m}, r^M e^{r_2 - r^M}\} \\
&\geq \min\left\{r_1, r_2, e^{2r_2 - 1 - e^{r_2 - 1}}, r_1 e^{r_2 - r_1}, g(e^{2r_2 - 1 - e^{r_2 - 1}}), g(r_1 e^{r_2 - r_1})\right\} \\
&= \min\left\{r^m, g(e^{2r_2 - 1 - e^{r_2 - 1}}), g(r_1 e^{r_2 - r_1})\right\}.
\end{aligned}$$

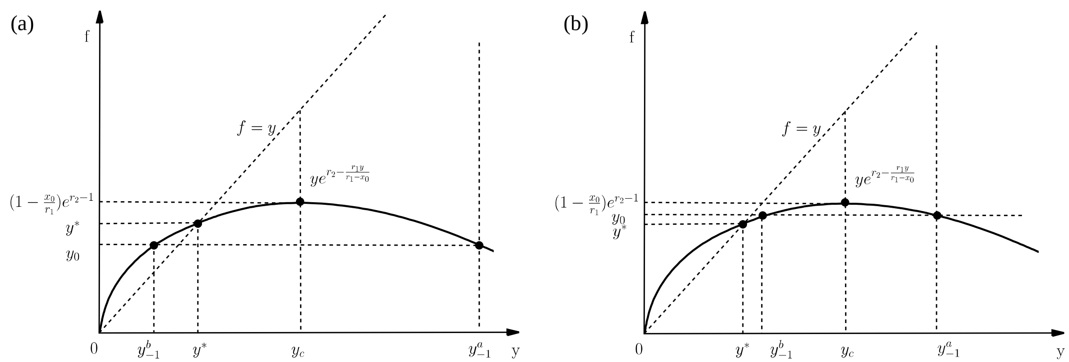

Figure 5. The relationship between $y^i_{-1}, i = a, b$ and y_0, y^*, y_c when $0 < r_2 \leq 1$ where $f\left(y^i_{-1}\right) = y_0, i = a, b$. (a) When $0 < r_2 \leq 1$ and $y_0 < y^*$. (b) When $0 < r_2 \leq 1$ and $y^* < y_0 < (1 - x_0/r_1)\mathrm{e}^{r_2 - 1}$.

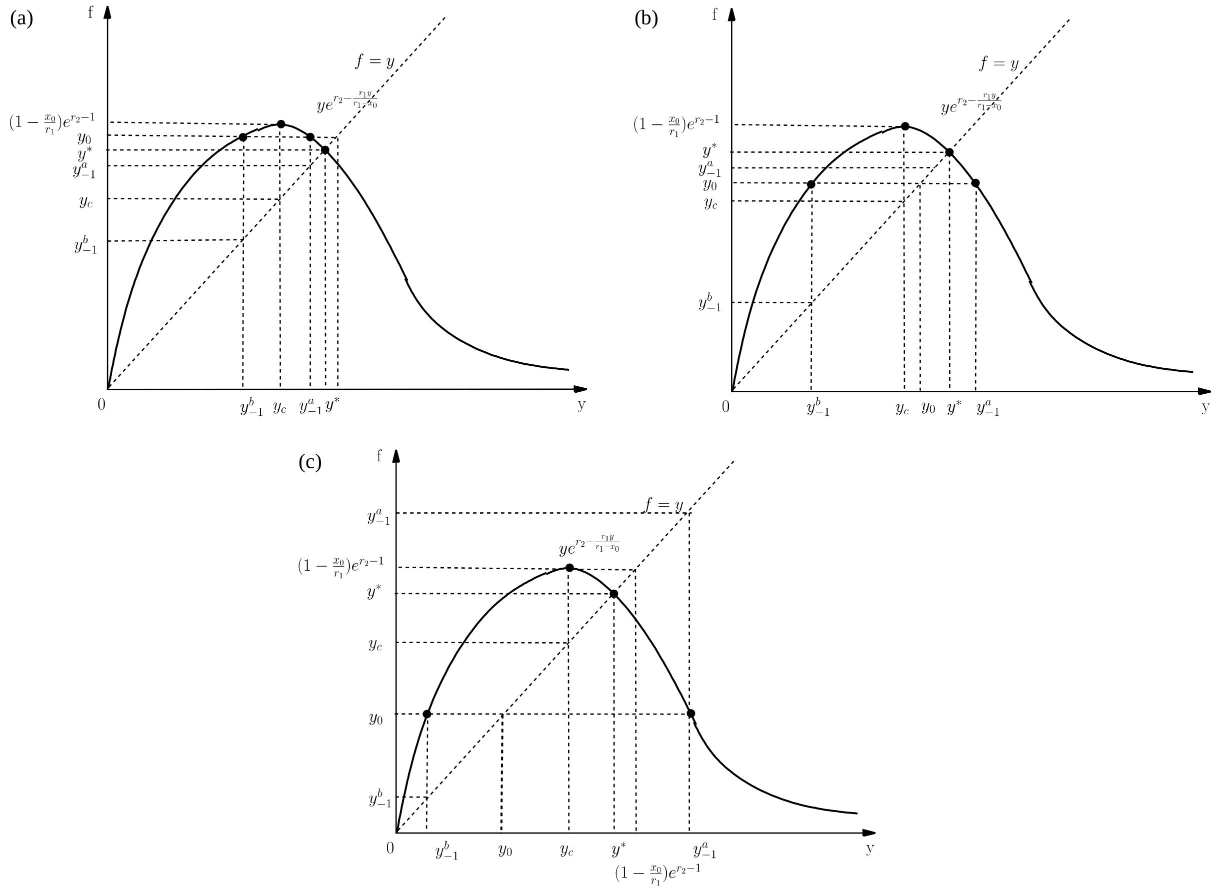

Figure 6. The relationship between $y_{-1}^i, i = a, b$ and y_0, y^*, y_c when $r_2 > 1$ where $f\left(y_{-1}^i\right) = y_0$, $i = a, b$. In Figure 6c, we can see that if $y_0 \leq f(f(y_c))$, then $y_{-1}^a > (1 - x_0/r_1)e^{r_2-1}$. (a) When $r_2 > 1$ and $y^* < y_0 < (1 - x_0/r_1)e^{r_2-1}$, (b) when $0 < r_2 \leq 1$ and $y_c < y_0 \leq y^*$ and (c) when $0 < r_2 \leq 1$ and $y_0 \leq y_c$.

Now assume that $r_2 > 1$, then $g \circ g(e^{r_2-1}) = g(e^{2r_2-1-e^{r_2-1}}) \in [e^{2r_2-1-e^{r_2-1}}, e^{r_2-1}]$, thus

$$g(e^{2r_2-1-e^{r_2-1}}) \geq e^{2r_2-1-e^{r_2-1}} \geq r^m.$$

While if $r_2 \leq 1$, then $g \circ g(e^{r_2-1}) = g(e^{2r_2-1-e^{r_2-1}}) \in \left[r_2, e^{r_2-1}\right]$, thus

$$g(e^{2r_2-1-e^{r_2-1}}) \geq r_2 \geq r^m.$$

Therefore, $\min\left\{r^m, g(e^{2r_2-1-e^{r_2-1}}), g(r_1 e^{r_2-r_1})\right\} = \min\{r^m, g(r_1 e^{r_2-r_1})\}$.

Suppose that $r_1 e^{r_2-r_1} \leq r_2$, then $g(r_1 e^{r_2-r_1}) \geq r_1 e^{r_2-r_1} \geq r^m$. Otherwise if $r_1 e^{r_2-r_1} > r_2$, then we have $r_2 < r_1 e^{r_2-r_1} < e^{r_2-1}$. This implies that

- If $r_2 \geq 1$, then $g(r_1 e^{r_2-r_1}) \geq g(e^{r_2-1}) = e^{2r_2-1-e^{r_2-1}} \geq r^m$.
- If $r_2 < 1$, then $g(r_1 e^{r_2-r_1}) \geq r_2 \geq r^m$.

Hence, we have $g(r_1 e^{r_2-r_1}) \geq r^m$. Therefore,

$$\min_{r^m \leq m \leq r^M} \min_{y=m} \{r_1, y e^{r_2-y}\} \geq \min\{r^m, g(r_1 e^{r_2-r_1})\} = r^m. \qquad \square$$