The Context of the Game *

Amanda Friedenberg† Martin Meier‡

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Abstract

Here, we study games of incomplete information and argue that it is important to correctly specify the "context" within which hierarchies of beliefs lie. We consider a situation where the players understand more than the analyst, in the following sense: It is transparent to the players—but not to the analyst—that certain hierarchies of beliefs are precluded. In particular, the players' type structure can be viewed as a strict subset of the analyst's type structure. How does this affect a Bayesian equilibrium analysis? One natural conjecture is that this doesn't change the analysis—i.e., every equilibrium of the players' type structure can be associated with an equilibrium of the analyst's type structure. We show that this conjecture is wrong. Bayesian equilibrium may fail an Extension Property. Specifically, this failure can occur in the case of a finite game—even when the analyst uses the so-called universal structure to analyze the game. We go on to discuss specific situations in which the Extension Property is satisfied. This involves restrictions on both the game and the type structures.

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†Department of Economics, W.P. Carey School of Business, Arizona State University, amanda.friedenberg@asu.edu, http://www.public.asu.edu/ afrieden.

‡Instituto de Análisis Económico - CSIC, Barcelona, and Institut für Höhere Studien, Vienna, martin.meier@iae.csic.es

Taken out of context, I must seem so strange.

— Ani Difranco, from the song Fire Door.

1 Introduction

This paper is concerned with the analysis of incomplete information games. For these games, the analyst must specify the players' choices, payoff functions, and hierarchies of beliefs (about the payoffs of the game). The importance of correctly specifying players' actual payoff functions and/or hierarchies of beliefs is well understood. (See, for instance, Kreps and Wilson (1982), Milgrom and Roberts (1982), Geanakoplos and Polemarchakis (1982), Monderer and Samet (1989), Rubinstein (1989), Carlsson and Van Damme (1993), Aumann and Brandenburger (1995), Kajii and Morris (1997), Oyama and Tercieux (2009), and Weinstein and Yildiz (2007), among many others.) Here, we argue that it is also important to correctly specify the "context" within which the given hierarchies lie.

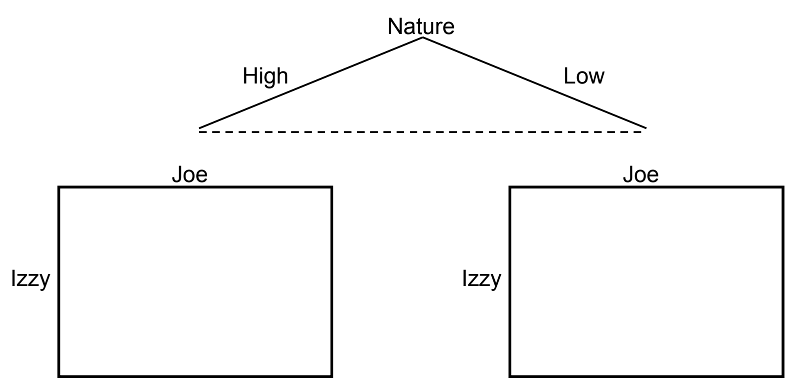

Figure 1

To understand this idea, let us take an example. Refer to Figure 1. Nature tosses a coin, whose realization is either High or Low. (This can, for instance, reflect a buyer having a High or Low valuation.) The realization of this toss results in distinct matrices (or payoff functions). Each of two players, resp. Izzy (i) and Joe (j), face uncertainty about the realization of this coin toss.

What choices should Izzy and Joe make here? Presumably, Izzy's choice will depend on her belief about the realization of the coin toss—after all, the realization influences which matrix is being played. But, presumably, Izzy's choice will also depend on what she thinks about Joe's belief about the realization of the coin toss. After all, Joe's belief (about the realization of the coin toss) should influence his action, too. And, Izzy is concerned not only with what matrix is being played, but also with what choice Joe is making within the matrix.

To analyze the situation, we must add to the description of the game, so that it also reflects these hierarchies of beliefs. In particular, we append a type structure to the game. One such type structure is given in Figure 2. Here, there are two possible types of Izzy, viz. t_i and u_i, and one possible type of Joe, viz. t_j. Type t_i (resp. u_i) of Izzy assigns probability one to Nature choosing High (resp. Low) and Joe's type being t_j. Type t_j of Joe assigns probability $\frac{1}{2}$ to "Nature choosing High and Izzy being type t_i" and probability $\frac{1}{2}$ to "Nature choosing Low and Izzy being type u_i." So, type t_j of Joe assigns probability $\frac{1}{2}$ to "Nature choosing High and Izzy assigning probability one to High" and probability $\frac{1}{2}$ to "Nature choosing Low and Izzy assigning probability one to Low." And so on.

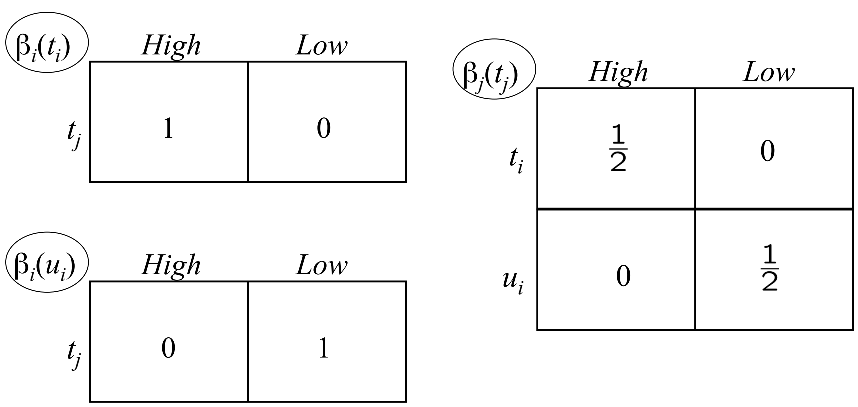

Figure 2

For a given type structure, as in Figure 2, we can analyze the game associated with Figure 1 We defer an analysis for now. Instead, we point to a particular feature of the type structure in Figure 2. Here, there are only two possible hierarchies of beliefs that Izzy can hold and only one possible hierarchy of beliefs that Joe can hold. In particular, the type structure does not contain all hierarchies of beliefs.

What is the rationale for limiting the type structure in this way? The specified game is only one part of the picture—a small piece of a larger story. The game sits within a broader strategic situation. That is, there is a history to the game, and this history influences the players. As Brandenburger, Friedenberg and Keisler (2008, p. 319) put it:

> We think of a particular . . . structure as giving the "context" in which the game is played. In line with Savage's Small-Worlds idea in decision theory, who the players are in the given game can be seen as a shorthand for their experiences before the game. The players' possible characteristics—including their possible types—then reflect the prior history or context.

Under this view, the type structure, taken as a whole, reflects the context of the game. (Section 9b expands on this point and discusses the relationship to other views of game theory.)

Here, we are concerned with the case where the players understand more than the analyst, in a particular sense. We imagine the following scenario: The analyst looks at the strategic situation and the history. Perhaps the analyst even deduces that certain hierarchies are inconsistent with the history. But, to the players, it is transparent that other—that is, even more—hierarchies are inconsistent with the history. Put differently, players rule out hierarchies the analyst hasn't ruled out.

Return to the earlier example. Consider the case in which the players' type structure is as given in Figure 2. Suppose the analyst misspecifies the type structure and instead studies the structure in Figure 3. Here, there is one extra type of Joe, viz. u_j. Type u_j is associated with some belief, distinct from type t_j's belief. The particular belief is immaterial. What is important is that each of Izzy's types assigns zero probability to this type of Joe. More to the point, each of Izzy's types is associated with the exact same beliefs as in the players' type structure. So, the players' type structure can be viewed as a subset (or substructure) of the analyst's type structure.

How does this affect an analysis? Take the solution concept of Bayesian Equilibrium, applied to the game in Figure 1 and the type structure in Figure 3. For a given Bayesian Equilibrium, the analyst will have a prediction associated with the type u_j—i.e., a type that the players have ruled out. But the analyst will also have a prediction for the types t_i, u_i, and t_j. These are types in the players' structure, namely Figure 2.

The question is: How does the analyst's predictions for these types relate to the predictions he would have, if he had analyzed the game using the players' type structure? Presumably, the analyst's predictions shouldn't change. After all, the beliefs associated with t_i, u_i, and t_j have not changed at all. So, we can associate any equilibrium of the

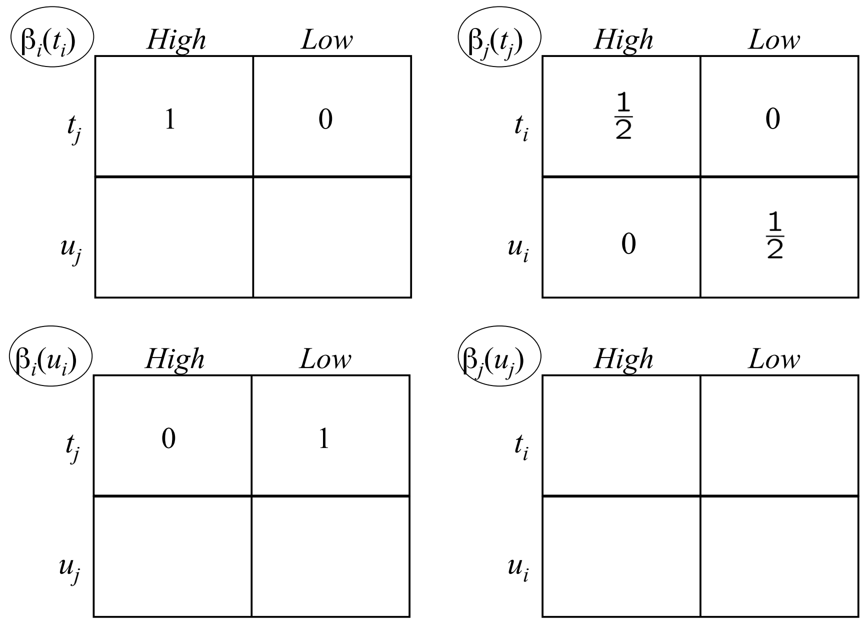

Figure 3

players' actual type structure with an equilibrium of the analyst's type structure, and vice versa.

Implicit in the above is that Bayesian Equilibrium satisfies Extension and Pull-Back Properties. Let us state these properties semi-formally.

Fix a type structure, viz. $\mathcal{T}$, associated with type sets T_i and T_j. We will think of $\mathcal{T}$ as the players' type structure. Now, consider another type structure $\mathcal{T}^*$, associated with type sets T_i^* and T_j^*. Suppose there is a map $h_i : T_i \to T_i^*$ (resp. $h_j : T_j \to T_j^*$) so that each t_i and $h_i(t_i)$ (resp. t_j and $h_j(t_j)$) induce the same hierarchies of beliefs. We will think of $\mathcal{T}^*$ as the analyst's structure. In our setting, we can then view the players' type structure $\mathcal{T}$ as a subset (or a substructure) of the analyst's structure $\mathcal{T}^*$.[1] (See Lemmata 6.1, 6.2, and D.1.) Now, we can state the Extension and Pull-back Properties.

The Equilibrium Extension Problem. Fix an equilibrium of $\mathcal{T}$. Does there exist an equilibrium of $\mathcal{T}^*$ so that each $h_i(t_i) \in T_i^*$ and each $h_j(t_j) \in T_j^*$ plays

[1]Formally, the main result (in Section 5) assumes that no two types induce the same hierarchies of beliefs. Sections 4 and 6 discuss what this assumption delivers formally. Section 9a discusses what this assumption delivers conceptually.

the same strategy as do t_i and t_j (under the original equilibrium of $\mathcal{T}$)?

The Equilibrium Pull-Back Problem. Fix an equilibrium of $\mathcal{T}^*$. Does there exist an equilibrium of $\mathcal{T}$ so that each $t_i \in T_i$ and each $t_j \in T_j$ plays the same strategy as do $h_i(t_i)$ and $h_j(t_j)$ (under the original equilibrium of $\mathcal{T}^*$)?

Return to the question of whether the analyst can study the Bayesian game in Figure 3. The answer is yes, provided that the analyst won't lose any predictions and won't introduce any new predictions. The question of losing predictions is the Extension Problem. The question of introducing new predictions is the Pull-Back Problem.

We will see that the answer to the Pull-Back Problem is yes. But, the answer to the Extension Problem is no. This is somewhat surprising, as "types associated with the players' structure," viz. $h_i(T_i)$ (resp. $h_j(T_j)$), assign zero probability to "types that are in the analyst's structure but not associated with the players' structure," viz. $T_j^* \backslash h_j(T_j)$ (resp. $T_i^* \backslash h_i(T_i)$). (See Lemma 6.2.)

What, then, goes wrong? The problem arises from the types that are in the analyst's structure, but not in the players' structure. Specifically, when we attempt to extend a particular equilibrium, the types in the players' structure impose an equilibrium restriction on types in the analyst's structure. This is where a conflict can arise. The next section gives a preview of this result.

2 Preview

Let us begin by giving the idea of the main result—a failure of Equilibrium Extension. We construct an example of a Bayesian game that fails Equilibrium Extension. To do so, we will fix a particular parameter set Θ (i.e., a set of choices of Nature) and an associated game matrix Γ. In Figure 1.1, Θ is the set $\{Heads, Tails\}$ and Γ specifies two matrices, one for Heads and one for Tails. Next, we append to the game two particular type structures, viz. $\mathcal{T}$ and $\mathcal{T}^*$, associated with the parameter set Θ. (These type structures both induce hierarchies of beliefs about Θ.) The structure $\mathcal{T}$ can be viewed as the players' type structure, and the structure $\mathcal{T}^*$ can be viewed as the analyst's structure—in a sense, $\mathcal{T}$ is contained in $\mathcal{T}^*$.[2] We will show that there exists an equilibrium of the Bayesian game $(\Gamma, \mathcal{T})$ that cannot be extended to an equilibrium of the Bayesian game $(\Gamma, \mathcal{T}^*)$. (There will be a second equilibrium of $(\Gamma, \mathcal{T})$ that can be extended to an equilibrium of $(\Gamma, \mathcal{T}^*)$. So, in particular, there is an equilibrium of the Bayesian game $(\Gamma, \mathcal{T}^*)$.)

[2]We have this containment property because we choose both structures so that no two types induce the same hierarchies of beliefs. See Lemma 6.1.

The details of the construction will be given in Section 5. But, for now, let us ask: Is the constructed example pathological? After all, at some level, any negative result ought to be pathological. Nonetheless, we will argue that the example is not pathological: We build an example of an Extension Failure from, arguably, "standard" ingredients—that is, ingredients which are well-understood and for which we would very much expect no problem to arise. Let us point to some features of the example:

- The parameter set Θ is finite.
- The game Γ has a finite number of players and each player has a finite number of choices.
- The players' type structure $\mathcal{T}$ is so that each player has a single type.
- The analyst's type structure $\mathcal{T}^*$ is the canonical construction of the universal type structure based on the (finite) parameter set Θ.

So we have a finite parameter set, a finite game, a finite players' type structure, and a universal analyst's structure—arguably, standard ingredients.

The case of a universal type structure is of particular interest. It is often presumed that the analyst should necessarily take the universal structure to applications, even if the current state of applied work does not do so. See, e.g., Morris and Shin (2003) who say "optimal strategic behavior should be analyzed in the space of all possible infinite hierarchies of beliefs." The Extension failure tells us that—while perhaps appealing—such an general principle may, in fact, be problematic.

Along the way, we will see that there is a second problem with such a general principle: Fix a finite non-singleton parameter set $\overline{\Theta}$ and the associated canonical construction of the universal type structure based on $\overline{\Theta}$, viz. $\overline{\mathcal{T}}^*$. There is a finite game based on $\overline{\Theta}$, viz. $\overline{\Gamma}$, so that there is no equilibrium of the Bayesian game $(\overline{\Gamma}, \overline{\mathcal{T}}^*)$. This step uses an important result due to Simon (2003) plus Lemma 3.1 below.

Notice, this latter fact already gives us a failure of Equilibrium Extension. In particular, fix a Bayesian game, viz. $(\overline{\Gamma}, \overline{\mathcal{T}})$, based on $\overline{\Theta}$ that does have an equilibrium. We will not be able to extend the equilibrium of $(\overline{\Gamma}, \overline{\mathcal{T}})$ to an equilibrium of the universal Bayesian game $(\overline{\Gamma}, \overline{\mathcal{T}}^*)$ because there is no equilibrium of this latter Bayesian game.

Arguably, this latter situation is not built from "standard" ingredients. In our main example, we have a failure of Extension, despite the fact that both the players' Bayesian game $(\Gamma, \mathcal{T})$ and the analyst's universal Bayesian game $(\Gamma, \mathcal{T}^*)$ both have an equilibrium. (Indeed, we will see that each $(\Gamma, \mathcal{T}')$ has an equilibrium, for each Θ-based type structure

$\mathcal{T}'$.) Indeed, precisely because the analyst's universal Bayesian game, viz. $(\Gamma, \mathcal{T}^*)$, does have an equilibrium, the analyst may be misled into thinking that he has captured all possible predictions, when he has not.

With this in mind, let us ask: What structure should the analyst use, if (perhaps) not the universal structure? That is, if the goal is to have a "large" type structure that captures all possible predictions associated with a Bayesian equilibrium analysis, what must a "large" structure look like? The results here shed light on this question. In particular, we will see that this structure must contain both fewer and more types than the universal structure. See Section 9c.

Let us reiterate: The main result provides an example of Extension failure with a finite parameter set Θ, a finite game Γ, the players' structure is finite, and the analyst's structure is the Θ-based universal structure. Moreover, precisely because the analyst's Bayesian game has "natural features"—and, in particular, does have an equilibrium—he may be misled to think that, indeed, there is no possible Extension failure for the particular Bayesian game studied. Here, we see otherwise.

This raises another question. Are there situations in which the analyst can be guaranteed that his analysis will not fail the Extension property? We provide two sets of conditions under which the answer is yes. First, we can extend any measurable equilibrium in compact continuous games, provided there are (at most) a countable number of types that are in the analyst's structure but not the players' structure. (In our example, we cannot extend a measurable equilibrium of the players' structure to an equilibrium of the universal structure. In that case, there are an uncountable number of types that are in the analyst's structure but not the players' structure.) Second, we have an Extension property if the analyst's structure satisfies a common prior plus a positivity requirement. (In our example, the analyst's structure is the universal structure, which cannot satisfy these conditions.) See Sections 7-8.

These positive results get at—but do not answer—an important question. To what extent do the Bayesian games studied in applications satisfy or fail Extension? The positive results tell us that, for certain applications, we do indeed satisfy Extension. But, they do not cover all applications. At the theoretical level, addressing this question requires answering a more fundamental question: Can we characterize the set of Bayesian games that satisfy or fail Extension? We don't know the answer and leave this as an open question.

The paper proceeds as follows. Section 3 gives notation. The Extension and Pull-Back Properties are formally defined in Section 4. There, we also show the Pull-Back result. Section 5 shows the negative results. Sections 7-8 provide positive results—conditions on the game and on the type structure which guarantee the Extension property. Finally, Section 9 concludes by discussing some conceptual and formal aspects of the paper.

3 Bayesian Games

Throughout the paper, we adopt the following conventions. We will endow the product of topological spaces with the product topology, and a subset of a topological space with the induced topology. Given a metrizable space Ω, endow Ω with the Borel sigma-algebra. Write $\Delta(\Omega)$ for the set of probability measures on Ω. Endow $\Delta(\Omega)$ with the topology of weak convergence, so that it is again metrizable. (If Ω is Polish, $\Delta(\Omega)$ is also Polish.)

Given sets $\Omega_1, \ldots, \Omega_I$, write $\Omega = \times_i \Omega_i$ and $\Omega_{-i} = \times_{j \neq i} \Omega_j$. Likewise, if $(\omega_1, \ldots, \omega_I) \in \Omega$, write $\omega = (\omega_1, \ldots, \omega_I)$ and $\omega_{-i} = (\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_I)$. Fix maps $f_1, \ldots, f_I$, where each $f_i : \Omega_i \to \Phi_i$. Write f for the product map $f_1 \times \cdots \times f_I : \Omega \to \Phi$, with $f(\omega_1, \ldots, \omega_I) = (f_1(\omega_1), \ldots, f_I(\omega_I))$. Define f_{-i} analogously.

Let Θ be a Polish set, to be interpreted as a set of **payoff types** or the **parameter set**. Throughout, we fix players $1, \ldots, I$, and write i for a particular player from $1, \ldots, I$. A **Θ-based game** is then some $\Gamma = \langle \Theta; C_1, \ldots, C_I; \pi_1, \ldots, \pi_I \rangle$. The set C_i is a **choice** or an **action** set for player i, which is taken to be Polish. A **payoff function** for player i is a measurable map, viz. $\pi_i : \Theta \times C \to \mathbb{R}$, whose range is bounded from above and below. Extend π_i to $\Theta \times \times_{j=1}^{I} \Delta(C_j)$ in the usual way. (Note, the extended functions are measurable and bounded.) A special case will be of particular interest—namely, a **finite game**, i.e., a game where the parameter set Θ and the choice sets $C_1, \ldots, C_I$ are each finite.

To analyze the game, we will need to append to the game a Θ-based interactive type structure.

Definition 3.1 *A **Θ -based interactive metrizable type structure** is some $\mathcal{T} = \langle \Theta; T_1, \ldots, T_I; \beta_1, \ldots, \beta_I \rangle$, where each T_i is a metrizable set and each β_i is a measurable map $\beta_i : T_i \to \Delta(\Theta \times T_{-i})$. A **$\Theta$-based interactive Polish type structure** is some Θ-based interactive metrizable type structure, viz. $\mathcal{T} = \langle \Theta; T_1, \ldots, T_I; \beta_1, \ldots, \beta_I \rangle$, where each T_i is a Polish set. We call $T_1, \ldots, T_I$ **type sets** and the maps $\beta_1, \ldots, \beta_I$ **belief maps**.*

A Θ-based interactive Polish type structure is a Θ-based interactive metrizable type structure. For most of our analysis, we will be interested in the Polish case. However, at times, it will be technically convenient to consider the case of a Θ-based interactive metrizable structure that need not be Polish. When we do so, we will be careful to qualify the type structure with the word "metrizable." When we simply say a **Θ-based interactive type structure**, without qualification, we mean a Θ-based interactive Polish type structure.

A **Θ-based metrizable Bayesian game** consists of a pair $(\Gamma, \mathcal{T})$, where Γ is a Θ-based game and $\mathcal{T}$ is a Θ-based interactive metrizable type structure. Again, when $\mathcal{T}$ is a Θ-based Polish interactive type structure, we call $(\Gamma, \mathcal{T})$ a **Θ-based Bayesian game**. Notice, when we refer to a Θ-based game, we mean some Γ. We reserve the term Θ-based (metrizable) Bayesian game for a pair $(\Gamma, \mathcal{T})$.

The (metrizable) Bayesian game induces strategies. A strategy for i, viz. s_i, is a map from T_i to $\Delta(C_i)$. Let S_i be the set of strategies for player i. Write $\mathrm{id} : \Theta \to \Theta$ for the identity map.

Definition 3.2 *Call a $E_{-i} \subseteq \Theta \times T_{-i}$ a **measurable support** of $\beta_i(t_i)$ if E_{-i} is a measurable set with $\beta_i(t_i)(E_{-i}) = 1$.*

Definition 3.3 *Call a strategy profile, viz. $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I)$, **$\mathbf{t_i}$-measurable** if there exists a measurable support, viz. $E_{-i} \subseteq \Theta \times T_{-i}$, of $\beta_i(t_i)$ so that $(\mathrm{id} \times s_{-i})$ is measurable when the domain is restricted to E_{-i}. Write $S^m_{-i}[t_i]$ for the set of strategy profiles s_{-i} that are t_i-measurable.*

Remark 3.1 *If $\beta_i(t_i)$ has (at most) countable support, then each strategy profile s_{-i} is t_i-measurable.*

Fix some type $t_i \in T_i$ and a strategy profile, viz. $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I)$, that is t_i-measurable. In this case, for each $\sigma_i \in \Delta(S_i)$, $\pi_i(\cdot, \sigma_i, s_{-i}(\cdot))$ is measurable when the domain is restricted to a measurable support of $\beta_i(t_i)$. (See Lemma A.1.) More generally, given a type t_i we can extend π_i to a mapping $\Pi_i[t_i] : S^m_{-i} \to \mathbb{R}$ so that

$$\Pi_i[t_i](s_i, s_{-i}) = \int_{\Theta \times T_{-i}} \pi_i(\theta, s_i(t_i), s_{-i}(t_{-i})) d\beta_i(t_i).$$

(This is well-defined, since for each $t_i \in T_i$ and $s_{-i} \in S^m_{-i}[t_i]$, $\pi_i(\cdot, s_i(t_i), s_{-i}(\cdot))$ is bounded and measurable when the domain is restricted to a measurable support of $\beta_i(t_i)$. This holds whether or not s_i is itself measurable.)

Definition 3.4 *Say $(s_1, \ldots, s_I)$ is a **Bayesian equilibrium** if, for each $i = 1, \ldots, I$ and each $t_i \in T_i$, the following hold:*

(i) s_{-i} is t_i-measurable; and

(ii) for each $r_i \in S_i$, $\Pi_i[t_i](s_i, s_{-i}) \geq \Pi_i[t_i](r_i, s_{-i})$.

Let us review the conditions for $(s_1, \ldots, s_I)$ to be a Bayesian equilibrium. Condition (i) is the minimal requirement so that each $\Pi_i[\cdot](\cdot, s_{-i})$ is well-defined. As such, is appears to be the minimal requirement so that each type $t_i \in T_i$ can compute its expected payoffs. Condition (ii) requires that each type maximize its expected payoffs, given its associated belief. (Note, in condition (ii), there is no requirement that r_i is measurable, since each $\Pi_i[t_i](r_i, s_{-i})$ is well-defined independent of whether or not r_i is measurable.)

Note, in certain instances, condition (i) is trivially satisfied:

Remark 3.2 *Fix a Θ-based Bayesian Game $(G, \mathcal{T})$, where each $\beta_i(t_i)$ has at most countable support. Then, $(s_1, \ldots, s_I)$ is a Bayesian equilibrium if and only if condition (ii) of Definition 3.4 is met.*

A consequence of this remark is: If Θ is (at most) countable and each T_i is (at most) countable, then $(s_1, \ldots, s_I)$ is a Bayesian equilibrium if and only if condition (ii) is met.

We will want to consider a special case, where the analyst studies a Θ-based Bayesian game $(\Gamma, \mathcal{T}^*)$, where $\mathcal{T}^*$ induces all hierarchies of beliefs. Mertens and Zamir (1985), Brandenburger and Dekel (1993), and Heifetz and Samet (1998) each provide (different) canonical constructions of (Θ-based) type structures that contain all hierarchies of beliefs. Here, we will not need to make use of the details a particular construction. Instead, we can focus on certain properties that each of these constructions satisfy. To state a key property, we will need to introduce some terminology.

Given a mapping $f : \Omega \to \Phi$, write $\underline{f} : \Delta(\Omega) \to \Delta(\Phi)$ for the map that takes each measure $\mu \in \Delta(\Omega)$ to its image measure under f.

Definition 3.5 (Mertens and Zamir (1985)) *Fix two Θ-based structures $\mathcal{T} = \langle \Theta; T_1, \ldots, T_I; \beta_1, \ldots, \beta_I \rangle$ and $\mathcal{T}^* = \langle \Theta; T_1^*, \ldots, T_I^*; \beta_1^*, \ldots, \beta_I^* \rangle$ and measurable maps $h_1, \ldots, h_I$, where each $h_i : T_i \to T_i^*$. Call $(h_1, \ldots, h_I)$ a **type morphism (from $\mathcal{T}$ to $\mathcal{T}^*$)** if, for each i, $\underline{\mathrm{id} \times h_{-i}} \circ \beta_i = \beta_i^* \circ h_i$.*

Definition 3.5 says that $(h_1, \ldots, h_I)$ is a type morphism if it preserves the belief maps $\beta_1, \ldots, \beta_I$. Specifically, it requires that the diagram in Figure 4 commutes. Proposition 5.1 in Heifetz and Samet (1998) shows that each type morphism is a mapping that preserves hierarchies of beliefs, i.e., **a hierarchy morphism**.

Each of the canonical constructions of a "universal type structure" in Mertens and Zamir (1985), Brandenburger and Dekel (1993), Heifetz and Samet (1998) induce all hierarchies of beliefs. More precisely, each of these constructions satisfy a terminality property.

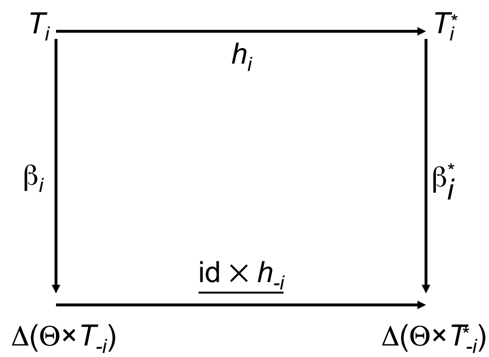

Figure 4

Definition 3.6 *Fix a parameter set Θ and $i = 1, \ldots, I$. Call a Θ-based interactive type structure, viz. $\mathcal{T}^*$, **terminal** if, for each Θ-based interactive metrizable structure $\mathcal{T}$, there is a unique type morphism from $\mathcal{T}$ to $\mathcal{T}^*$.*

If a (Θ-based) type structure is terminal, then it induces each (Θ-based) hierarchy of beliefs—and so it is "terminal" in the sense of Böge and Eisele's (1979) original definition.

Definition 3.7 *Fix a parameter set Θ and $i = 1, \ldots, I$. Call a Θ-based interactive type structure, viz. $\mathcal{T}^*$, **universal** if:*

(i) the type sets $T_1^*, \ldots, T_I^*$ are Polish;

(ii) $\mathcal{T}^*$ it is terminal; and

*(iii) $\mathcal{T}^*$ is **non-redundant**, i.e., no two types induce the same hierarchies of beliefs.*[3]

Write $\mathcal{U}(\Theta)$ for some particular Θ-based interactive universal structure. Notice, we can take $\mathcal{U}(\Theta)$ to be the top-up construction of Mertens and Zamir (1985), the bottom-down construction of Brandenburger and Dekel (1993), the embedding construction of Heifetz and Samet (1998) or some other construction of a Θ-based terminal structure. We will be interested in the particular case where the analyst studies a universal Bayesian game:

[3]We will not need a formal definition of non-redundancy, since we will only make use of consequences of the property.

Definition 3.8 *Call a Θ-based Bayesian game, viz. $(\Gamma, \mathcal{T})$, a* ***universal Bayesian game*** *if $\mathcal{T}$ is universal.*

Bayesian equilibria of universal Bayesian games necessarily involve measurable strategies.

Definition 3.9 *Call a Bayesian equilibrium, viz. $(s_1, \ldots, s_I)$, a* ***measurable equilibrium*** *if each s_i is measurable.*

Lemma 3.1 *Fix a Θ-based universal Bayesian game, viz. $(\Gamma, \mathcal{U}(\Theta))$. Any Bayesian equilibrium of $(\Gamma, \mathcal{U}(\Theta))$ is a measurable equilibrium.*

Proof. Fix some Θ-based universal Bayesian game, viz. $(\Gamma, \mathcal{U}(\Theta))$, where $\mathcal{U}(\Theta) = \langle\Theta; U_1, \ldots, U_I; \gamma_1, \ldots, \gamma_I\rangle$. It follows that $\mathcal{U}(\Theta)$ is complete in the sense of Brandenburger (2003), i.e., that each $\gamma_1, \ldots, \gamma_I$ is onto. (See Proposition 4.1 in Friedenberg, 2010 or, alternatively, use Theorem 4 in Meier, 2010.) As such, for each i, there is a type $u_i \in U_i$ so that the support of $\gamma_i(u_i)$ is $\Theta \times U_{-i}$. Applying condition (i) of a Bayesian equilibrium, for each i, s_{-i} is measurable. ■

4 The Extension and Pull-Back Properties

Fix two Θ-based structures $\mathcal{T} = \langle\Theta; T_1, \ldots, T_I; \beta_1, \ldots, \beta_I\rangle$ and $\mathcal{T}^* = \langle\Theta; T_1^*, \ldots, T_I^*; \beta_1^*, \ldots, \beta_I^*\rangle$. We want to capture the idea that there is a **hierarchy morphism** from $\mathcal{T}$ to $\mathcal{T}^*$, i.e., for each player i and each type t_i in T_i, there is a type t_i^* in T_i^* that induces the same hierarchy of beliefs. As we have seen, the type morphism concept allows us to capture this idea without explicitly describing hierarchies of beliefs. We will state the Extension and Pull-Back properties relative to the type morphism concept. (We discuss this choice further below.) Given a Θ-based game Γ, write s_i for a strategy of player i in the Bayesian game $(\Gamma, \mathcal{T})$, and write s_i^* for a strategy of player i in the Bayesian game $(\Gamma, \mathcal{T}^*)$. Now, we can state the Equilibrium Extension and Pull-Back Properties.

Definition 4.1 *Let $\mathcal{T}$ and $\mathcal{T}^*$ be two Θ-based interactive (metrizable) type structures. Say $\mathcal{T}$ can be* ***mapped to $\mathcal{T}^*$ (via $\mathbf{h_1}, \ldots, \mathbf{h_I}$)*** *if $(h_1, \ldots, h_I)$ is a type morphism from $\mathcal{T}$ to $\mathcal{T}^*$.*

Definition 4.2 *Let $\mathcal{T}$ and $\mathcal{T}^*$ be two Θ-based interactive (metrizable) type structures, so that $\mathcal{T}$ can be mapped to $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$. Say a Bayesian Equilibrium, viz. $(s_1, \ldots, s_I)$,* *of* $(\Gamma, \mathcal{T})$ *can be* ***extended*** *to a Bayesian Equilibrium of* $(\Gamma, \mathcal{T}^*)$ *if there exists a Bayesian Equilibrium, viz.* $(s_1^*, \ldots, s_I^*)$*, of* $(\Gamma, \mathcal{T}^*)$ *so that* $(s_1^* \circ h_1, \ldots, s_I^* \circ h_I) = (s_1, \ldots, s_I)$*. The pair* $\langle \mathcal{T}, \mathcal{T}^* \rangle$ *satisfies the* ***Equilibrium Extension Property for the*** $\boldsymbol{\Theta}$***-based game*** $\boldsymbol{\Gamma}$ *if each Bayesian Equilibrium of* $(\Gamma, \mathcal{T})$ *can be extended to a Bayesian Equilibrium of* $(\Gamma, \mathcal{T}^*)$*. Say the pair* $\langle \mathcal{T}, \mathcal{T}^* \rangle$ *satisfies the* ***Equilibrium Extension Property*** *if it satisfies the Equilibrium Extension Property for each* Θ*-based game* Γ*.*

Definition 4.3 *Let* $\mathcal{T}$ *and* $\mathcal{T}^*$ *be two* Θ*-based interactive (metrizable) type structures, so that* $\mathcal{T}$ *can be mapped to* $\mathcal{T}^*$ *via* $(h_1, \ldots, h_I)$*. Say a Bayesian Equilibrium, viz.* $(s_1^*, \ldots, s_I^*)$*, of* $(\Gamma, \mathcal{T}^*)$ *can be* ***pulled-back*** *to a Bayesian Equilibrium of* $(\Gamma, \mathcal{T})$ *if* $(s_1^* \circ h_1, \ldots, s_I^* \circ h_I)$ *is a Bayesian Equilibium of* $(\Gamma, \mathcal{T})$*. The pair* $\langle \mathcal{T}, \mathcal{T}^* \rangle$ *satisfies the* ***Equilibrium Pull-Back Property*** *if, for each* Θ*-based game* Γ*, any Bayesian Equilibrium of* $(\Gamma, \mathcal{T}^*)$ *can be pulled-back to a Bayesian Equilibrium of* $(\Gamma, \mathcal{T})$*.*

Section 5 will show that the Equilibrium Extension Property may fail (for structures that are Polish). Sections 7-8 will provide conditions under which the Equilibrium Extension Property is satisfied. On the other hand, the Equilibrium Pull-Back Property is always satisfied, even for metrizable type structures.

Proposition 4.1 *Let* $\mathcal{T}$ *and* $\mathcal{T}^*$ *be two* Θ*-based interactive (metrizable) type structures, where* $\mathcal{T}$ *can be mapped to* $\mathcal{T}^*$ *via* $(h_1, \ldots, h_I)$*. The pair* $\langle \mathcal{T}, \mathcal{T}^* \rangle$ *satisfies the Equilibrium Pull-Back Property.*

The arguments to prove Proposition 4.1 are well-known and so relegated to Appendix A.

How does the Pull-Back property fit with what is known from the literature? Ely and Peski (2006) and Dekel, Fudenberg and Morris (2007) give an example of a Θ-based game Γ and Θ-based structures $\mathcal{T}$ and $\mathcal{T}^*$, so that there is (a) a hierarchy morphism, viz. $(h_1, \ldots, h_I)$, from $\mathcal{T}$ to $\mathcal{T}^*$, i.e., maps that preserve hierarchies of beliefs, (b) $(s_1^*, \ldots, s_I^*)$ is a Bayesian Equilibrium of $(\Gamma, \mathcal{T}^*)$, but (c) $(s_1^* \circ h_1, \ldots, s_I^* \circ h_I)$ is not a Bayesian Equilibrium of $(\Gamma, \mathcal{T})$. (Such an example is reproduced in Section 9d below.) This would appear to contradict Proposition 4.1. However, we note that while, in that example, there is a hierarchy morphism from $\mathcal{T}$ to $\mathcal{T}^*$, there is no type morphism from $\mathcal{T}$ to $\mathcal{T}^*$. We defined the Pull-Back Property in terms of type morphisms and not hierarchy morphisms. Now we see why.

Back to the relationship between hierarchy and type morphisms. Consider two Θ-based type structures, viz. $\mathcal{T}$ to $\mathcal{T}^*$. (So, in particular, $\mathcal{T}^*$ has Polish type sets.) Suppose that the structure $\mathcal{T}^*$ is non-redundant (i.e., no two types induce the same hierarchy of beliefs).[4] In this case, any hierarchy morphism from $\mathcal{T}$ to $\mathcal{T}^*$ is also a type morphism. (See Corollary 6.2 in Friedenberg and Meier (forthcoming).) So, with this in mind, Proposition 4.1 says that, if the (analyst's) structure $\mathcal{T}^*$ is non-redundant and there is a hierarchy morphism from the (players') structure $\mathcal{T}$ to the (analyst's) structure $\mathcal{T}^*$, then any equilibrium of the (analyst's) Bayesian game $(\Gamma, \mathcal{T}^*)$ induces an equilibrium of the (players') Bayesian game $(\Gamma, \mathcal{T})$ whether or not the (players') structure $\mathcal{T}$ is non-redundant. For the example in Ely and Peski (2006) and Dekel, Fudenberg and Morris (2007), $\mathcal{T}^*$ is redundant.

More generally, Definitions 4.2-4.3 capture the Extension and Pull-Back ideas, when we restrict attention to the case where the analyst's structure $\mathcal{T}^*$ is Polish and non-redundant. Indeed, we will construct an example that fails Equilibrium Extension, where both the players' and analyst's structures are Polish and non-redundant. (In Section 9a, we explain why non-redundancy is conceptually important.) The positive Extension results in Sections 7-8 also presume non-redundancy.

Back to the idea of a universal Bayesian game. We have the following property:

Lemma 4.1 *Fix a Θ-based Bayesian game $(\Gamma, \mathcal{T})$. The pair $\langle \mathcal{T}, \mathcal{U}(\Theta)\rangle$ satisfies the Equilibrium Extension Property for Γ if and only if the following condition is satisfied: For every Θ-based interactive metrizable structure $\mathcal{T}^*$ so that $\mathcal{T}$ can be mapped to $\mathcal{T}^*$, the pair $\langle \mathcal{T}, \mathcal{T}^*\rangle$ satisfies the Equilibrium Extension Property for Γ.*

5 Extension Failure

In this section, we construct an example of a finite Θ-based game Γ and a Θ-based interactive type structures $\mathcal{T}$, so that $\langle \mathcal{T}, \mathcal{U}(\Theta)\rangle$ fails the Equilibrium Extension Property for Γ.

Let $\Theta = \{\theta_1, \theta_2, \theta_3\}$. Construct a three-player Θ-based game $\Gamma = \langle \Theta; C_1, C_2, C_3; \pi_1, \pi_2, \pi_3\rangle$, where each $C_i = \{c_i, d_i, e_i\}$ and the payoff functions are depicted in Figure 5 below.

For each θ-section of the game Γ, the choice profile (e_1, e_2, e_3) is a pure strategy Nash equilibrium. As such, we have the following:

Remark 5.1 *For any Θ-based structure $\mathcal{T}'$, $(\Gamma, \mathcal{T}')$ has a Bayesian equilibrium. In particular, for each $(\Gamma, \mathcal{T}')$, there is a Bayesian equilibrium where each type of each player chooses e_i with probability one.*

[4] To preserve notation, we don't provide a formal definition. We instead make use of consequences of non-redundancy.

θ_1-section

P3 c_3	P2 c_2	P2 d_2	P2 e_2
P1 c_1	1, 4, 3	2, 1, 3	1, 0, 1
P1 d_1	3, 1, 1	2, 3, 1	1, 0, 1
P1 e_1	0, 1, 1	0, 1, 1	1, 1, 1

P3 d_3	P2 c_2	P2 d_2	P2 e_2
P1 c_1	3, 3, 1	3, 1, 1	1, 0, 1
P1 d_1	2, 1, 3	1, 3, 4	1, 0, 1
P1 e_1	0, 1, 1	0, 1, 1	1, 1, 1

P3 e_3	P2 c_2	P2 d_2	P2 e_2
P1 c_1	1, 1, 0	1, 1, 0	1, 1, 1
P1 d_1	1, 1, 0	1, 1, 0	1, 1, 1
P1 e_1	1, 1, 1	1, 1, 1	1, 1, 1

θ_2-section

P3 c_3	P2 c_2	P2 d_2	P2 e_2
P1 c_1	1, 4, 3	2, 1, 3	1, 0, 1
P1 d_1	3, 1, 1	2, 3, 1	1, 0, 1
P1 e_1	0, 1, 1	0, 1, 1	1, 1, 1

P3 d_3	P2 c_2	P2 d_2	P2 e_2
P1 c_1	2, 3, 1	3, 1, 1	1, 0, 1
P1 d_1	3, 1, 3	1, 3, 4	1, 0, 1
P1 e_1	0, 1, 1	0, 1, 1	1, 1, 1

P3 e_3	P2 c_2	P2 d_2	P2 e_2
P1 c_1	1, 1, 0	1, 1, 0	1, 1, 1
P1 d_1	1, 1, 0	1, 1, 0	1, 1, 1
P1 e_1	1, 1, 1	1, 1, 1	1, 1, 1

θ_3-section

P3 c_3	P2 c_2	P2 d_2	P2 e_2
P1 c_1	1, 1, 1	1, 1, 1	0, 0, 0
P1 d_1	1, 1, 1	1, 1, 1	0, 0, 0
P1 e_1	0, 0, 0	0, 0, 0	0, 0, 0

P3 d_3	P2 c_2	P2 d_2	P2 e_2
P1 c_1	1, 1, 1	1, 1, 1	0, 0, 0
P1 d_1	1, 1, 1	1, 1, 1	0, 0, 0
P1 e_1	0, 0, 0	0, 0, 0	0, 0, 0

P3 e_3	P2 c_2	P2 d_2	P2 e_2
P1 c_1	0, 0, 0	0, 0, 0	0, 0, 0
P1 d_1	0, 0, 0	0, 0, 0	0, 0, 0
P1 e_1	0, 0, 0	0, 0, 0	1, 1, 1

Figure 5

Nonetheless, we construct a Θ-based structure $\mathcal{T}$, where some (measurable) equilibrium of $(\Gamma, \mathcal{T})$ cannot be extended to an equilibrium of $(\Gamma, \mathcal{U}(\Theta))$. (Of course, there will be another equilibrium of $(\Gamma, \mathcal{T})$ that can be extended to an equilibrium of $(\Gamma, \mathcal{U}(\Theta))$—i.e., the one just mentioned above.)

The Θ-based structure $\mathcal{T} = \langle \Theta; T_1, T_2, T_3; \beta_1, \beta_2, \beta_3 \rangle$ will be such that each $T_i = \{t_i\}$ and each $\beta_i(t_i)$ is concentrated on (θ_3, t_{-i}). The structure $\mathcal{T}$ is non-redundant. Moreover, there is an equilibrium of $(\Gamma, \mathcal{T})$ where each type of each player chooses e_i with probability one. But, note, there is also an equilibrium of $(\Gamma, \mathcal{T})$, viz. (s_1, s_2, s_3), where each t_i plays $\{c_i, d_i\}$ with probability one. In fact, there are many such Bayesian equilibria. We will see that no such equilibrium can be extended to an equilibrium of $(\Gamma, \mathcal{U}(\Theta))$.

We will construct a Θ-based type structure, viz. $\mathcal{T}^*$, so that we can map $\mathcal{T}$ to $\mathcal{T}^*$ but we cannot extend the equilibrium (s_1, s_2, s_3) (i.e., where each t_i plays $\{c_i, d_i\}$ with probability one) of $(\Gamma, \mathcal{T})$ to an equilibrium of $(\Gamma, \mathcal{T}^*)$. With this, it follows from Lemma 4.1 that we cannot extend the equilibrium (s_1, s_2, s_3) (i.e., where each t_i plays $\{c_i, d_i\}$ with probability one) of $(\Gamma, \mathcal{T})$ to an equilibrium of $(\Gamma, \mathcal{U}(\Theta))$.

To construct the Θ-based structure $\mathcal{T}^*$, we begin with a subset of the parameters, viz.

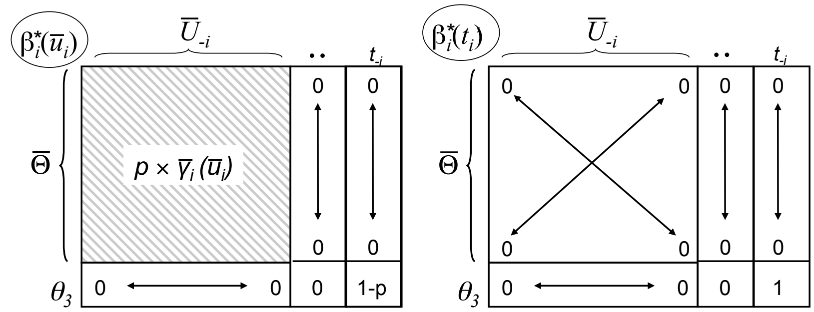

Figure 6

$\overline{\Theta} = \{\theta_1, \theta_2\} \subseteq \Theta$ and some particular universal structure $\mathcal{U}(\overline{\Theta}) = \langle \overline{\Theta}; \overline{U}_1, \overline{U}_2, \overline{U}_3; \overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3 \rangle$. For the Θ-based structure $\mathcal{T}^* = \langle \Theta; T^*_1, T^*_2, T^*_3; \beta^*_1, \beta^*_2, \beta^*_3 \rangle$ refer to Figure 6: Each T^*_i is the disjoint union of T_i and $\overline{U}_i$. For $t_i \in T_i \subseteq T^*_i$, take $\beta^*_i(t_i)(\{(\theta_3, t_{-i})\}) = 1$. (The profile t_{-i} is the one contained in T_{-i}.) For types $u_i \in \overline{U}_i \subseteq T^*_i$, define $\beta^*_i(u_i)$ as follows. Fix some $p \in (0, 1)$. (Note, p will be chosen to be the same for each type $u_i \in \overline{U}_i$.) Take $\beta^*_i(u_i)(E_{-i}) = p\overline{\gamma}_i(u_i)(E_{-i} \cap (\overline{\Theta} \times \overline{U}_{-i})) + (1 - p)$ for each event E_{-i} in $\Theta \times T^*_{-i}$ with $(\theta_3, t_{-i}) \in E_{-i}$. Take $\beta^*_i(u_i)(E_{-i}) = p\overline{\gamma}_i(u_i)(E_{-i} \cap (\overline{\Theta} \times \overline{U}_{-i}))$ for each event E_{-i} in $\Theta \times T^*_{-i}$.

Let us point to two properties. First:

Property 5.1 *Both $\mathcal{T}$ and $\mathcal{T}^*$ are non-redundant.*

This statement is immediate for $\mathcal{T}$. For $\mathcal{T}^*$, recall that $\mathcal{U}(\overline{\Theta})$ is non-redundant, so that types in $\overline{U}_i$ continue to induce distinct hierarchies of beliefs in $\mathcal{T}^*$. We also have that $\beta^*_i(u_i)(\overline{\Theta} \times T^*_{-i}) > 0$, for each $u_i \in \overline{U}_i$, and $\beta^*_i(t_i)(\overline{\Theta} \times \overline{T}^*_{-i}) = 0$. As such, types in $\overline{T}_i$ have distinct first-order beliefs from the type t_i. Second:

Property 5.2 *The structure $\mathcal{T}$ can be mapped into $\mathcal{T}^*$ via $(\mathrm{id}_1, \ldots, \mathrm{id}_I)$, where $\mathrm{id}_i : T_i \to T^*_i$ denotes the identity map.*

Now, we turn to Equilibrium Extension. The Bayesian game $(\Gamma, \mathcal{T})$ has some equilibrium—in fact, many. Referring to Remark 5.1, one such equilibrium is where strategies map each t_i into a measure that assigns probability one to e_i, and this equilibrium can be extended to an equilibrium of the Bayesian game $(\Gamma, \mathcal{T}^*)$. But there are also many (measurable) equilibria of $(\Gamma, \mathcal{T})$, where the strategies map each t_i into a measure that assigns probability one to $\{c_i, d_i\}$. We will see that no such equilibrium can be extended to an equilibrium of $(\Gamma, \mathcal{T}^*)$.

Why is this the case? Consider a $\overline{\Theta}$-based game, viz. $\overline{\Gamma} = \langle \overline{\Theta}; \overline{C}_1, \overline{C}_2, \overline{C}_3; \overline{\pi}_1, \overline{\pi}_2, \overline{\pi}_3 \rangle$, that is in a sense a restriction of the game Γ. The game $\overline{\Gamma}$ is given in Figure 7. Note, we restrict both the parameter set and the choice sets, so that each $\overline{C}_i = \{c_i, d_i\}$. Now we will see two seemingly contradictory facts: First, there is no Bayesian equilibrium of the game $(\overline{\Gamma}, \mathcal{U}(\overline{\Theta}))$. Second, if there is an equilibrium of $(\Gamma, \mathcal{T}^*)$ so that, for each i, $t_i \in T_i \subseteq T_i^*$ plays $\{c_i, d_i\}$ with probability one, then there is a Bayesian equilibrium of $(\overline{\Gamma}, \mathcal{U}(\overline{\Theta}))$. Putting these two together, we get that there is no equilibrium of $(\Gamma, \mathcal{T}^*)$ so that, for each i, $t_i \in T_i \subseteq T_i^*$ plays $\{c_i, d_i\}$ with probability one. As such, we cannot extend an equilibrium of $(\Gamma, \mathcal{T})$ to an equilibrium of $(\Gamma, \mathcal{T}^*)$.

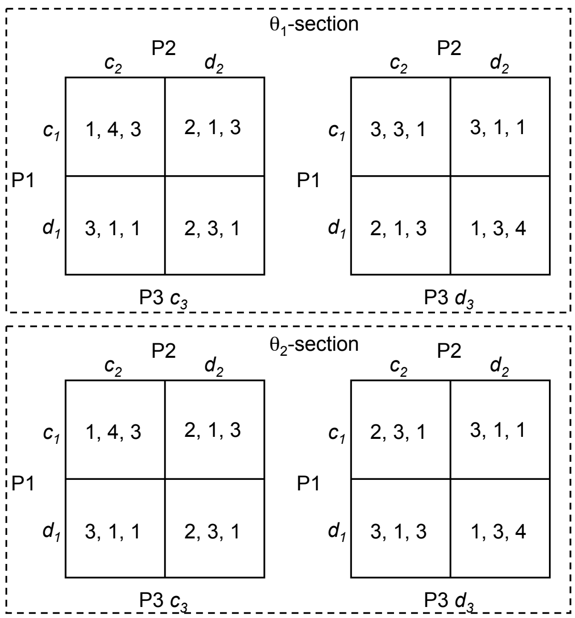

Figure 7

Let's review each of these steps. The first step is:

Proposition 5.1 *There is no equilibrium of the (finite) $\overline{\Theta}$-based Bayesian game $(\overline{\Gamma}, \mathcal{U}(\overline{\Theta}))$.*

To see why this is the case, let us recall a result from Simon (2003), who studies the $\overline{\Theta}$-based game $\overline{\Gamma}$. He constructs a $\overline{\Theta}$-based metrizable structure $\overline{\mathcal{T}}$ to that there is no measurable equilibrium of the Bayesian game $(\overline{\Gamma}, \overline{\mathcal{T}})$. More specifically:

Proposition 5.2 (Simon, 2003) *There exists a $\overline{\Theta}$-based metrizable structure $\overline{\mathcal{T}}$ so that:*

(i) there is an equilibrium of the Bayesian game $(\overline{\Gamma}, \overline{\mathcal{T}})$*, but*

(ii) there is no measurable equilibrium of the Bayesian game $(\overline{\Gamma}, \overline{\mathcal{T}})$*.*

Appendix B reviews the result and specifically points to our use of the phrase metrizable. We will only make use of part (ii) and include part (i) only for completeness. From part (ii) and our earlier results, we can see Proposition 5.1.[5]

Proof of Proposition 5.1. Suppose, contra hypothesis, that there is an equilibrium of $(\overline{\Gamma}, \mathcal{U}(\overline{\Theta}))$. By Lemma 3.1, the equilibrium is measurable. But then, using the Pull-Back property, there is a measurable equilibrium of each $\overline{\Theta}$-based metrizable Bayesian Game, viz. $(\overline{\Gamma}, \overline{\mathcal{T}}^*)$. But this contradicts Simon's (2003) result (Proposition 5.2(ii)). ■

Now turn to the second step. It is:

Lemma 5.1 *If there is an equilibrium of* $(\Gamma, \mathcal{T}^*)$ *so that, for each* i*,* $t_i \in T_i \subseteq T_i^*$ *plays* $\{c_i, d_i\}$ *with probability one, then there is a Bayesian equilibrium of* $(\overline{\Gamma}, \mathcal{U}(\overline{\Theta}))$*.*

Proof. Fix a Bayesian Equilibrium, viz. (s_1^*, s_2^*, s_3^*) so that, for each i, $s_i^*(t_i)(\{c_i, d_i\}) = 1$.

Consider a type $u_i \in \overline{U}_i \subseteq T_i^*$. For this type, the expected payoffs from choosing some $f_i \in \{c_i, d_i\}$ are

$$\mathbb{E}(u_i, f_i) = p \int_{\overline{\Theta} \times \overline{U}_{-i}} \pi_i(\theta, f_i, s_{-i}^*(t_{-i}^*)) d\beta_i^*(u_i) + (1-p)x.$$

This type's expected payoffs from choosing e_i are

$$\mathbb{E}(u_i, e_i) = p \int_{\overline{\Theta} \times \overline{U}_{-i}} \pi_i(\theta, e_i, s_{-i}^*(t_{-i}^*)) d\beta_i^*(u_i).$$

Also, note that, for each $(\theta, u_{-i}) \in \overline{\Theta} \times \overline{U}_{-i}$,

$$\pi_i(\theta, f_i, s_{-i}^*(u_{-i})) \geq \pi_i(\theta, e_i, s_{-i}^*(u_{-i})) \quad \text{for each } f_i \in \{c_i, d_i\}.$$

(If $s_{-i}^*(u_{-i})(\{c_i, d_i\}) > 0$, then the inequality is strict.) So, for each $f_i \in \{c_i, d_i\}$,

$$p \int_{\overline{\Theta} \times \overline{U}_{-i}} \pi_i(\theta, f_i, s_{-i}^*(t_{-i}^*)) d\beta_i^*(u_i) \geq p \int_{\overline{\Theta} \times \overline{U}_{-i}} \pi_i(\theta, e_i, s_{-i}^*(t_{-i}^*)) d\beta_i^*(u_i).$$

[5] Certainly, formally, this result is not a far leap from Proposition 5.2. Nonetheless, we note that Simon himself does not make the leap. In our view, this leap is conceptually important for the literature, since there is an equilibrium of the Bayesian game studied in Simon (2003).

Indeed, since $1 > p$ and $x > 0$,

$$\max\{\mathbb{E}(u_i, c_i), \mathbb{E}(u_i, d_i)\} > \mathbb{E}(u_i, e_i).$$

By condition (ii) of a Bayesian equilibrium, for each $u_i \in \overline{U}_i$, $s_i^*(u_i)(\{c_i, d_i\}) = 1$.

Now turn to the $\overline{\Theta} = \{\theta_1, \theta_2\}$-based game Bayesian $(\overline{\Gamma}, \mathcal{U}(\overline{\Theta}))$. Restrict the maps (s_1^*, s_2^*, s_3^*) to this Bayesian game. So, now, we have maps $\overline{s}_i : \overline{U}_i \to \Delta(\overline{C}_i)$ where, for each $u_i \in \overline{U}_i$ and each event $\overline{E}_i$ in $\overline{C}_i$, $\overline{s}_i(u_i)(\overline{E}_i) = s_i^*(u_i)(\overline{E}_i)$. We will show that $(\overline{s}_1, \overline{s}_2, \overline{s}_3)$ also constitutes an equilibrium of the Bayesian game $(\overline{\Gamma}, \mathcal{U}(\overline{\Theta}))$.

To show condition (i): Recall from the proof of Lemma 3.1 that $\overline{\gamma}_1, \overline{\gamma}_2$, and $\overline{\gamma}_3$ are each onto. (Again, see Proposition 4.1 in Friedenberg (2010) or, alternatively, use Theorem 4 in Meier (2010).) It follows that, for each i, there exists some $u_i \in \overline{U}_i$ so that $\text{supp}\,\beta_i^*(u_i) = \Theta \times T_{-i}^*$. Using condition (i) of Definition 3.4 applied to (s_1^*, s_2^*, s_3^*), it follows that each of s_1^*, s_2^*, and s_3^* are measurable. Now, fix some event $\overline{E}_i$ in $\{c_i, d_i\}$. Note, that $\overline{E}_i$ is also an event in $\{c_i, d_i, e_i\}$ and so $(s_i^*)^{-1}(\overline{E}_i)$ is measurable. Using the fact that $(\overline{s}_i)^{-1}(\overline{E}_i) = \overline{U}_i \cap (s_i^*)^{-1}(\overline{E}_i)$, we get that $(\overline{s}_i)^{-1}(\overline{E}_i)$ is measurable. This implies condition (i).

To show condition (ii): Fix a strategy of player i for the Bayesian game $(\overline{\Gamma}, \mathcal{U}(\overline{\Theta}))$, viz. $\overline{r}_i : \overline{U}_i \to \Delta(\overline{C}_i)$. This strategy can be extended to a strategy for the Bayesian game $(\Gamma, \mathcal{T}^*)$, viz. $r_i^* : T_i^* \to \Delta(C_i)$. Specifically, for each type $u_i \in \overline{U}_i$ and each event E_i in C_i, set $r_i^*(u_i)(E_i) = \overline{r}_i(u_i)(E_i \cap \overline{C}_i)$. Also, choose $r_i^*(t_i)$ to be any element of $\Delta(C_i)$ with $r_i^*(t_i)(\overline{C}_i) = 1$. Moreover, under this extension,

$$\Pi_i[u_i](r_i^*, s_{-i}^*) = p\overline{\Pi}_i(u_i, \overline{r}_i, \overline{s}_{-i}) + (1-p)x \quad \text{for all } u_i \in \overline{U}_i.$$

Return to the fact that (s_1^*, s_2^*, s_3^*) is an equilibrium for the Bayesian game $(\Gamma, \mathcal{T}^*)$. Then, using the above, for each $u_i \in \overline{U}_i$ and each $\overline{r}_i \in \overline{S}_i$,

$$\begin{aligned} p\overline{\Pi}_i(u_i, \overline{s}_i, \overline{s}_{-i}) + (1-p)x &= \Pi_i(u_i, s_i^*, s_{-i}^*) \\ &\geq \Pi_i(u_i, r_i^*, s_{-i}^*) \\ &= p\overline{\Pi}_i(u_i, \overline{r}_i, \overline{s}_{-i}) + (1-p)x, \end{aligned}$$

where r_i^* is defined as above. It follows that, for each $u_i \in \overline{U}_i$,

$$\overline{\Pi}_i(u_i, \overline{s}_i, \overline{s}_{-i}) \geq \overline{\Pi}_i(u_i, \overline{r}_i, \overline{s}_{-i}) \quad \text{for all } \overline{r}_i \in \overline{S}_i,$$

i.e., condition (ii) is satisfied. ■

As a corollary of Proposition 5.1 and Lemma 5.1, we have:

Corollary 5.1 *There is no equilibrium of* $(\Gamma, \mathcal{T}^*)$ *so that, for each* i, $t_i \in T_i \subseteq T_i^*$ *plays* $\{c_i, d_i\}$ *with probability one.*

The negative result is an immediate implication of Corollary 5.1.

Theorem 5.1 *The pair* $\langle \mathcal{T}, \mathcal{U}(\Theta) \rangle$ *fails the Equilibrium Extension Property.*

Proof. Consider a strategy profile (s_1, s_2, s_3) of the Bayesian game $(\Gamma, \mathcal{T})$, with $s_i(t_i)(\{c_i, d_i\}) = 1$ for each $i = 1, 2, 3$. This is an equilibrium of that Bayesian game. Fix a Bayesian equilibrium $(s_1^{**}, s_2^{**}, s_3^{**})$ of the Bayesian game $(\Gamma, \mathcal{U}(\Theta))$. By Corollary 5.1, $s_i \neq (s_i^{**} \circ \mathrm{id}_i)$ for some i. ∎

Before concluding this Section, we note that this extension failure is *not* a result of an attempt to extend a non-measurable equilibrium to a measurable equilibrium. In particular:

Remark 5.2 *The proof of Theorem 5.1 shows that we may not be able to extend a measurable equilibrium of the Bayesian game* $(\Gamma, \mathcal{T})$ *to any equilibrium of* $(\Gamma, \mathcal{U}(\Theta))$.

6 Positive Results

In Section 5, we saw that the Equilibrium Extension Property may fail. Sections 7-8 we ask: Are there (interesting) situations where the Extension Property does obtain? In providing an answer to this question, we will gain a better understanding of when there can vs. cannot be a failure of the Extension Property.

We focus on providing positive results for the case of non-redundant Θ-based type structures. We do not give a formal definition of non-redundancy. Instead, we will make use of a single property that follows from it. The property is manifested in Lemma 6.1 below.[6]

Recall, a measurable map is bimeasurable if the image of each measurable set is itself measurable. It is an embedding if it is an injective bimeasurable map. Now, the property:

Lemma 6.1 *Fix non-redundant* Θ*-based type structures* $\mathcal{T}$ *to* $\mathcal{T}^*$. *If* $\mathcal{T}$ *can be mapped to* $\mathcal{T}^*$ *via* $(h_1, \ldots, h_I)$, *then each* h_i *is an embedding from* T_i *to* T_i^*. *If, in addition,* $\mathcal{T}^*$ *can be mapped to* $\mathcal{T}$ *via* $(h_1^*, \ldots, h_I^*)$, *then each map* h_i *is a measurable isomorphism with* $h_i = (h_i^*)^{-1}$.

[6] The proofs for this section can be found in the Online Appendix.

With Lemma 6.1 in mind, we give the following definitions.

Definition 6.1 *Say $\mathcal{T}$ can be* ***embedded*** *into $\mathcal{T}^*$ (via $(h_1, \ldots, h_I)$) if $\mathcal{T}$ can be mapped into $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$, and each h_i is an embedding from T_i to T_i^*. Say $\mathcal{T}$ and $\mathcal{T}^*$ are* ***isomorphic*** *if $\mathcal{T}$ can be embedded into $\mathcal{T}^*$ and $\mathcal{T}^*$ can be embedded into $\mathcal{T}$. Say $\mathcal{T}$ can be* ***properly embedded*** *into $\mathcal{T}^*$ if $\mathcal{T}$ can be embedded into $\mathcal{T}^*$ but $\mathcal{T}^*$ cannot be embedded into $\mathcal{T}$.*

Fix Θ-based structures $\mathcal{T}$. Say $\times_{i=1}^{I} E_i$ is a **belief-closed subset** of $T = \times_{i=1}^{I} T_i$ if each E_i is measurable in T_i and, for each $t_i \in E_i$, $\beta_i(t_i)(\Theta \times E_{-i}) = 1$.

Lemma 6.2 *Fix Θ-based structures $\mathcal{T}$ and $\mathcal{T}^*$. If $\mathcal{T}$ can be embedded into $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$, then $\times_{i=1}^{I} h_i(T_i)$ is a belief-closed subset of $T^* = \times_{i=1}^{I} T_i^*$.*

Now, by the Pull-Back Property, we have:

Corollary 6.1 *Let $\mathcal{T}$ and $\mathcal{T}^*$ be two isomorphic Θ-based interactive type structures. Then, the pair $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the Equilibrium Extension Property.*

In light of Corollary 6.1, we will focus on the case in which $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$. In this case, we have the following:

Lemma 6.3 *Fix Θ-based structures $\mathcal{T}$ and $\mathcal{T}^*$, so that $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$. Then, for some $i = 1, \ldots, I$, $h_i(T_i) \subsetneq T_i^*$.*

7 Compact and Continuous Games

In Section 5, we saw that, even in the case of a finite game, the Extension property may fail. Note two features of the example: First, because the players' type structure was finite, each equilibrium of the players' Bayesian game was a measurable equilibrium. Second, there were an uncountable number of types that are in the analyst's structure but not in the players' structure—so, the analyst's structure is "large" relative to the players' type structure.

This section picks up on these two features. In particular, fix a finite game, viz. Γ, and an associated "players' Bayesian game," viz. $(\Gamma, \mathcal{T})$. We will see that, if there are (at most) a countable number of types that are in the analyst's structure but not the players' structure, then we will be able to extend any measurable equilibrium of the players' structure to the analyst's structure. Thus, we can only have an extension failure for measurable equilibria, if there are an uncountable number of types that are in the analyst's structure but not the players' structure.

Definition 7.1 *Say a Θ-based game, viz. $\Gamma = \langle \Theta; C_1, \ldots, C_I; \pi_1, \ldots, \pi_I \rangle$, is **compact and continuous** if, for each player $i = 1, \ldots, I$, C_i is compact and π_i is continuous.*

(Note, there is no requirement that Θ be compact.)

Proposition 7.1 *Fix Θ-based structures $\mathcal{T}$ and $\mathcal{T}^*$, so that $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$ and each $T_1^* \backslash h_1(T_1), \ldots, T_I^* \backslash h_I(T_I)$ is (at most) countable. If Γ is compact and continuous, then any measurable equilibrium of $(\Gamma, \mathcal{T})$ can be extended to a measurable equilibrium of $(\Gamma, \mathcal{T}^*)$.*

The proof can be found in Appendix C. Here, we give the idea. In so doing, we will see the role of the requirement that each $T_i^* \backslash h_i(T_i)$ is countable.

Suppose $\mathcal{T}$ can be embedded into $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$. Fix a measurable equilibrium $(s_1, \ldots, s_I)$ of the Bayesian game $(\Gamma, \mathcal{T})$. We want to show that there is an equilibrium of the Bayesian Game $(\Gamma, \mathcal{T}^*)$, viz. $(s_1^*, \ldots, s_I^*)$, that extends the equilibrium $(s_1, \ldots, s_I)$, i.e., that satisfies $(s_1, \ldots, s_I) = (s_1^* \circ h_1, \ldots, s_I^* \circ h_I)$.

We will begin by constructing a certain game of complete information, viz. G, that depends on the game Γ and the equilibrium $(s_1, \ldots, s_I)$. There will be many players in this game, each corresponding to a type in $T_i^* \backslash h_i(T_i)$ for some player i. As such, there are (at most) a countable number of players in this game. Each such player $t_i \in T_i^* \backslash h_i(T_i)$ gets to make a choice from C_i, as in Γ. The payoff functions will be constructed in a specific way. In particular, they will depend on Γ and the equilibrium $(s_1, \ldots, s_I)$.

The complete information game G is compact and continuous. Compactness follows from the fact that the underlying game is compact. Continuity uses the fact that the underlying game is continuous, but it does not follow immediately from this fact. There are two issues: First, the payoff functions depend on the equilibrium and the equilibrium may be discontinuous. Second, there may be an infinite (but countable) number of players in the game and, when there are a countable number of players, payoff functions may be discontinuous even if the choice set is finite. See Peleg (1969).[7]

Now we have a compact and continuous complete information game G, with a countable number of players. As such, we can apply Glicksberg's (1952) fixed-point theorem to show that there exists a mixed-strategy equilibrium of G.

[7] Satoru Takahashi pointed us to the fact that, if a game with a countable number of players is (in a sense) "generated" by a compact and continuous game of incomplete information, then the payoff functions are nonetheless continuous. In so doing, Takahashi generalized a result in a previous version of this paper. We are very much indebted to Satoru for this contribution.

Finally, we return to the Bayesian game $(\Gamma, \mathcal{T}^*)$. We consider strategies that extend the equilibrium $(s_1, \ldots, s_I)$ of $(\Gamma, \mathcal{T})$. We show that, in a certain sense, these strategies correspond to the mixed strategies of the complete information game G. As such, we can use the fact that there is a mixed strategy equilibrium of G to show that there is an equilibrium of $(\Gamma, \mathcal{T}^*)$ that extends the equilibrium $(s_1, \ldots, s_I)$ of $(\Gamma, \mathcal{T})$.

Notice, it is important, for this argument, that we begin with a measurable equilibrium of the Bayesian game $(\Gamma, \mathcal{T})$. To see why, suppose that we begin instead with an equilibrium, viz. $(s_1, \ldots, s_I)$, where say s_1 is not measurable. Notice, there may be a type of some player $i \neq 1$, viz. $t_i^* \in T_i^* \backslash h_i(T_i)$, so that any extension of s_{-i} is not t_i^* measurable. In this case, we cannot associate an equilibrium of the complete information game with an equilibrium of the Bayesian game $(\Gamma, \mathcal{T}^*)$.

Of course, for a type structure with (at most) a countable number of types, all strategies are measurable. As such, an equilibrium is a measurable equilibrium. With this, we have the following corollary:

Corollary 7.1 *Fix Θ-based structures $\mathcal{T}$ and $\mathcal{T}^*$, so that $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$. If Γ is compact and continuous and $T_1^*, \ldots, T_I^*$ is (at most) countable, then any equilibrium of $(\Gamma, \mathcal{T})$ can be extended to an equilibrium of $(\Gamma, \mathcal{T}^*)$.*

8 The Common Prior Assumption

In this section, we will see that, if the analyst's structure satisfies the common prior assumption, then we have the Extension Property. Note, this holds independent of the underlying game Γ.

To see why, let us begin by reviewing the analysis in Section 5. There, we had Θ-based structures $\mathcal{T} = (\Theta; T_1, T_2, T_3; \beta_1, \beta_2, \beta_3)$ and $\mathcal{U}(\Theta) = (\Theta; U_1, U_2, U_3; \gamma_1, \gamma_2, \gamma_3)$ The structure $\mathcal{T}$ can be viewed as a belief-closed subset of $\mathcal{U}(\Theta)$. Write $\times_{i=1}^3 h_i(T_i) \subseteq \times_{i=1}^3 U_i$ for this belief closed subset. Note, types in this belief closed subset impose an equilibrium restriction on (some) types outside of this subset. This is because there are types in $U_i \backslash h_i(T_i)$ that assign strictly positive probability to types in $h_{-i}(T_{-i})$. This problem would not arise if the only types in the analyst's structure that assigned positive probability to types in $h_{-i}(T_{-i})$ are types that are in $h_{-i}(T_{-i})$. (Of course, this is not be the case in a universal type structure.)

With the above in mind, consider the following scenario: Suppose we have a type structure, viz. $\mathcal{T}^*$, that can be viewed as the union of two type structures. For a given game, can we extend an equilibrium associated with one of these structures to an equilibrium associated with $\mathcal{T}^*$? The answer will be yes if and only if there exists an equilibrium associated with the other structure.

Let us first formalize the idea that a type structure $\mathcal{T}^*$ can be viewed as the union of some structure $\mathcal{T}$ and some 'remaining structure,' which we'll call the difference structure.

Definition 8.1 *Fix Θ-based structures $\mathcal{T}$ and $\mathcal{T}^*$. Say $\mathcal{T}$* ***induces a decomposition of $\mathcal{T}^*$ (via $(\mathbf{h_1}, \ldots, \mathbf{h_I})$)*** *if $\mathcal{T}$ can be embedded into $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$ and $\times_{i=1}^I T_i^* \backslash h_i(T_i)$ is a belief-closed subset of T^*.*

Note, if $\mathcal{T}$ induces a decomposition of $\mathcal{T}^*$ (via $(h_1, \ldots, h_I)$), then both $\times_{i=1}^I h_i(T_i)$ and $\times_{i=1}^I (T_i^* \backslash h_i(T_i))$ are belief-closed subsets of T^*. Any belief-closed subset of $\mathcal{T}^*$ induces a metrizable type structure in its own right. (See Lemma D.1.) So, we can view the Θ-based structure $\mathcal{T}^*$ as the union of two Θ-based structures: **the structure induced by** $\mathcal{T}$ (which corresponds to the belief-closed set $\times_{i=1}^I h_i(T_i)$) and **the difference structure** (which corresponds to the belief-closed set $\times_{i=1}^I (T_i^* \backslash h_i(T_i))$). Write

$$(\mathcal{T}^* \backslash \mathcal{T}) = \langle \Theta; T_1^* \backslash h_1(T_1), \ldots, T_I^* \backslash h_I(T_I); \gamma_1^*, \ldots, \gamma_I^* \rangle,$$

for this difference structure. (Here, $\gamma_i^*(t_i^*)(E^*) = \beta_i^*(t_i^*)(E^*)$ for each event E^* in $\Theta \times \times_{j \neq i}(T_j^* \backslash h_j(T_j))$. Again, refer to Lemma D.1 for details.)

Now, we can state the result.[8]

Lemma 8.1 *Fix Θ-based structures $\mathcal{T}$ and $\mathcal{T}^*$, so that $\mathcal{T}$ induces a decomposition of $\mathcal{T}^*$. Fix, also, a Θ-based game Γ so that $(\Gamma, \mathcal{T})$ has an equilibrium. Then, $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the Equilibrium Extension Property for Γ if and only if there is an equilibrium of the difference game $(\Gamma, (\mathcal{T}^* \backslash \mathcal{T}))$.*

As a consequence of Lemma 8.1 and the Pull-Back Property, we have the following:

Proposition 8.1 *Fix Θ-based structures $\mathcal{T}$ and $\mathcal{T}^*$, so that $\mathcal{T}$ induces a decomposition of $\mathcal{T}^*$. Fix, also, a Θ-based game Γ, so that $(\Gamma, \mathcal{T})$ has an equilibrium. Then, $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the Equilibrium Extension Property for Γ if and only if there is an equilibrium for the Bayesian game $(\Gamma, \mathcal{T}^*)$.*

Taken together, Propositions 4.1 and 8.1 say: If $\mathcal{T}$ induces a decomposition of $\mathcal{T}^*$, then $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the Equilibrium Extension Property for Γ if and only if either both $(\Gamma, \mathcal{T})$

[8]Proofs for this section can be found in Appendix D.

and $(\Gamma, \mathcal{T}^*)$ have an equilibrium or both $(\Gamma, \mathcal{T})$ and $(\Gamma, \mathcal{T}^*)$ do not have an equilibrium. So, here, we cannot have the Extension failure of Section 5—i.e., an Extension failure where there is an equilibrium for $(\Gamma, \mathcal{T}^*)$.

Let's now ask: Is it of interest to consider the case where $\mathcal{T}$ induces a decomposition of $\mathcal{T}^*$? We will show that there is a notable case in which $\mathcal{T}$ does induce a decomposition of $\mathcal{T}^*$. Specifically, if the analyst's structure, viz. $\mathcal{T}^*$, admits a common prior, then the players' structure must induce a decomposition of the analyst's structure.

Why is this the case? Recall, the common prior assumption (CPA) reflects the idea that differences in beliefs reflect only differences in information. That is, if an outside observer looks at the situation, he can understand the different beliefs (i.e., associated with different types) as reflecting some underlying belief, common to both players. Each type of each player reflects the conditional of this belief on certain information.

Under a common prior, what does Izzy think Joe thinks about Izzy? Can a type of Izzy consider it possible that Joe considers that type of Izzy impossible? The answer would seem to be no. In particular, this appears to require that Izzy considers it possible that Joe has learned certain information that is inconsistent with the information she herself learned. This suggests that, if a type structure satisfies the CPA, then it also satisfies a mutual absolute continuity condition—i.e., if a type of Izzy considers a type of Joe possible (i.e., if $\beta_i^*(t_i^*)(\Theta \times \{t_j^*\} \times T_{-i-j}^*) > 0$), then that type of Joe also considers the given type of Izzy possible (i.e., then $\beta_j^*(t_j^*)(\Theta \times \{t_i^*\} \times T_{-i-j}^*) > 0$). Note, here, we write T_{-i-j}^* for $\times_{k \neq i,j} T_k^*$.

Going back to the structures $\mathcal{T}$ and $\mathcal{T}^*$, suppose the analyst's structure satisfies the common prior assumption. We have just argued that it also satisfies a mutual absolute continuity condition. Consider a type t_i^* that is not contained in the structure induced by $\mathcal{T}$. Can the type t_i^* assign strictly positive probability to a type of Joe in the structure induced by $\mathcal{T}$? No. The structure induced by $\mathcal{T}$ is a belief-closed subset. So, types in this structure cannot assign positive probability to the type t_i^*, which is what mutual absolute continuity would require. As such, the type t_i^* must assign probability one to types in (what will be) the difference structure. That is, $\mathcal{T}$ induces a decomposition of $\mathcal{T}^*$.

So, we have informally argued that, if the analyst's structure satisfies the CPA, then the players' structure must induce a decomposition of the analyst's structure. Let's now state this formally.

Definition 8.2 *Say a Θ-based interactive type structure $\mathcal{T} = \langle \Theta; T_1, \ldots, T_I; \beta_1, \ldots, \beta_I \rangle$ is **countable** if each $T_1, \ldots, T_I$ is countable.*

Fix a Θ-based type structure $\mathcal{T} = \langle \Theta; T_1, \ldots, T_I; \beta_1, \ldots, \beta_I \rangle$. Write $[t_i]$ for the event $\Theta \times \{t_i\} \times T_{-i}$. Given a measure $\mu \in \Delta(\Theta \times T)$ with $\mu([t_i]) > 0$, write $\mu(\cdot || [t_i])$ for conditional of μ on $[t_i]$ and write $\text{marg}_{\Theta \times T_{-i}} \mu$ for the marginal of μ on $\Theta \times T_{-i}$.

Definition 8.3 *Fix a Θ-based interactive type structure $\mathcal{T} = \langle \Theta; T_1, \ldots, T_I; \beta_1, \ldots, \beta_I \rangle$. Call $\mu \in \Delta(\Theta \times T)$ a* ***common prior (for $\mathcal{T}$)*** *if $\mathcal{T}$ is countable and, for each player i and each $t_i \in T_i$,*

(i) $\mu([t_i]) > 0$,

(ii) $\beta_i(t_i) = \text{marg}_{\Theta \times T_{-i}} \mu(\cdot || [t_i])$.

Say the structure $\mathcal{T}$ ***admits a common prior*** *if there is a common prior for $\mathcal{T}$.*

Definition 8.4 (Stuart, 1997) *Say a Θ-based interactive type structure $\mathcal{T} = \langle \Theta; T_1, \ldots, T_I; \beta_1, \ldots, \beta_I \rangle$ is* ***mutually absolutely continuous*** *if $\mathcal{T}$ is countable and, for every pair of (distinct) players $i, j = 1, \ldots, I$, $\beta_i(t_i)(\Theta \times \{t_j\} \times T_{-i-j}) > 0$ implies $\beta_j(t_j)(\Theta \times \{t_i\} \times T_{-i-j}) > 0$.*

Now, the connections. First, the CPA implies mutual absolute continuity.

Lemma 8.2 *Fix a Θ-based interactive type structure $\mathcal{T} = \langle \Theta; T_1, \ldots, T_I; \beta_1, \ldots, \beta_I \rangle$, where $\mathcal{T}$ admits a common prior. Then, $\mathcal{T}$ is mutually absolutely continuous.*

And:

Lemma 8.3 *Fix non-redundant Θ-based structures $\mathcal{T}$ and $\mathcal{T}^*$, so that $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$. If $\mathcal{T}^*$ is mutually absolutely continuous, then $\mathcal{T}$ induces a decomposition of $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$.*

Now, as a Corollary of Lemma 8.3 and Proposition 8.1, we have:

Corollary 8.1 *Fix Θ-based structures $\mathcal{T}$ and $\mathcal{T}^*$, so that $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$ and so that $\mathcal{T}^*$ satisfies mutual absolute continuity. Fix, also, a Θ-based game Γ, so that $(\Gamma, \mathcal{T})$ has an equilibrium. Then, $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the Equilibrium Extension Property for Γ if and only if there is an equilibrium for the Bayesian game $(\Gamma, \mathcal{T}^*)$.*

And, as a corollary of Lemma 8.2 and Corollary 8.1, we have:

Proposition 8.2 *Fix Θ-based structures $\mathcal{T}$ and $\mathcal{T}^*$, so that $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$ and so that $\mathcal{T}^*$ admits a common prior. Fix, also, a Θ-based game Γ, so that $(\Gamma, \mathcal{T})$ has an equilibrium. Then, $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the Equilibrium Extension Property for Γ if and only if there is an equilibrium for the Bayesian game $(\Gamma, \mathcal{T}^*)$.*

This says that, if the analyst's structure satisfies the common prior assumption, then the only way we can have an Extension failure is if there is any equilibrium of the players' Bayesian game but not the analyst's Bayesian game. Of course, if the analyst's structure satisfies the common prior assumption, then both the players and analyst's structure have (at most) a countable number of types. So, in this case, if Γ is compact and continuous, then both the players' and analyst's Bayesian game do have an equilibrium.[9] As such, we have the following corollary.

Corollary 8.2 *Fix Θ-based structures $\mathcal{T}$ and $\mathcal{T}^*$, so that $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$ and so that $\mathcal{T}^*$ admits a common prior. If Γ is a compact and continuous Θ-based game, then the pair $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the Equilibrium Extension Property for Γ.*

9 Discussion

This section discusses the relationship to the literature, and further discusses the results.

a. Misspecifying the Type Structure: The idea that the analyst can misspecify the type structure is not new to this paper. It can also be found in the robustness and interim rationalizability literatures. Let us review the canonical questions in these areas:

(i) What if the analyst misspecifies players' actual higher-order beliefs?

(ii) What if the analyst misspecifies the players' parameter set?

For (i): The sensitivity of an analysis to players' higher-order beliefs has a long history in game theory. This insight goes back to Geanakoplos and Polemarchakis (1982), followed by Monderer and Samet (1989), Rubinstein (1989) Carlsson and Van Damme (1993), and Aumann and Brandenburger (1995). Kajii and Morris (1997) shows how misspecifying these beliefs can matter for a Bayesian Equilibrium analysis.

[9] Here we use the following result: If $(G, \mathcal{T})$ is a Bayesian game where G is compact and continuous and $\mathcal{T}$ has countable type sets, then there is a Bayesian equilibrium of $(G, \mathcal{T})$. This result appears to be well-known in the literature. Takahashi (2009) pointed out to us why Glicksberg's (1952) Fixed Point Theorem can be applied to this case.

The idea is that players may think that only parameters in E (a subset of Θ) are possible; they may think that others think the same, etc...; up to some m^{th}-order belief. In this case, we will say (informally) that the event E satisfies mutual belief up to level m. The analyst looks at this situation and incorrectly deduces that the event E satisfies mutual belief at all levels. But, in fact, Izzy's $(m+1)^{th}$-order belief considers the possibility that E is not mutually believed up to level m. As such, the analyst misspecifies players' hierarchies of beliefs—even if only by a little bit.

Here, we do not change players' hierarchies of beliefs. Return to Section 5. There we had two Θ-based type structures, viz. $\mathcal{T}$ and $\mathcal{U}(\Theta)$, and each type in the structure $\mathcal{T}$ induces the same hierarchy of beliefs as some type in $\mathcal{U}(\Theta)$. The only difference between the two situations is the context within which these hierarchies lie.

For (ii): This question is discussed in papers such as Battigalli and Siniscalchi (2003), Ely and Peski (2006), and Dekel, Fudenberg and Morris (2007).

The idea is that players may observe signals external to the game as specified. By conditioning their choices on these external signals, new choices may be consistent with equilibrium play. Formally, this is the idea that the analyst's parameter set is Θ, while the players' parameter set is $\Theta \times \Sigma$, where Σ is a payoff-irrelevant extra dimension of uncertainty. Liu (2009) shows that this is equivalent to the case in which the analyst uses a non-redundant Θ-based structure, but the players, in fact, use a redundant Θ-based structure (as formalized in Battigalli and Siniscalchi (2003), Ely and Peski (2006), and Dekel, Fudenberg and Morris (2007)).[10]

Here, we implicitly assume that the analyst "understands" the players' parameter set (or the signals the players may see). To see this, return to our negative result in Sections 5. Note that the Θ-based structures $\mathcal{T}$ and $\mathcal{U}(\Theta)$ are non-redundant.[11] By focusing on non-redundant structures, we separate two difficulties. We see that, even if the analyst correctly specifies the signals that the players may observe, the analyst's predictions may still differ from the players'. This can happen if the analyst fails to "understand" the context of the game.

[10] Penta (2008) discusses a different way of misspecifying the parameter set, which he calls model dependence.

[11] The structure $\mathcal{T}$ also induces a $\{\theta_3\}$-based structure, but the structure $\mathcal{U}(\Theta)$ does not induce a $\{\theta_3\}$-based structure. So, it would be incorrect to conclude that this problem is equivalent to one of redundancy on $\{\theta_3\}$-based structures.

b. The Context of the Game: There are two distinct views of a game. Under the first view, the game itself is a complete description of all interactions past, present, and future. See, for instance, the discussion in Kohlberg and Mertens (1986). Under the second view, it is impractical to write down "the big game." Instead, the game studied represents a snapshot of the strategic situation. This is a game-theoretic analog to Savage's (1972) Small Worlds view in decision theory.

Our position is that each of these views is of interest—both deserve to be studied. Here, we focus on the second view, where there is a history prior to the given game. As such, it seems natural to consider the case where the history influences which hierarchies of beliefs players can hold. That is, it seems natural to consider the case in which the history determines the context of the game.

In this case, two robustness questions arise. First, what if the players know more than the analyst? This is the question we focused on here. But we can also address a second question. What if the analyst rules out more hierarchies than the players? In this case, the main result here says that the analyst will not lose any predictions—instead, the analyst may introduce extraneous predictions.

c. Constructing Large Type Structures: Can the analyst use a "larger" type structure to analyze the game and maintain predictions associated with the players' actual "smaller" type structure? We have seen that the answer is no. In fact, Section 5 points to two problems:

> *(i)* There may be an equilibrium of a Bayesian game $(\Gamma, \mathcal{T})$ that cannot be extended to an equilibrium of a Bayesian game $(\Gamma, \mathcal{T}^*)$, despite the fact that there is an equilibrium of $(\Gamma, \mathcal{T}^*)$.
>
> *(ii)* There may be an equilibrium of a Bayesian game $(\Gamma, \mathcal{T})$ that cannot be extended to an equilibrium of a Bayesian game $(\Gamma, \mathcal{T}^*)$, precisely because there is no equilibrium of $(\Gamma, \mathcal{T}^*)$.

If the analyst recognizes that he misspecified the players' type structure (in a particular way), he can simply use the players' type structure to analyze the game. However, we are concerned with the case in which the analyst doesn't recognize this. What can be done?

One possibility is to compute Bayesian equilibria for additional type structures. For instance, suppose the analyst studies a Θ-based Bayesian game $(\Gamma, \mathcal{T}^*)$ and is concerned that the players may have actually ruled out certain hierarchies of beliefs. The analyst can ensure that he has preserved all predictions by also analyzing equilibrium behavior in all Θ-based games $(\Gamma, \mathcal{T})$ so that $\mathcal{T}$ is a substructure of $\mathcal{T}^*$. Indeed, this is what Sadzik (2009) refers to as a "local analysis."

But, notice, if $\mathcal{T}^*$ is a universal type structure than such an analysis may be cumbersome. An alternate approach would be to construct a large type structure that can maintain all possible predictions associated with the players' actual type structure (whatever structure that may be). What does such a type structure look like? We leave the construction of such a type structure as an open question. Here, we point to some features and/or difficulties in constructing such a type structure.

Too Few Types Focus on the Extension problem in *(i)*. Specifically, consider the Θ-based Bayesian games there of $(\Gamma, \mathcal{T})$ and $(\Gamma, \mathcal{U}(\Theta))$ in Section 5. The structure $\mathcal{T}$ can be embedded into $\mathcal{U}(\Theta) = \langle \Theta; U_1, \ldots, U_I; \gamma_1, \ldots, \gamma_I \rangle$ via some $(h_1, \ldots, h_I)$. We want to have a prediction associated with each type in $\mathcal{U}(\Theta)$ and there are types u_i in that structure that assign strictly positive probability to $\Theta \times h_{-i}(T_{-i})$. But we also want to maintain each prediction associated with $(\Gamma, \mathcal{T})$. A problem arises precisely because there are types in $U_i \backslash h_i(T_i)$ that assign strictly positive probability to $\Theta \times h_{-i}(T_{-i})$.

We have two conflicting goals: to have a prediction for each type associated with $\mathcal{U}(\Theta)$ and to maintain each of the predictions associated with $\mathcal{T}$. It seems that the way to resolve this conflict is to have two copies of the types in $h_i(T_i)$ in our large structure: one copy that can get strictly positive probability under types in $U_i \backslash h_i(T_i)$ and another copy that gets zero probability under types in $U_i \backslash h_i(T_i)$.

Notice, we begin by looking at predictions associated with non-redundant structures. We, then ask for a large type structure—i.e., one that can capture all of these predictions. What we see is that the structure may need to contain two belief-closed subsets that are "identical," and so induce the same hierarchies of beliefs. The output is a particular form of a redundant structure—with two essentially identical belief-closed subsets.

Too Many Types Focus on the Extension problem in *(ii)*. Specifically, consider the $\overline{\Theta}$-based universal Bayesian game $(\overline{\Gamma}, \mathcal{U}(\overline{\Theta}))$ in Section 5. This Bayesian game had no equilibrium and so, for each $\overline{\Theta}$-based type structure, viz. $\overline{\mathcal{T}}$, there is no equilibrium of $(\overline{\Gamma}, \overline{\mathcal{T}})$ that can be extended to an equilibrium of $(\overline{\Gamma}, \mathcal{U}(\overline{\Theta}))$. To remedy this Extension problem, it seems that we will want to remove types from $\mathcal{U}(\overline{\Theta})$. In particular, we will want to remove types associated with types in Simon's (2003) type structure, since such types are a cause of non-existence and so non-extension.

d. Redundancy, Again: In part c above, we said that, if we want a type structure that captures all possible predictions, this structure may need to be redundant—i.e., may need to contain two types that induce the same hierarchies of beliefs. Redundant structures have also played a role in other papers. Refer to Section 9a*(ii)* above and note that redundancy may also be appropriate if the analyst misspecifies the players' parameter sets.

It is important to note that these two types of redundancies are quite different. If the question is misspecifying the context of the game, as here, we would introduce redundancies by adding belief-closed subsets that are "identical" to ones already present. (Refer to Section 9c above.) This would not be appropriate if the question is misspecifying the players' parameter sets.

θ_1-section

Izzy \ Joe	*L*	*R*
U	5, 0	0, 0
D	3, 0	3, 0

θ_2-section

Izzy \ Joe	*L*	*R*
U	0, 0	5, 0
D	3, 0	3, 0

Figure 8

To see this last claim, refer to Figure 8, which is essentially Example 1 in Dekel, Fudenberg and Morris (2007). First, consider the structure $\mathcal{T}$, with $T_i = \{t_i\}$, $T_j = \{t_j\}$, $\beta_i(t_i)(\{(\theta_1, t_j)\}) = \beta_i(t_i)(\{(\theta_2, t_j)\}) = \frac{1}{2}$, and $\beta_j(t_j)(\{(\theta_1, t_i)\}) = \beta_j(t_j)(\{(\theta_2, t_i)\}) = \frac{1}{2}$. In any Bayesian Equilibrium here, t_i plays Down. If we add a belief-closed subset to this structure, as suggested in c above, we have new types u_i and u_j with $\beta_i(u_i)(\{(\theta_1, u_j)\}) = \beta_i(u_i)(\{(\theta_2, u_j)\}) = \frac{1}{2}$, and $\beta_j(u_j)(\{(\theta_1, u_i)\}) = \beta_j(u_j)(\{(\theta_2, u_i)\}) = \frac{1}{2}$. Again, each type of Izzy plays Down.

Consider instead the structure $\mathcal{T}^*$, with $T_i^* = \{t_i^*\}$, $T_j^* = \{t_j^*, u_j^*\}$, $\beta_i^*(t_i^*)(\{(\theta_1, t_j^*)\}) = \beta_i^*(t_i^*)(\{(\theta_2, u_j^*)\}) = \frac{1}{2}$, $\beta_j^*(t_j^*)(\{(\theta_1, t_i^*)\}) = \beta_j^*(t_j^*)(\{(\theta_2, t_i^*)\}) = \frac{1}{2}$, and $\beta_j^*(u_j^*)(\{(\theta_1, t_i^*)\}) = \beta_j^*(u_j^*)(\{(\theta_2, t_i^*)\}) = \frac{1}{2}$. Note, the type t_i^* induces the same hierarchies of beliefs as the type t_i above. And, likewise, for types t_j^*, u_j^* relative to the type t_j above. But, now there is an equilibrium where t_i^* plays Up, t_j^* plays Left, and u_j^* plays Right. (Again, Liu (2009) tells us that we can interpret this equilibrium as an equilibrium where the parameter space is, in fact, $\{\theta_1, \theta_2\} \times \Sigma$, for some payoff-irrelevant parameter space Σ.)

In sum: If the question is robustness to misspecifying the context of the game, we would add redundancies so that the "new" types assign zero probability to the "old" types. However, if the question is robustness to misspecifying the parameter set, we would add redundancies so that the "new" types assign strictly positive probability to the "old" types.

e. The Ingredients of the Extension Failure: The negative result in Section 5 makes use of a Bayesian game without an equilibrium. But, it is important to note that the result is not simply a corollary of the fact that there is some Bayesian game that does not have an equilibrium. In particular, we have seen an example of a (finite) Θ-based Bayesian game $(\Gamma, \mathcal{T})$ where we cannot extend an equilibrium of this game to an equilibrium of $(\Gamma, \mathcal{U}(\Theta))$, despite the fact that there is an equilibrium of this Bayesian game. Moreover, each belief closed subset of $\mathcal{U}(\Theta)$ induces a Bayesian game that does have an equilibrium.

To better understand the connection between Extension failures and non-existence, it may be useful to compare this analysis to another solution concept, namely the correlated rationalizability solution concept. There are games—albeit, perhaps, pathological games—for which the set of rationalizable strategies is empty. (See Example 2 in Dufwenberg and Stegeman, 2002.) Yet, we have the following result.

> *Result:* Fix Θ-based structures, $\mathcal{T}$ and $\mathcal{T}^*$, so that $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$. Fix also a Θ-based game Γ. If the rationalizable strategies are non-empty in both the Bayesian games $(\Gamma, \mathcal{T})$ and $(\Gamma, \mathcal{T}^*)$, then $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies rationalizable extension and pull-back properties for Γ. [12]

As such, a non-existence example (as in Dufwenberg and Stegeman, 2002) cannot be used to get an Extension failure where the analyst's Bayesian game satisfies existence.

To sum up: Certainly, we can have an Extension failure that stems from the fact that there is a prediction associated with the players' Bayesian game but not the analyst's Bayesian game. Such an Extension failure necessarily stems from an Existence problem. But, the case of interest is the case where there is a prediction associated with the analyst's game. In this case, whether we do vs. do not have such an Extension failure depends on the particular solution concept studied. In particular, for Bayesian equilibrium there is such an Extension failure while for correlated rationalizability there is no such Extension

[12] The proof can be found in the Online Appendix. Dekel, Fudenberg and Morris (2007) show an analogous result, when the parameter and action sets are finite.

failure—this is the case despite the fact that, for both solution concepts, there are Bayesian games that fail existence.

f. Compact and Continuous Games: We have seen that there are instances where we may fail the Extension Property for compact and continuous games (e.g., as in Section 5) and that there are also instances where we will satisfy the Extension Property for such games (i.e., as in Corollary 7.1). Specifically, we saw that, for such games, if the analyst's structure has, at most, a countable number of types, then we will have an Extension property.

Yildiz (2009) also provides conditions under which there is an Extension Property. There, each C_i is finite (but, again, Θ may not be finite). The players' type structure is taken to be finite and each type's belief has finite support. The analyst's structure may be uncountably infinite, but is the union of such player structures. So, in particular, there may be an uncountably infinite number of types that are in the analyst's structure but not the players' structure, but (unlike a universal structure) each type has finite support. Proposition 3 in Yildiz (2009) shows an Extension Property for this case.

g. The Common Prior Assumption: Definition 8.3 states that the CPA reflects two requirements, a common prior requirement and a positivity requirement.

Consider the sets $[t_i] = \Theta \times \{t_i\} \times T_{-i}$ and note that these sets form a partition of $\Theta \times T$. Write τ_i for the subalgebra generated by this partition. Given a measure $\mu \in \Delta(\Theta \times T)$ and an event E in $\Theta \times T$, write $\mu(E, \cdot||\tau_i)$ for a version of μ-conditional probability of E given τ_i. (Note, since the conditioning events for Izzy and Joe are distinct, the versions of conditional probability will also be distinct.) The **common prior requirement** is: There exists a measure $\mu \in \Delta(\Theta \times T)$ and a version of μ-conditional probability of E given τ_i so that, for any type t_i and any event E in $[t_i]$, $\beta_i(t_i)(\text{proj}_{\Theta \times T_{-i}} E) = \mu(E, [t_i]||\tau_i)$. (Note, $\mu(E, \cdot||\tau_i)$ is constant on $[t_i]$.) **Positivity** requires that, in addition, $\mu([t_i]) > 0$, for each type $t_i \in T_i$.

The positivity requirement is important for Proposition 8.2. To see this, return to the type structure $\mathcal{T}^*$ in Section 5 and note that this structure satisfies the common prior requirement. In particular, the measure $\mu \in \Delta(\Theta \times T^*)$ with $\mu(\{(\theta_3, t_1, t_2, t_3)\}) = 1$ is a common prior for $\mathcal{T}^*$. Of course, it is not positive. Thus, we can see that the common prior requirement alone does not suffice for Proposition 8.2. We also need the positivity requirement.

The need for the positivity requirement is important from the perspective of generalizing Proposition 8.2. In particular, if T_i is uncountably infinite, there is no probability measure that assigns strictly positive probability to each event $[t_i]$. This suggests a limitation to Proposition 8.2. Alternatively, this might suggest that other tools are needed to study the case of uncountably infinite spaces—i.e., lexicographic probability systems (Blume, Brandenburger and Dekel, 1991), conditional probability systems (Rényi, 1955), or non-standard probabilities.

There is an interesting connection to be made at the conceptual level. Does a non-positive common prior fit with the CPA? Arguably not. Recall, the idea of the CPA is that differences in probabilities only reflect differences in information. As a consequence, the only personalistic features of probability should come from informational differences. But, there may be many (regular and proper) versions of conditional probability. Given this, the common prior requirement (as specified above) need not pin down the beliefs (i.e., each $\beta_i(t_i)$). Indeed, in the example above, there are many Θ-based structures $\mathcal{T}$ corresponding to the common prior μ. In fact, choosing distinct probabilities p gives just such structures.

Appendix A Proofs for Sections 3-4

Lemma A.1 *Fix a Θ-based (metrizable) Bayesian game $(\Gamma, \mathcal{T})$, strategies $s_1, \ldots, s_I$, and a measure $\sigma_i \in \Delta(C_i)$. Define $\overrightarrow{\pi}_i : \Theta \times T_{-i} \to \mathbb{R}$ so that $\overrightarrow{\pi}_i(\theta, t_{-i}) = \pi_i(\theta, \sigma_i, s_{-i}(t_{-i}))$, for each (θ, t_{-i}). If $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I)$ is t_i-measurable, then $\overrightarrow{\pi}_i$ is t_i-measurable.*

Proof. Define the map $\overrightarrow{s}_{-i} : \Theta \times \{\sigma_i\} \times T_{-i} \to \Theta \times \{\sigma_i\} \times \times_{j \neq i} \Delta(C_j)$ so that each $\overrightarrow{s}_{-i}(\theta, \sigma_i, t_{-i}) = (\theta, \sigma_i, s_{-i}(t_{-i}))$. Note, there exists a measurable support of $\beta_i(t_i)$, viz. E_{-i}, so that $\overrightarrow{s}_{-i}$ is measurable when the domain is restricted to $\{\sigma_i\} \times E_{-i}$. Now, define a map $\overrightarrow{t}_i : \Theta \times T_{-i} \to \Theta \times \{\sigma_i\} \times T_{-i}$ so that $\overrightarrow{t}_i(\theta, t_{-i}) = (\theta, \sigma_i, t_{-i})$. Of course, $\overrightarrow{t}_i$ is measurable. Finally, note that $\overrightarrow{\pi}_i = \pi_i \circ \overrightarrow{s}_{-i} \circ \overrightarrow{t}_i$ and so is measurable when the domain is restricted to E_{-i} ■

Proof of Proposition 4.1. Fix a Bayesian equilibrium, viz. $(s_1^*, \ldots, s_I^*)$, of $(\Gamma, \mathcal{T}^*)$. We will show that $(s_1^* \circ h_1, \ldots, s_I^* \circ h_I)$ is a Bayesian Equilibrium of $(\Gamma, \mathcal{T})$.

Begin with condition (i) of Definition 3.4. Fix some $t_i \in T_i$ and note that there exists a measurable support of $h_i(t_i)$, viz. $\operatorname{supp} \beta_i^*(h_i(t_i))$, so that $\mathrm{id} \times s_{-i}^*$ is measurable when restricting the domain to $\operatorname{supp} \beta_i^*(h_i(t_i))$. Note, too, that $(\mathrm{id} \circ h_{-i})^{-1}(\operatorname{supp} \beta_i^*(h_i(t_i)))$ is a measurable support of $\beta_i(t_i)$. So, it suffices to show that $\mathrm{id} \times s_{-i} = \mathrm{id} \times (s_{-i}^* \circ h_{-i})$ is measurable when restricting the domain to $(\mathrm{id} \circ h_{-i})^{-1}(\operatorname{supp} \beta_i^*(h_i(t_i)))$. For this, fix some E measurable in $\Theta \times \prod_{j \neq i} \Delta(C_j)$ and note that $(\mathrm{id} \times s_{-i}^*)^{-1}(E) \cap \operatorname{supp} \beta_i^*(h_i(t_i))$ is measurable. It follows that

$$(\text{id} \times h_{-i})^{-1}((\text{id} \times s^*_{-i})^{-1}(E) \cap \operatorname{supp} \beta^*_i(h_i(t_i)))$$

is measurable. But now notice that

$$(\text{id} \times h_{-i})^{-1}((\text{id} \times s^*_{-i})^{-1}(E) \cap \operatorname{supp} \beta^*_i(h_i(t_i)))$$

is simply

$$(\text{id} \times (s^*_{-i} \circ h_{-i}))^{-1}(E) \cap (\text{id} \times h_{-i})^{-1}(\operatorname{supp} \beta^*_i(h_i(t_i)))$$

or

$$(\text{id} \times s_{-i})^{-1}(E) \cap (\text{id} \times h_{-i})^{-1}(\operatorname{supp} \beta^*_i(h_i(t_i))),$$

as required.

Now turn to Condition (ii) of Definition 3.4. Fix some type $t_i \in T_i$ and some strategy r_i of the Bayesian game $(\Gamma, \mathcal{T})$. Choose some strategy r^*_i of the Bayesian game $(\Gamma, \mathcal{T}^*)$ with $r^*_i(h_i(t_i)) = r_i(t_i)$. For this strategy, we have that

$$\begin{aligned}
\Pi_i(t_i, s^*_i \circ h_i, s^*_{-i} \circ h_{-i}) &= \int_{\Theta \times T_{-i}} \pi_i(\theta, s^*_i(h_i(t_i)), s^*_{-i}(h_{-i}(t_{-i}))) d\beta_i(t_i) \\
&= \int_{\Theta \times T^*_{-i}} \pi_i(\theta, s^*_i(h_i(t_i)), s^*_{-i}(t^*_{-i})) d\beta^*_i(h_i(t_i)) \\
&\geq \int_{\Theta \times T^*_{-i}} \pi_i(\theta, r^*_i(h_i(t_i)), s^*_{-i}(t^*_{-i})) d\beta^*_i(h_i(t_i)) \\
&= \int_{\Theta \times T_{-i}} \pi_i(\theta, r^*_i(h_i(t_i)), s^*_{-i}(h_{-i}(t_{-i}))) d\beta_i(t_i) \\
&= \int_{\Theta \times T_{-i}} \pi_i(\theta, r_i(t_i), s^*_{-i}(h_{-i}(t_{-i}))) d\beta_i(t_i) \\
&= \Pi_i(t_i, r_i, s^*_{-i} \circ h_{-i}),
\end{aligned}$$

where the second and fourth lines use the Change of Variables Theorem (e.g., Billingsley, 2008, Theorem 16.13) plus the fact that $(h_1, \ldots, h_I)$ is a type morphism, the third line uses the fact that $(s^*_1, \ldots, s^*_I)$ is a Bayesian equilibrium of $(\Gamma, \mathcal{T}^*)$, and the fifth line uses the fact that $r^*_i(h_i(t_i)) = r_i(t_i)$. This establishes condition (ii). ■

Proof of Lemma 4.1. Fix a Θ-based Bayesian game $(\Gamma, \mathcal{T})$. If, for each Θ-based interactive metrizable structure $\mathcal{T}^*$ so that $\mathcal{T}$ can be mapped to $\mathcal{T}^*$, the pair $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the Equilibrium Extension Property for Γ, then certainly this is the case when $\mathcal{T}^* = \mathcal{U}(\Theta)$. We show that, if the pair $\langle \mathcal{T}, \mathcal{U}(\Theta)\rangle$ satisfies the Equilibrium Extension Property for Γ, then the pair $\langle \mathcal{T}, \mathcal{T}^*\rangle$ also satisfies the Equilibrium Extension Property for Γ, where $\mathcal{T}^*$ is some Θ-based interactive metrizable structure $\mathcal{T}^*$ so that $\mathcal{T}$ can be mapped to $\mathcal{T}^*$.

To show this, it will be useful to begin with properties of the mappings between these structures. Since $\mathcal{U}(\Theta)$ is terminal, there is a type morphism $(k_1, \ldots, k_I)$ from $\mathcal{T}$ to $\mathcal{U}(\Theta)$ and a type morphism $(l_1, \ldots, l_I)$ from $\mathcal{T}^*$ to $\mathcal{U}(\Theta)$. Note, the map $(l_1 \circ h_1, \ldots, l_I \circ h_I)$ is also a type morphism from $\mathcal{T}$ to $\mathcal{U}(\Theta) = \langle \Theta; U_1, \ldots, U_I; \gamma_1, \ldots, \gamma_I\rangle$. To see this, fix an event E in $\Theta \times U_{-i}$ and note that

$$\begin{aligned}\gamma_i(l_i(h_i(t_i)))(E) &= \beta_i^*(h_i(t_i))((\mathrm{id} \times l_{-i})^{-1}(E)) \\ &= \beta_i(t_i)((\mathrm{id} \times h_{-i})^{-1}((\mathrm{id} \times l_{-i})^{-1}(E))),\end{aligned}$$

where the first line uses the fact that $(l_1, \ldots, l_I)$ is a type morphism from $\mathcal{T}^*$ to $\mathcal{U}(\Theta)$ and the second line uses the fact that $(h_1, \ldots, h_I)$ is a type morphism from $\mathcal{T}$ to $\mathcal{T}^*$. So, $\gamma_i(l_i(h_i(t_i)))$ is the image measure of $\beta_i(t_i)$ under $(\mathrm{id} \times l_{-i}) \circ (\mathrm{id} \times h_{-i}) = \mathrm{id} \times (l_{-i} \circ h_{-i})$, as required.

Fix a Θ-based game Γ, and suppose $\langle \mathcal{T}, \mathcal{U}(\Theta)\rangle$ satisfies the Extension Property for Γ. We will show that $\langle \mathcal{T}, \mathcal{T}^*\rangle$ also satisfies the Extension Property for Γ.

Let $(s_1, \ldots, s_I)$ be a Bayesian equilibrium of $(\Gamma, \mathcal{T})$. Since $\langle \mathcal{T}, \mathcal{U}(\Theta)\rangle$ satisfies the Extension Property for Γ, there exists an equilibrium $(r_1, \ldots, r_I)$ of $(\Gamma, \mathcal{U}(\Theta))$ so that $(s_1, \ldots, s_I) = (r_1 \circ k_1, \ldots, r_I \circ k_I)$. Notice, $(k_1, \ldots, k_I)$ and $(l_1 \circ h_1, \ldots, l_I \circ h_I)$ are both type morphisms from $\mathcal{T}$ to $\mathcal{U}(\Theta)$—and, since there is a unique type morphism from $\mathcal{T}$ to $\mathcal{U}(\Theta)$, $(k_1, \ldots, k_I) = (l_1 \circ h_1, \ldots, l_I \circ h_I)$. So,

$$\begin{aligned}(s_1, \ldots, s_I) &= (r_1 \circ k_1, \ldots, r_I \circ k_I) \\ &= (r_1 \circ l_1 \circ h_1, \ldots, r_I \circ l_I \circ h_I).\end{aligned}$$

Finally, note that, since $(r_1, \ldots, r_I)$ is an equilibrium of $(\Gamma, \mathcal{U}(\Theta))$, the Pull-Back Property (Proposition 4.1) gives that $(r_1 \circ l_1, \ldots, r_I \circ l_I)$ is an equilibrium of $(\Gamma, \mathcal{T}^*)$. Thus, we have found an equilibrium $(r_1 \circ l_1, \ldots, r_I \circ l_I)$ of $(\Gamma, \mathcal{T}^*)$, so that $(r_1 \circ l_1 \circ h_1, \ldots, r_I \circ l_I \circ h_I)$ is indeed our original equilibrium $(s_1, \ldots, s_I)$. This establishes the claim. ■

Appendix B Review of Simon

This appendix provides a review of the architecture and main result in Simon (2003). As will become clear, his structure is a bit different from the one adopted in this paper. We map Simon's result into a result that can be stated within the current framework. This is no more than bookkeeping.

Fix $\Theta = \{\theta_1, \theta_2\}$ and consider the set $\Omega = \Theta^{\mathbb{Z}}$. So, Ω is Polish. Each element of Ω, viz. $\omega = (\ldots, \omega^{-2}, \omega^{-1}, \omega^0, \omega^1, \omega^2, \ldots)$, is called a state. (Of course, each $\omega^k \in \Theta$.) There is a continuous mapping $\phi_0 : \Omega \to \Theta$ so that $\phi_0(\omega) = \omega^0$, i.e., ϕ_0 maps ω into its zeroth coordinate ω^0.

For each player $i \in \{1, 2, 3\}$, there is a continuous involution $\phi_i : \Omega \to \Omega$, i.e., a continuous map with $\phi_i(\phi_i(\omega)) = \omega$. For Player 1, this map takes each ω and gives its reflection around zero, i.e., $\phi_1(\ldots, \omega^{-1}, \omega^0, \omega^1, \ldots) = (\ldots, \overline{\omega}^{-1}, \overline{\omega}^0, \overline{\omega}^1, \ldots)$ where each $\overline{\omega}^i = \omega^{-i}$. For Player 2, this map takes each ω, first reflects it around zero, and then shifts it up by one, i.e., $\phi_2(\ldots, \omega^{-1}, \omega^0, \omega^1, \ldots) = (\ldots, \overline{\omega}^{-1}, \overline{\omega}^0, \overline{\omega}^1, \ldots)$ where each $\overline{\omega}^i = \omega^{-i+1}$. For Player 3, we have $\phi_3 = \phi_2$.

The next step is to define continuous maps from Ω to beliefs on Ω, i.e. $\varphi_i : \Omega \to \Delta(\Omega)$. For each $i \in \{1, 2, 3\}$, and each event E in Ω, $\varphi_i(\omega)(E) = 1$ if $\{\omega, \phi_i(\omega)\} \subseteq E$, $\varphi_i(\omega)(E) = 0$ if $\{\omega, \phi_i(\omega)\} \cap E = \emptyset$, and $\varphi_i(\omega)(E) = \frac{1}{2}$ otherwise. (So, if $\omega \neq \phi_i(\omega)$, $\varphi_i(\omega)(\{\omega\}) = \varphi_i(\omega)(\{\phi_i(\omega)\}) = \frac{1}{2}$. Otherwise, $\varphi_i(\omega)(\{\omega\}) = 1$.) Notice, $\varphi_i(\omega) = \varphi_i(\phi_i(\omega))$ for each $\omega \in \Omega$, since ϕ_i is an involution. The fact that each φ_i is continuous is shown as Lemma 0 in Simon (2003).

For Simon (2003), a Θ-based structure is given by $(\Theta; \Omega; \phi_0; \varphi_1, \varphi_2, \varphi_3)$, so that, if ω' is contained in the support of $\varphi_i(\omega)$, then $\varphi_i(\omega') = \varphi_i(\omega)$. We will see that, in the terminology of our paper, ω and $\phi_i(\omega)$ represent the same type of player i. Indeed, Simon asks that player i play the same way at these two states. Specifically, given a Θ-based game Γ, let Σ_i be the set of all mappings $\sigma_i : \Omega \to \Delta(C_i)$ so that $\sigma_i(\omega) = \sigma_i(\phi_i(\omega))$ for all $\omega \in \Omega$. Then, the main result is as follows:

Main Result in Simon, 2003: Consider the Θ-based game Γ in Figure 7.

(i) There exists $(\sigma_1, \sigma_2, \sigma_3) \in \Sigma_1 \times \Sigma_2 \times \Sigma_3$ so that, for each player i and each $\omega \in \Omega$,

$$\int_\Omega \pi_i(\phi_0(\omega), \sigma_i(\omega), \sigma_{-i}(\omega)) d\varphi_i(\omega) \geq \int_\Omega \pi_i(\phi_0(\omega), \rho_i(\omega), \sigma_{-i}(\omega)) d\varphi_i(\omega), \tag{1}$$

for all $\rho_i \in \Sigma_i$.

(ii) Each triple $(\sigma_1, \sigma_2, \sigma_3) \in \Sigma_1 \times \Sigma_2 \times \Sigma_3$ satisfying Equation (1) has some i so that σ_i is not measurable.

We now translate this result into our framework.

Begin by recalling two facts, which hold for each $i \in \{1, 2, 3\}$ and $\omega \in \Omega$. First, the beliefs $\varphi_i(\omega)$ and $\varphi_i(\phi_i(\omega))$ coincide. Second, $\sigma_i(\omega)$ and $\sigma_i(\phi_i(\omega))$ are required to play the same. So, we can view a type of i as some $\{\omega, \phi_i(\omega)\}$. Set $T_i = \{\{\omega, \phi_i(\omega)\} : \omega \in \Omega\}$. Note, since ϕ_i is an involution, T_i forms a partition of Ω. Write $t_i[\omega]$ for the partition member $\{\omega, \phi_i(\omega)\}$. Let $q_i : \Omega \to T_i$ be the quotient map, i.e., $q_i(\omega) = t_i[\omega]$. Equip T_i with the quotient Borel space, i.e., all subsets E_i of T_i so that $(q_i)^{-1}(E_i)$ is Borel in Ω. Then T_i is isomorphic to an analytic set in a Polish space, and so metrizable. (See Proposition 5.1.11 and Exercise 5.1.14 in Srivastava, 1998.)

Take i, j, k to be three distinct players. Now, we can use the map φ_i to construct the map β_i. Fix some $t_i[\omega] \in T_i$, and write $t_j[\omega']$ and $t_k[\omega']$ for the types associated with players j and k when the state is ω'. Fix some event E_{-i} in $\Theta \times T_{-i}$. Set $\beta_i(t_i[\omega])(E_{-i}) = 1$ if E_{-i} contains both $(\omega^0, t_j([\omega]), t_k([\omega]))$ and $((\phi_i(\omega))^0, t_j([\phi_i(\omega)]), t_k([\phi_i(\omega)]))$, set $\beta_i(t_i[\omega])(E_{-i}) = 0$ if E_{-i} does not contain either $(\omega^0, t_j([\omega]), t_k([\omega]))$ or $((\phi_i(\omega))^0, t_j([\phi_i(\omega)]), t_k([\phi_i(\omega)]))$; and, otherwise, set $\beta_i(t_i[\omega])(E_{-i}) = \frac{1}{2}$.

Note, β_i is measurable. To see this, recall that $\Delta(\Theta \times T_{-i})$ is metrizable and so generated by sets of the form $\{\mu \in \Delta(\Theta \times T_{-i}) : \mu(E_{-i}) \geq p\}$ for E_{-i} Borel in $\Theta \times T_{-i}$ and $p \in [0, 1]$. (See the proof of Lemma 14.16 in Aliprantis and Border, 2007.) Fix an event E_{-i} in $\Theta \times T_{-i}$ and some $p \in [0, 1]$. Construct a map $r_{-i} : \Omega \to \Theta \times T_{-i}$ so that $r_{-i}(\omega) = (\phi_0(\omega), q_{-i}(\omega))$. Since ϕ_0 and q_{-i} are measurable, it follows that r_{-i} is measurable, and so $(r_{-i})^{-1}(E_{-i})$ is Borel in Ω. Now, note that, by construction, $\beta_i(t_i[\omega])(E_{-i}) \geq p$ if and only if $g_i(\omega)((r_{-i})^{-1}(E_{-i})) \geq p$. Since g_i is measurable, we have that $\{\omega \in \Omega : g_i(\omega)((r_{-i})^{-1}(E_{-i})) \geq p\}$ is Borel (using, again, the proof of Lemma 14.16 in Aliprantis and Border, 2007) and so $\{t_i \in T_i : \beta_i(t_i)(E_{-i}) \geq p\}$ is Borel. This establishes that β_i is measurable.

With this, we can apply Simon's result.

Application of Main Result in Simon, 2003: Consider the Θ-based game Γ in Figure 7 and let $\mathcal{T}$ be the Θ-based game described above.

(i) There exists a Bayesian Equilibrium of $(\Gamma, \mathcal{T})$.

(ii) Any Bayesian Equilibrium of $(\Gamma, \mathcal{T})$ is not a measurable equilibrium.

Proof: First, fix a profile $(\sigma_1, \sigma_2, \sigma_3)$ satisfying Equation (1). Construct (s_1, s_2, s_3) so that, each $s_i(t_i[\omega]) = \sigma_i(\omega)$. (This is well-defined since σ_i is ϕ_i-measurable.)

We can see that (s_1, s_2, s_3) is a Bayesian equilibrium: Each $\beta_i(t_i[\omega])$ has finite support and so condition (i) of a Bayesian equilibrium is satisfied. By construction, for each $t_i[\omega]$,

$$\int_{\Theta \times T_{-i}} \pi_i(\theta, \cdot, s_{-i}(t_{-i})) d\beta_i(t_i[\omega]) = \int_{\Omega} \pi_i(\phi_0(\omega), \cdot, \sigma_{-i}(\omega)) dg_i(\omega).$$

As such, condition (ii) of a Bayesian equilibrium follows from Equation (1).

Next, suppose contra hypothesis, (s_1, s_2, s_3) is a measurable Bayesian equilibrium. Let $\sigma_i = s_i \circ q_i$, and note that σ_i is measurable since s_i and q_i are measurable. Also, $\sigma_i(\omega) = \sigma_i(\phi_i(\omega))$, by construction. Finally note that, by construction,

$$\int_{\Theta \times T_{-i}} \pi_i(\theta, \cdot, s_{-i}(t_{-i})) d\beta_i(t_i[\omega]) = \int_{\Omega} \pi_i(\phi_0(\omega), \cdot, \sigma_{-i}(\omega)) dg_i(\omega).$$

But, then, using condition (ii) of a Bayesian equilibrium, we contradict Simon's result.

We conclude by noting that Simon's structure is not terminal. Recall, type morphisms preserve hierarchies of beliefs. So, if a structure is terminal, then, for each possible hierarchy of beliefs (satisfying a certain coherency condition), there is a type in the terminal structure that induces the exact same hierarchy of beliefs. Note, in the Simon structure, each type assigns either probability $\frac{1}{2} : \frac{1}{2}$, $1 : 0$, or $0 : 1$ to $\theta_1 : \theta_2$. So, we certainly do not have all first-order beliefs—let alone all hierarchies of beliefs.

Appendix C Proofs for Section 7

This appendix is devoted to proving Proposition 7.1. Throughout, we make use of the following notational conventions: Given sets $\Omega_1, \ldots, \Omega_I$ and some subset $K \subseteq \{1, \ldots, I\}$, write $\Omega_K = \times_{k \in K} \Omega_k$ and write ω_K for a profile in Ω_K. Likewise, given maps $f_1, \ldots, f_I$, where each $f_i : \Omega_i \to \Phi_i$, write $f_K : \Omega_K \to \Phi_K$ for the associated product map.

Fix two (non-redundant) Θ-based type structures $\mathcal{T} = \langle \Theta; T_1, \ldots, T_I; \beta_1, \ldots, \beta_I \rangle$ and $\mathcal{T}^* = \langle \Theta; T_1^*, \ldots, T_I^*; \beta_1^*, \ldots, \beta_I^* \rangle$. Suppose, further, that $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$, so that each $T_1^* \backslash h_1(T_1), \ldots, T_I^* \backslash h_I(T_I)$ is (at most) countable (and possibly empty). By Lemma 6.3, there is some $i = 1, \ldots, I$, so that $T_i^* \backslash h_i(T_i)$ is nonempty. Order players so that (a) for each $i = 1, \ldots, J$, $T_i^* \backslash h_i(T_i) \neq \emptyset$ and (b) for each $i = J+1, \ldots, I$, $T_i^* \backslash h_i(T_i) = \emptyset$ (if $J < I$). For each $i = 1, \ldots, J$, write $M(i)$ for the cardinality of $T_i^* \backslash h_i(T_i)$and $m(i)$ for some element of $T_i^* \backslash h_i(T_i)$. We take $M(i)$ to be (at most) countable.

Consider a Θ-based compact and continuous game $\Gamma = \langle \Theta; C_1, \ldots, C_I; \pi_1, \ldots, \pi_I \rangle$. Throughout this appendix, we fix an equilibrium of the Bayesian game $(\Gamma, \mathcal{T})$, viz. $(s_1, \ldots, s_I)$. We want to show that there is a equilibrium of the Bayesian game $(\Gamma, \mathcal{T}^*)$, viz. $(s_1^*, \ldots, s_I^*)$, with $(s_1, \ldots, s_I) = (s_1^* \circ h_1, \ldots, s_I^* \circ h_I)$.

Section 7 gives the idea of the proof. In particular, we begin by constructing the game of complete information, namely G. The game has a finite or countable number of players, corresponding to $\bigcup_{i=1}^J T_i^* \backslash h_i(T_i)$. The choice set for a player $m(i) \in T_i^* \backslash h_i(T_i)$ is C_i. Write $\mathcal{C}_i$ for the set $[C_i]^{M(i)}$, so that $\mathcal{C} = \times_{i=1}^J \mathcal{C}_i$ is the set of choice profiles in this game. Note, we can think of $\overrightarrow{c}_i = (c_i^1, c_i^2, \ldots) \in \mathcal{C}_i$ as a mapping $\overrightarrow{c}_i : T_i^* \backslash h_i(T_i) \to C_i$. So, when we write $\overrightarrow{c}_i(t_i^*)$ we mean the t_i^*-th component of $\overrightarrow{c}_i = (c_i^1, c_i^2, \ldots)$. Likewise, given a subset of players $K \subseteq \{1, \ldots, J\}$, we can think of the mapping $\overrightarrow{c}_K : \times_{i \in K} (T_i^* \backslash h_i(T_i)) \to C_K$. Write $\overrightarrow{c}_K(t_K^*)$ for the profile in C_K with $\overrightarrow{c}_K(t_K^*) = (\overrightarrow{c}_i(t_i^*) : i \in K)$.

Let us point out: We endow the set of players, viz. $\bigcup_{i=1}^J T_i^* \backslash h_i(T_i)$, with the discrete topology. Note carefully that the discrete topology is finer than the induce topology. With this choice, the mapping $\overrightarrow{c}_i$ defined above is continuous.

We now want to define a payoff function $u_{m(i)} : \mathcal{C} \to \mathbb{R}$ for player $m(i)$ (in the game G). To do so, it will be useful to first define auxiliary (payoff) functions for $m(i)$ that depend on subsets of players. The function $u_{m(i)}$ will be, effectively, the sum of these auxiliary functions.

Fix some player i and consider a subset K of players not containing i, i.e., some $K \subseteq \{1, \ldots, J\} \backslash \{i\}$. Write $K^c = \{1, \ldots, I\} \backslash (K \cup \{i\})$, i.e., all players that are not in $K \cup \{i\}$. Let us give the loose idea: We will construct a function $v_{m(i)}[K]$ that takes choice profiles for members of K and maps it into a payoff for player $m(i)$. When we do so, we will assume that players in K^c (if there are any!) play according to the equilibrium profile. For instance, if $I = J = 3$ and $i = 1$, then we can have K be either $\emptyset$, $\{2\}$, $\{3\}$, or $\{2, 3\}$. Consider the case of $K = \{2\}$. We will have $v_{m(1)}[\{2\}] : C_1 \times \mathcal{C}_2 \to \mathbb{R}$, so that we are computing expected payoffs for $m(1)$ when types for player 2 are in $T_2^* \backslash h_2(T_2)$ and types for player 3 are in $h_3(T_3)$. Because (for this subset K) types for player 2 are in $T_2^* \backslash h_2(T_2)$, the domain maps a choice for player $m(1)$ plus choices players in $T_2^* \backslash h_2(T_2)$, i.e., $C_1 \times \mathcal{C}_2$, into a payoff. Because (for this subset K) types for player 3 are in $h_3(T_3)$, we assume they play according to the given equilibrium.

Once we have the functions $v_{m(i)}[K]$ for all subsets $K \subseteq \{1,\ldots,J\}\backslash\{i\}$, we can extend these functions to a function $u_{m(i)} : \mathcal{C} \to \mathbb{R}$. Specifically, set $u_{m(i)} = \sum_{K\subseteq J}[v_{m(i)}[K] \circ \text{proj}_{C_i\times\mathcal{C}_K}]$, where we write $\text{proj}_{C_i\times\mathcal{C}_K} : \mathcal{C} \to C_i \times \mathcal{C}_K$ for the projection map. The functions $u_{m(i)}$ are the payoff functions for the game G.

Now, let's specify the functions $v_{m(i)}[K]$. To do so, it will be useful to recall that, for each $j = 1,\ldots,I$, $h_j : T_j \to T^*_j$ is injective and bimeasurable. (See Lemma 6.1.) As such, we can define a bimeasurable map $g_j : h_j(T_j) \to T_j$ so that $g_j(h_j(t_j)) = t_j$. Now, fix a $K \subseteq \{1,\ldots,J\}\backslash\{i\}$. Let $v_{m(i)}[K] : C_i \times \mathcal{C}_K \to \mathbb{R}$ be such that

$$v_{m(i)}[K](c_i, \overrightarrow{c}_K) = \int_{\Theta\times\times_{j\in K}(T^*_j\backslash h_j(T_j))\times h_{K^c}(T_{K^c})} \pi_i(\theta, c_i, \overrightarrow{c}_K(t^*_K), s_{K^c}(g_{K^c}(t^*_{K^c})))d\beta^*_i(m(i)).$$

(Note, if $K = \emptyset$, then we take the convention that $\Theta \times (T^*_K\backslash h_K(T_K)) \times h_{K^c}(T_{K^c}) = \Theta \times h_{K^c}(T_{K^c})$ so that $v_{m(i)}[K]$ reduces to a mapping from C_i to $\mathbb{R}$. If $K^c = \emptyset$, then we take the convention that $\Theta \times (T^*_K\backslash h_K(T_K)) \times h_{K^c}(T_{K^c}) = \Theta \times T^*_{-i}$, so that $v_{m(i)}[K]$ reduces with $s_{K^c}(g_{K^c}(t^*_{K^c}))$ no longer being a factor.)

We begin by showing that each $v_{m(i)}[K]$ is continuous. For this, we will need a mathematical result.

Lemma C.1 *Fix metrizable spaces Ω_1, Ω_2. Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a bounded function so that each $f(\omega_1, \cdot) : \Omega_2 \to \mathbb{R}$ is measurable and each $f(\cdot, \omega_2) : \Omega_1 \to \mathbb{R}$ is continuous. Define $F : \Omega_1 \to \mathbb{R}$ so that*

$$F(\omega_1) = \int_{E_2} f(\omega_1, \omega_2)d\mu,$$

where E_2 is some event in Ω_2 and $\mu \in \Delta(\Omega_2)$. Then, F is a bounded continuous function.

Proof. The fact that F is bounded follows directly from the fact that f is bounded and $\mu(E_2) \leq 1$. We focus on showing that F is continuous. For this, fix a sequence $(\omega^n_1 : n = 1, 2, \ldots)$ contained in Ω_1 and suppose $\omega^n_1 \to \omega^*_1$. To show that F is continuous, it suffices to show that $F(\omega^n_1) \to F(\omega^*_1)$.

Write $f^*(\cdot) : \Omega_2 \to \mathbb{R}$ for the ω^*_1-section of the map f. Also, for each n, write $f^n(\cdot) : \Omega_2 \to \mathbb{R}$ for the ω^n_1-section of the map f. By assumption, each of $f^*, f^1, f^2, \ldots$ is measurable. Moreover, since f is bounded, f^* is bounded and the sequence $(f^n : n = 1, 2, \ldots)$ is uniformly bounded. Given this, it suffices to show that $f^n \to f^*$. If so, then, by the Bounded Convergence Theorem, $F(\omega^n_1) \to F(\omega^*_1)$. (See Doob, 1994, pages 83-84.)

To show that $f^n \to f^*$: Note that $\omega_1^n \to \omega_1^*$. So, for any ω_2, $(\omega_1^n, \omega_2) \to (\omega_1^*, \omega_2)$. It follows from the fact that each $f(\cdot, \omega_2)$ is continuous that $f^n \to f^*$. ■

Lemma C.2 *For each $m(i) \in T_i^* \backslash h_i(T_i)$ and each $K \subseteq \{1, \ldots, J\} \backslash \{i\}$, $v_{m(i)}[K] : C_i \times \mathcal{C}_K \to \mathbb{R}$ is continuous.*

Proof. Define a mapping $f_{m(i)}[K] : C_i \times \mathcal{C}_K \times \Theta \times \times_{j \in K}(T_j^* \backslash h_j(T_j)) \times h_{K^c}(T_{K^c}) \to \mathbb{R}$ so that

$$f_i[K](c_i, \overrightarrow{c}_K, \theta, t_K^*, t_{K^C}^*) = \pi_i(\theta, c_i, \overrightarrow{c}_K(t_K^*), s_{K^c}(g_{K^c}(t_{K^c}^*))).$$

Certainly, then, $f_i[K]$ is bounded. We will show that each $f_i[K](c_i, \overrightarrow{c}_K, \cdot)$ is measurable and each $f_i[K](\cdot, \theta, t_K^*, t_{K^C}^*)$ is continuous. Then the result follows from Lemma C.1 and the fact that

$$v_{m(i)}[K](c_i, \overrightarrow{c}_K) = \int_{\Theta \times \times_{j \in K}(T_j^* \backslash h_j(T_j)) \times h_{K^c}(T_{K^c})} f_i[K](c_i, \overrightarrow{c}_K, \theta, t_K^*, t_{K^C}^*) d\beta_i^*(m(i)).$$

First we show that, for each $(c_i, \overrightarrow{c}_K)$, $f_i[K](c_i, \overrightarrow{c}_K, \cdot)$ is measurable. To see this, note that there is a measurable mapping $(\theta, t_K^*, t_K^*) \mapsto (\theta, c_i, \overrightarrow{c}_K(t_K^*), s_{K^c}(g_{K^c}(t_{K^c}^*)))$. (Here, we use the fact that $\overrightarrow{c}_K$, s_{K^c}, and g_{K^c} are measurable.) Moreover, $f_i[K](c_i, \overrightarrow{c}_K, \cdot)$ is the composite of this function with π_i. As such, $f_i[K](c_i, \overrightarrow{c}_K, \cdot)$ is measurable.

Next we show that, for each $(\theta, t_K^*, t_{K^C}^*)$, $f_i[K](\cdot, \theta, t_K^*, t_{K^C}^*)$ is continuous. For this, suppose that $(c_i^n, \overrightarrow{c}_K^n) \to (c_i, \overrightarrow{c}_K)$. Then, note that $(\theta, c_i^n, \overrightarrow{c}_K^n(t_K^*), s_{K^c}(g_{K^c}(t_{K^c}^*))) \to (\theta, c_i, \overrightarrow{c}_K(t_K^*), s_{K^c}(g_{K^c}(t_{K^c}^*)))$. So, using the continuity of π_i, $f_i[K](c_i^n, \overrightarrow{c}_K^n, \theta, t_K^*, t_{K^C}^*) \to f_i[K](c_i, \overrightarrow{c}_K, \theta, t_K^*, t_{K^C}^*)$, as required. ■

Lemma C.3 *The map $u_{m(i)}$ is continuous.*

Proof. Note, each $\text{proj}_{C_i \times \mathcal{C}_K}$ is a continuous function. With this and Lemma C.2, each $v_{m(i)}[K] \circ \text{proj}_{C_i \times \mathcal{C}_K}$ is a continuous function. It follows that $u_{m(i)}$ is a finite sum of continuous functions and so continuous. ■

Write $\mathcal{D}_i$ for the set $[\Delta(C_i)]^{M(i)}$ write $\overrightarrow{\sigma}_i$ for an arbitrary element of $\mathcal{D}_i$. Take $\mathcal{D} = \times_{i=1}^J \mathcal{D}_i$ and write $\overrightarrow{\sigma} = (\overrightarrow{\sigma}_1, \ldots, \overrightarrow{\sigma}_J)$ for an arbitrary element of $\mathcal{D}$. For a given player $m(i)$, take $\mathcal{D}_{-m(i)}$ to be $[\Delta(C_i)]^{(M(i)-1)} \times_{j \neq i} \mathcal{D}_j$ if $M(i)$ is finite and $\mathcal{D}$ otherwise. Note, if $M(i)$ is (countably) infinite $\mathcal{D}_{-m(i)} = \mathcal{D}$. An arbitrary element of $\mathcal{D}_{-m(i)}$ will be denoted as $\overrightarrow{\sigma}_{-m(i)}$.

Extend payoff functions to $u_{m(i)} : \mathcal{D} \to \mathbb{R}$ in the usual way.

Lemma C.4 *There exists some mixed choice equilibrium for the game G.*

Proof. For each player $m(i)$, define a best response correspondence $\max_{m(i)} : \mathcal{D}_{-m(i)} \twoheadrightarrow \Delta(C_i)$ so that

$$\max_{m(i)}(\overrightarrow{\sigma}_{-m(i)}) = \{\sigma_{m(i)} \in \arg\max u_{m(i)}(\cdot, \overrightarrow{\sigma}_{-m(i)})\}.$$

Extend this correspondence to a best response correspondence $\mathrm{BR}_{m(i)} : \mathcal{D} \twoheadrightarrow \mathcal{D}$ so that

$$\mathrm{BR}_{m(i)}(\cdot, \overrightarrow{\sigma}_{-m(i)}) = \max_{m(i)}(\overrightarrow{\sigma}_{-m(i)}) \times \mathcal{D}_{-m(i)}.$$

Define $\mathrm{BR} : \mathcal{D} \twoheadrightarrow \mathcal{D}$ so that $\mathrm{BR}(\overrightarrow{\sigma}) = \bigcap_{i=1}^{J} \bigcap_{m(i)=1}^{M(i)} \mathrm{BR}_{m(i)}(\overrightarrow{\sigma})$. To show that there is a mixed strategy equilibrium of the game G, it suffices to show that there is a fixed point of BR.

To show that there is a fixed point of BR, we will apply the Glicksberg's (1952) Theorem. For this, it suffices to show that $\mathcal{D}$ is a non-empty, compact, convex subset of a convex Hausdorff linear topological space and that BR has a closed graph and is non-empty convex valued.

Note that each $\Delta(C_i)$ is a non-empty, compact, convex subset of a convex Hausdorff linear topological space. It follows that $\mathcal{D}$ satisfies the desired conditions. As such, we focus on the properties of BR.

First, we show that BR has a closed graph: By Berge's Maximum Theorem (see 17.31 in Aliprantis and Border (2007)), for each $m(i)$, $\max_{m(i)}$ is compact valued and upper-hemicontinuous. It follows that $\mathrm{BR}_{m(i)}$ is a compact valued and upper-hemicontinuous correspondence to a Hausdorff space. So, applying Theorem 17.10 in Aliprantis and Border (2007), it follows that $\mathrm{BR}_{m(i)}$ has a closed graph. It now follows from Theorem 17.25 in Aliprantis and Border (2007) that BR has a closed graph.

Next we show that BR is non-empty convex valued: By Berge's Maximum Theorem (see 17.31 in Aliprantis and Border (2007)), for each $m(i)$, $\max_{m(i)}$ is non-empty valued. It is standard that $\max_{m(i)}$ is convex valued. (This follows from the fact that a face of a polytope is convex.) It follows from construction then that $\mathrm{BR}_{m(i)}$ and BR are non-empty valued. ■

In what follows, we will consider strategies r_i^* of $(\Gamma, \mathcal{T}^*)$ satisfying $r_i^* \circ h_i = s_i$. Note, such strategies are well defined since h_i is injective. If $T_i^* \backslash h_i(T_i) \neq \emptyset$, then given some r_i^* we write $\overrightarrow{r}_i^*$ for $(r_i^*(1), r_i^*(2), \ldots)$, i.e., the associated element of $\mathcal{D}_i$ played by types in $T_i^* \backslash h_i(T_i)$ under r_i^*. A standard argument establishes the next remark.

Remark C.1 *Fix some $m(i) \in T_i^* \backslash h_i(T_i)$. For any $(r_1^*, \ldots, r_I^*) \in S^*$ with $(r_1^* \circ h_1, \ldots, r_I^* \circ h_I) = (s_1, \ldots, s_I)$,*

$$\Pi_i(m(i), r_i^*, r_{-i}^*) = u_{m(i)}(\overrightarrow{r}^*_1, \ldots, \overrightarrow{r}^*_J).$$

Conversely, given some $(\overrightarrow{\sigma}_1, \ldots, \overrightarrow{\sigma}_J) \in \mathcal{D}$, there is a unique strategy profile $(r_1^*, \ldots, r_I^*) \in S^*$ with $(\overrightarrow{r}^*_1, \ldots, \overrightarrow{r}^*_J) = (\overrightarrow{\sigma}_1, \ldots, \overrightarrow{\sigma}_J)$ and $(r_1^* \circ h_1, \ldots, r_I^* \circ h_I) = (s_1, \ldots, s_I)$. In this case,

$$\Pi_i(m(i), r_i^*, r_{-i}^*) = u_{m(i)}(\overrightarrow{r}^*_1, \ldots, \overrightarrow{r}^*_J).$$

Proof of Proposition 7.1. Fix a measurable equilibrium $(s_1, \ldots, s_I)$ of the Bayesian game $(\Gamma, \mathcal{T})$. As above, construct the game G (based on $(s_1, \ldots, s_I)$). By Lemma C.4, there exists a mixed choice profile, viz. $(\overrightarrow{\sigma}_1, \ldots, \overrightarrow{\sigma}_J)$, that is an equilibrium for the game G. Now, by Remark C.1, we can find a strategy profile $(s_1^*, \ldots, s_I^*)$ so that $(\overrightarrow{s}^*_1, \ldots, \overrightarrow{s}^*_J) = (\overrightarrow{\sigma}_1, \ldots, \overrightarrow{\sigma}_J)$ and $(s_1^* \circ h_1, \ldots, s_I^* \circ h_I) = (s_1, \ldots, s_I)$. We will show that $(s_1^*, \ldots, s_I^*)$ is an equilibrium for the Bayesian game $(\Gamma, \mathcal{T}^*)$.

First, we show condition (i). For this, it suffices to show that each s_i^* is measurable: Fix a measurable E_i in $\Delta(C_i)$ and note that

$$(s_i^*)^{-1}(E_i) = h_i((s_i)^{-1}(E_i)) \cup \{t_i^* \in T_i^* \backslash h_i(T_i) : s_i^*(t_i^*) \in E_i\}.$$

Note, since s_i is measurable, $(s_i)^{-1}(E_i)$ is Borel and, so, using the bimeasurability of h_i, $h_i((s_i)^{-1}(E_i))$ is Borel. Next, notice that $T_i^* \backslash h_i(T_i)$ is countable (and possibly empty) and so $\{t_i^* \in T_i^* \backslash h_i(T_i) : s_i^*(t_i^*) \in E_i\}$ is Borel. It follows that $(s_i^*)^{-1}(E_i)$ is the union of two measurable sets and so measurable.

Now, turn to condition (ii). First, fix some type $h_i(t_i) \in h_i(T_i)$. Notice that, for each $r_i^* : T_i^* \to \Delta(C_i)$,

$$\begin{aligned}
\int_{\Theta \times T_{-i}^*} \pi_i(\theta, s_i^*(h_i(t_i)), s_{-i}^*(t_{-i}^*)) d\beta_i^*(h_i(t_i)) &= \int_{\Theta \times T_{-i}} \pi_i(\theta, s_i^*(h_i(t_i)), s_{-i}^*(h_{-i}(t_{-i}))) d\beta_i(t_i) \\
&\geq \int_{\Theta \times T_{-i}} \pi_i(\theta, r_i^*(h_i(t_i)), s_{-i}^*(h_{-i}(t_{-i}))) d\beta_i(t_i) \\
&= \int_{\Theta \times T_{-i}^*} \pi_i(\theta, r_i^*(h_i(t_i)), s_{-i}^*(t_{-i}^*)) d\beta_i^*(h_i(t_i)),
\end{aligned}$$

where the first and last lines use the Change of Variables Theorem (e.g., Billingsley (2008, Theorem 16.13) plus the fact that h_{-i} is injective, and the second line uses the fact that the fact that $(s_1, \ldots, s_I) = (s_1^* \circ h_1, \ldots, s_I^* \circ h_I)$ is an equilibrium for the Bayesian game

$(\Gamma, \mathcal{T})$. Next, fix some type $t_i^* \in T_i^* \backslash h_i(T_i)$, if one exists. Note, for any strategy r_i^* of the Bayesian game $(\Gamma, \mathcal{T}^*)$, there exists a strategy q_i^* with $q_i^* \circ h_i = s_i$ and $q_i^*(t_i^*) = r_i^*(t_i^*)$. So, $\Pi_i(t_i^*, r_i^*, s_{-i}^*) = \Pi_i(t_i^*, q_i^*, s_{-i}^*)$. With this, it follows from Remark C.1 and the fact that $(\overrightarrow{\sigma}_1, \ldots, \overrightarrow{\sigma}_I)$ is an equilibrium that $\Pi_i(t_i^*, s_i^*, s_{-i}^*) \geq \Pi_i(t_i^*, r_i^*, s_{-i}^*)$, for all strategies r_i^* of the Bayesian game $(\Gamma, \mathcal{T}^*)$.

Thus, we see that $(s_1^*, \ldots, s_I^*)$ is a Bayesian equilibrium of $(\Gamma, \mathcal{T}^*)$. Moreover, $(s_1^* \circ h_1, \ldots, s_I^* \circ h_I) = (s_1, \ldots, s_I)$, as required. ■

Appendix D Proofs for Section 8

Lemma D.1 *Fix a Θ-based interactive type structure $\mathcal{T} = \langle \Theta; T_1, \ldots, T_I; \beta_1, \ldots, \beta_I \rangle$. Let $\times_{i=1}^I \overline{T}_i$ be a belief-closed subset of T. Then, there is a Θ-based interactive metrizable type structure*

$$\langle \Theta; \overline{T}_1, \ldots, \overline{T}_I; \overline{\beta}_1, \ldots, \overline{\beta}_I \rangle,$$

where, for each $t_i \in \overline{T}_i$ and each event E_{-i} in $\Theta \times \overline{T}_{-i}$, $\overline{\beta}_i(t_i)(E_{-i}) = \beta_i(t_i)(E_{-i})$.

Proof. Since we endow each $\overline{T}_i$ with the relative topology, we have that each $\overline{T}_i$ is metrizable. Also note that $\overline{\beta}_i(t_i)$ is indeed a probability measure on $\Theta \times \overline{T}_{-i}$. To see this, recall that $\times_{i=1}^I \overline{T}_i$ is a belief-closed subset of T, so that each $\overline{T}_{-i}$ is Borel in T_{-i} with $\beta_i(t_i)(\Theta \times \overline{T}_{-i}) = 1$. So, if E_{-i} is an event in $\Theta \times \overline{T}_{-i}$, it is also an event in $\Theta \times T_{-i}$ and $\overline{\beta}_i(t_i)$ forms a probability measure.

Finally, we show that each $\overline{\beta}_i$ is measurable. Fix some F Borel in $\Delta(\Theta \times \overline{T}_{-i})$. Define $H \subseteq \Delta(\Theta \times T_{-i})$ so that $\nu \in H$ if and only if there exists some $\mu \in F$ so that $\mu(E_{-i}) = \nu(E_{-i})$ for each event E_{-i} in $\Theta \times \overline{T}_{-i}$. It follows from Lemma 14.4 in Aliprantis and Border (2007) that H is Borel in $\Delta(\Theta \times T_{-i})$. It is immediate from the construction that $(\overline{\beta}_i)^{-1}(F) = (\beta_i)^{-1}(H) \cap \overline{T}_i$. So, using the fact that β_i is measurable, $(\overline{\beta}_i)^{-1}(F)$ is measurable, as required. ■

Proof of Lemma 8.1. Suppose $\mathcal{T}$ induces a decomposition of $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$. Then, both $\times_{i=1}^I h_i(T_i)$ and $\times_{i=1}^I (T_i^* \backslash h_i(T_i))$ are belief-closed subsets of T^*. So, using Lemma D.1, each of these induce a Θ-based interactive metrizable type structure. Write

$$\mathcal{T}(h_1, \ldots, h_I) = \langle \Theta; h_1(T_1), \ldots, h_I(T_I); \overline{\beta}_1, \ldots, \overline{\beta}_I \rangle$$

for the structure induced by $\mathcal{T}$, and write

$$(\mathcal{T}^*\backslash\mathcal{T}) = \langle\Theta; T_1^*\backslash h_1(T_1), \ldots, T_I^*\backslash h_I(T_I); \overline{\beta}_1^*, \ldots, \overline{\beta}_I^*\rangle$$

for the difference structure.

Fix a Θ-based game Γ and an equilibrium $(s_1, \ldots, s_I)$ for the Bayesian Game $(\Gamma, \mathcal{T})$. Suppose there exists an equilibrium for the difference game $(\Gamma, (\mathcal{T}^*\backslash\mathcal{T}))$, viz. $(s_1^\nabla, \ldots, s_I^\nabla)$. Construct a strategy, viz. s_i^*, for $(\Gamma, \mathcal{T}^*)$, as follows. For each $t_i \in T_i$, let $s_i^*(h_i(t_i)) = s_i(t_i)$. (This is well-defined since each h_i is injective.) For each $t_i^* \in T_i^*\backslash h_i(T_i)$, let $s_i^*(t_i^*) = s_i^\nabla(t_i^*)$. We now show that the constructed $(s_1^*, \ldots, s_I^*)$ is a Bayesian equilibrium for $(\Gamma, \mathcal{T}^*)$.

Begin with condition (i). First, fix $h_i(t_i) \in h_i(T_i)$. There is some measurable support of $\beta_i(t_i)$, viz. $\operatorname{supp}\beta_i(t_i)$, so that $\mathrm{id} \times s_{-i}$ is measurable when the domain is restricted to $\operatorname{supp}\beta_i(t_i)$. To see this, begin by noting that $(\mathrm{id} \times h_{-i})(\operatorname{supp}\beta_i(t_i))$ is measurable, since $(\mathrm{id} \times h_{-i})$ is bimeasurable. Moreover,

$$\beta_i^*(h_i(t_i))((\mathrm{id} \times h_{-i})(\operatorname{supp}\beta_i(t_i))) = \beta_i(t_i)(\operatorname{supp}\beta_i(t_i) = 1,$$

so that $(\mathrm{id} \times h_{-i})(\operatorname{supp}\beta_i(t_i))$ is a support of $\beta_i^*(h_i(t_i))$. Finally, fix some measurable E_{-i} in $\Delta(C_{-i})$ and note that

$$(s_{-i}^*)^{-1}(E_{-i}) \cap (\mathrm{id} \times h_{-i})(\operatorname{supp}\beta_i(t_i)) = (s_{-i})^{-1}(E_{-i}) \cap (\operatorname{supp}\beta_i(t_i)),$$

and so is measurable.

Next suppose $t_i^* \in T_i^*\backslash h_i(T_i)$. There is some measurable support of $\beta_i^*(t_i^*)$, viz. $\operatorname{supp}\beta_i^*(t_i^*)$, so that $\mathrm{id} \times s_{-i}^\nabla$ is measurable when the domain is restricted to $\operatorname{supp}\beta_i^*(t_i^*)$. Moreover, $\operatorname{supp}\beta_i^*(t_i^*) \subseteq \Theta \times \times_{j\neq i}(T_j\backslash h_j(T_j))$. So, for each measurable E_{-i} in $\Delta(C_{-i})$,

$$(s_{-i}^*)^{-1}(E_{-i}) \cap (\operatorname{supp}\beta_i^*(t_i^*)) = (s_{-i}^\nabla)^{-1}(E_{-i}) \cap (\operatorname{supp}\beta_i^*(t_i^*)),$$

and so is measurable.

Now, we turn to condition (ii). First, fix a type $h_i(t_i) \in h_i(T_i)$. Given a strategy $r_i^* : T_i^* \to \Delta(C_i)$, write r_i for the strategy $r_i : T_i \to \Delta(C_i)$ with $r_i(t_i) = r_i^*(h_i(t_i))$ for all $t_i \in T_i$. Then, using the Change of Variables Theorem (e.g., Billingsley, 2008, Theorem 16.13, for any strategy profile $(r_1^*, \ldots, r_I^*)$ with r_{-i}^* measurable,

$$\int_{\Theta\times T_{-i}^*} \pi_i(\theta, r_i^*(h_i(t_i)), r_{-i}^*(t_{-i}^*))d\beta_i^*(h_i(t_i)) = \int_{\Theta\times T_{-i}} \pi_i(\theta, r_i(t_i), r_{-i}(t_{-i}))d\beta_i(t_i).$$

So, using the fact that $(s_1, \ldots, s_I)$ is a Bayesian Equilibrium of $(\Gamma, \mathcal{T})$,

$$\int_{\Theta \times T^*_{-i}} \pi_i(\theta, s^*_i(h_i(t_i)), s^*_{-i}(t^*_{-i}))d\beta^*_i(h_i(t_i)) \geq \int_{\Theta \times T^*_{-i}} \pi_i(\theta, r^*_i(h_i(t_i)), s^*_{-i}(t^*_{-i}))d\beta^*_i(h_i(t_i)), \tag{2}$$

for all strategies r^*_i of the Bayesian game $(\Gamma, \mathcal{T}^*)$.

Turn to a type $t^*_i \in T^*_i \backslash h_i(T_i)$. Given a strategy $r^*_i : T^*_i \to \Delta(C_i)$, write r^∇_i for the strategy $r^\nabla_i : T^*_i \backslash h_i(T_i) \to \Delta(C_i)$ with $r^\nabla_i(t^*_i) = r^*_i(t^*_i)$ for all $t^*_i \in T^*_i \backslash h_i(T_i)$. Certainly,

$$\int_{\Theta \times T^*_{-i}} \pi_i(\theta, r^*_i(t^*_i), r^*_{-i}(t^*_{-i}))d\beta^*_i(t^*_i) = \int_{\Theta \times \times_{j \neq i} T^*_j \backslash h_j(T_j)} \pi_i(\theta, r^\nabla_i(t^*_i), r^\nabla_{-i}(t^*_{-i}))d\beta^*_i(t^*_i).$$

So, using the fact that $(s^\nabla_1, \ldots, s^\nabla_I)$ is a Bayesian Equilibrium of $(\Gamma, \mathcal{T}^* \backslash \mathcal{T})$,

$$\int_{\Theta \times T^*_{-i}} \pi_i(\theta, s^*_i(t^*_i), s^*_{-i}(t^*_{-i}))d\beta^*_i(t^*_i) \geq \int_{\Theta \times T^*_{-i}} \pi_i(\theta, r^*_i(t^*_i), s^*_{-i}(t^*_{-i}))d\beta^*_i(t^*_i), \tag{3}$$

for all strategies r^*_i of the Bayesian game $(\Gamma, \mathcal{T}^*)$.

Taking Equations 2-3, $(s^*_1, \ldots, s^*_I)$ is an equilibrium for the Bayesian game $(\Gamma, \mathcal{T}^*)$. The converse follows immediately from the Pull-Back Property (Proposition 4.1). ■

Proof of Proposition 8.1. If $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the Extension Property for Γ, then it is immediate that there is an equilibrium for the Bayesian game $(\Gamma, \mathcal{T}^*)$. Conversely, suppose there is an equilibrium for the Bayesian game $(\Gamma, \mathcal{T}^*)$. By the Pull-Back Property (Proposition 4.1), there is an equilibrium for the difference game $(\Gamma, (\mathcal{T}^* \backslash \mathcal{T}))$. Now, using Lemma 8.1, $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the Equilibrium Extension Property for Γ. ■

Proof of Lemma 8.2. Now, let μ be a common prior for $\mathcal{T}$. Fix distinct players i and j and note that

$$\beta_i(t_i)(\Theta \times \{t_j\} \times T_{-i-j}) = \frac{\mu(\Theta \times \{t_i\} \times \{t_j\} \times T_{-i-j})}{\mu(\Theta \times \{t_i\} \times T_{-i-j})}.$$

So, $\beta_i(t_i)(\Theta \times \{t_j\} \times T_{-i-j}) > 0$ if and only if $\mu(\Theta \times \{t_i\} \times \{t_j\} \times T_{-i-j}) > 0$. But, an analogous argument for j gives that $\beta_j(t_j)(\Theta \times \{t_i\} \times T_{-i-j}) > 0$ if and only if $\mu(\Theta \times \{t_i\} \times \{t_j\} \times T_{-i-j}) > 0$. This establishes the result. ■

Proof of Lemma 8.3. By Lemma 6.3, there exists some i with $T_i^* \backslash h_i(T_i)$ non-empty. In particular, fix $t_i^* \in T_i^* \backslash h_i(T_i)$. Recall, since $\mathcal{T}^*$ is mutually absolutely continuous, it is countable. As such, for each player $j \neq i$, we can find some $t_j^* \in T_j^*$ with $\beta_i^*(t_i^*)(\Theta \times \{t_j^*\} \times T_{-i-j}^*) > 0$. Again using the fact that $\mathcal{T}^*$ is mutually absolutely continuous, we also have that, for each such t_j^*, $\beta_j^*(t_j^*)(\Theta \times \{t_i^*\} \times T_{-i-j}^*) > 0$. This implies that $t_j^* \in T_j^* \backslash h_j(T_j)$. (If $t_j^* \in h_j(T_j)$, then there is some $t_j \in T_j$ with $\beta_j(t_j)(\Theta \times (h_i)^{-1}(\{t_i^*\}) \times T_{-i-j}) > 0$, contradicting the fact that $(h_i)^{-1}(\{t_i^*\}) = \emptyset$.)

Now, note that, since each h_j is bimeasurable, each $T_j^* \backslash h_j(T_j)$ is Borel. So, for each j, $\beta_i^*(t_i^*)(\Theta \times (T_j^* \backslash h_j(T_j)) \times T_{-i-j}^*) = 1$. Since this holds for each $j \neq i$, we have

$$\begin{aligned} 1 &= \beta_i^*(t_i^*)(\Theta \times \cap_{j \neq i}((T_j^* \backslash h_j(T_j)) \times T_{-i-j}^*)) \\ &= \beta_i^*(t_i^*)(\Theta \times \times_{j \neq i}(T_j^* \backslash h_j(T_j))), \end{aligned}$$

as required.

Finally, note that we showed that, for each $j \neq i$, $T_j^* \backslash h_j(T_j)$ is non-empty. So, applying the same argument to each $t_j^* \in T_j^* \backslash h_j(T_j)$, we get the desired result. ■

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