# Online Appendix: The Context of the Game

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This appendix proves some properties mentioned in the main text.

## 1 Proofs for Section 4

This appendix provides the proofs for Section 7. Throughout, we assume that type structures are non-redundant. Let us begin by recording two known properties.

**Property 1.1** *Fix non-redundant $\Theta$-based structures $\mathcal{T}$ to $\mathcal{T}^*$. If $(h_1, \ldots, h_I)$ is a type morphism from $\mathcal{T}$ to $\mathcal{T}^*$, then each $h_i$ is injective and uniquely defined.*

**Property 1.2** *Fix non-redundant $\Theta$-based structures $\mathcal{T}$ to $\mathcal{T}^*$. If $(h_1, \ldots, h_I)$ is a type morphism from $\mathcal{T}$ to $\mathcal{T}^*$, then each $h_i$ is bimeasurable.*

Property 1.1 follows from Proposition 5.1 in Heifetz-Samet [21, 1998]. If $(h_1, \ldots, h_I)$ is a type morphism from $\mathcal{T}$ to $\mathcal{T}^*$, then it is a mapping that preserves hierarchies of beliefs. Since $\mathcal{T}$ is non-redundant, each $h_i$ must be injective. Since $\mathcal{T}^*$ is non-redundant, there is a unique type morphism from $\mathcal{T}$ to $\mathcal{T}^*$. Property 1.2 foolows from Lemma 6.4 in Friedenberg-Meier [17, 2008].

**Proof of Lemma 7.1.** Suppose $(h_1, \ldots, h_I)$ is a type morphism from $\mathcal{T}$ to $\mathcal{T}^*$. Apply Properties 1.1-1.2 to get that $(h_1, \ldots, h_I)$ is an embedding. Suppose there is also a type morphism $(h_1^*, \ldots, h_I^*)$ from $\mathcal{T}^*$ to $\mathcal{T}$. Fix a type $t_i^* \in T_i^*$ and note that $t_i^*$ and $h_i^*(t_i^*)$ induce the same hierarchies of beliefs. (See Proposition 5.1 in [21, 1998].) So, since $\mathcal{T}^*$ is non-redundant, $h_i(h_i^*(t_i^*)) = t_i^*$. (That is, $h_i$ is surjective.) Now use Property 1.1 (i.e., that $h_i$ is injective) to get that $h_i^* = (h_i)^{-1}$. ■

**Proof of Lemma 7.2.** Suppose $(h_1, \ldots, h_I)$ is a type morphism from $\mathcal{T}$ to $\mathcal{T}^*$. By Property 1.2, $\Theta \times h_{-i}(T_{-i})$ is measurable. So, by definition of a type morphism,

$$\begin{aligned}
\beta_i^*(h_i(t_i))(\Theta \times h_{-i}(T_{-i})) &= \beta_i(t_i)\left((\mathrm{id} \times h_{-i})^{-1}(\Theta \times h_{-i}(T_{-i}))\right) \\
&= \beta_i(t_i)(\Theta \times T_{-i}) \\
&= 1,
\end{aligned}$$

where the second line uses the fact that $h_{-i}$ is injective. ■

**Proof of Lemma 7.3.** Suppose, contra hypothesis, for each $i = 1, \ldots, I$, $h_i(T_i) = T_i^*$. Define $h_i^* : T_i^* \to T_i$ so that $h_i^* = (h_i)^{-1}$. Then, $h_i^*$ is bijective, with $h_i(h_i^*(t_i^*)) = t_i^*$ for each $t_i^* \in T_i^*$. Also, by Property 1.2, $h_i$ is bimeasurable and, therefore, $h_i^*$ is bimeasurable. So, for any event $E \subseteq \Theta \times T_{-i}$, $((\mathrm{id} \times h_{-i})(E)) = ((\mathrm{id} \times h_{-i}^*)^{-1}(E))$. These facts will be used below.

We will show that $\mathcal{T}^*$ can be embedded into $\mathcal{T}$ via $(h_1^*, \ldots, h_I^*)$. Fix some $t_i^* \in T_i^*$ and consider $h_i^*(t_i^*) = t_i$. Also, fix an event $E \subseteq \Theta \times T_{-i}$. Note that

$$\begin{aligned}
\beta_i(h_i^*(t_i^*))(E) &= \beta_i(h_i^*(t_i^*))((\mathrm{id} \times h_{-i})^{-1}((\mathrm{id} \times h_{-i})(E))) \\
&= \beta_i^*(t_i^*)((\mathrm{id} \times h_{-i})(E)) \\
&= \beta_i^*(t_i^*)((\mathrm{id} \times h_{-i}^*)^{-1}(E)),
\end{aligned}$$

where the first line makes use of Properties 1.1-1.2, the second line makes use of the fact that $h_i(h_i^*(t_i^*)) = t_i$ and the fact that $(h_1, \ldots, h_I)$ is a type morphism from $\mathcal{T}$ to $\mathcal{T}^*$, and the last line uses the fact that $((\mathrm{id} \times h_{-i})(E)) = ((\mathrm{id} \times h_{-i}^*)^{-1}(E))$. ■

## 2 Rationalizability

In this appendix, we explore the Extension and Pull-back properties for the case of rationalizability. Specifically, we show that, we can only have a failure of Extension if the rationaizable set of the analyst's structure is empty. So, we cannot have the Extension failure constructed in Section 5.

We focus on correlated rationalizability. We extend the definitions in Dekel-Fudenberg-Morris [14, 2007] and Battigalli-Di Tillio-Grillo-Penta [1, 2008] to arbitrary (i.e., non-finite) games. Let us begin with the definition.

Fix a $\Theta$-based Bayesian game $(\Gamma, \mathcal{T})$. The $m$-rationalizable and rationalizable sets will be defined relative to this Bayesian game.

**Definition F1** *Set $R_i^0 = C_i \times T_i$. For each $m \geq 0$, put $(c_i, t_i) \in R_i^{m+1}$ if there exists a measure $\mu \in \Delta(\Theta \times C_{-i} \times T_{-i})$ so that*

*(i) for any $d_i \in C_i$, $\int_{\Theta \times C_{-i}} \pi_i(\theta, c_i, c_{-i})\, d\,\mathrm{marg}_{\Theta \times C_{-i}}\, \mu \geq \int_{\Theta \times C_{-i}} \pi_i(\theta, d_i, c_{-i})\, d\,\mathrm{marg}_{\Theta \times C_{-i}}\, \mu$;*

*(ii) $\mu(\Theta \times R_{-i}^m) = 1$; and*

*(iii) $\beta_i(t_i) = \mathrm{marg}_{\Theta \times T_{-i}}\, \mu$.*

*Call $R^m = \times_{i=1}^I R_i^m$ the set of* ***m-rationalizable*** *choice-type pairs.*

**Remark F1** *Suppose, for each $m$, $R_1^m, \ldots, R_I^m$ are non-empty and measurable. Then, for each $m$, $R^{m+1} \subseteq R^m$.*

**Definition F2** *Call $R_i = \bigcap_m R_i^m$ the set of i-**rationalizable** choice-type pairs and $R = \times_i R_i$ the **rationalizable** choice-type pairs.*

Battigalli-Di Tillio-Grillo-Penta [1, 2008] provide epistemic conditions for rationalizability when the parameter and choice sets are finite. Definitions F1-F2 appear to be natural extensions to the infinite case. (Of course, this is a far cry from a formal epistemic treatment.) We also note that rationalizability may not be the appropriate solution concept when the question is about "small type structures." The appropriate concept may correspond to best-response sets, i.e., as in Pearce [33, 1984].[1]

We begin by pointing out that there are Bayesian games (perhaps pathological) for which the rationalizable set is empty.

**Example F1 (Dufwenberg-Stegeman [15, 2002; Example 2])** *Let $\Theta = \{\theta\}$ and consider a $\Theta$-based type structure $\mathcal{T} = \langle \Theta; T_1, T_2; \beta_1, \beta_2 \rangle$ with each $T_1 = \{t_1\}$ and $T_2 = \{t_2\}$. Define a $\Theta$-based game as follows: Let each $C_1 = C_2 = [0,1]$. Set*

$$\pi_i(\theta, c_i, c_{-i}) = \begin{cases} 1 - c_i & \text{if } 2c_i \geq c_{-i} > 0 \\ c_i & \text{otherwise.} \end{cases}$$

*Then, for each $m \geq 1$, $R_i^m = (0, \frac{1}{2^{m-1}}] \times \{t_i\}$. So, $\bigcap_m [R_1^m \times R_2^m] = \emptyset$.*

Is every extension failure associated with such a pathological game? We do not know. We next point out that there could be a second existence problem.

**Lemma F1** *If $\times_{i=1}^I R_i \neq \emptyset$, then, for each $i$ and each $m$, $R_i^m$ are measurable.*

**Proof.** Suppose, for some player $j \neq i$ and some $m$, $R_j^m$ is not measurable. Then, there is no $\mu \in \Delta(\Theta \times C_{-i} \times T_{-i})$ with $\mu(\Theta \times R_{-i}^m) = 1$. As such, $R_i^{m+1} = \emptyset$ and so $R^i = \emptyset$. ■

Lemma F1 says that, if there is some $m$ so that $R_i^m$ is not measurable, then $\times_{i=1}^I R_i = \emptyset$. We don't know if, for a general game, the sets $R_i^m$ must be measurable. As such, we don't know if this non-existence problem is possible.

Fix a $\Theta$-based game $\Gamma$ and $\Theta$-based interactive structures $\mathcal{T}$ and $\mathcal{T}^*$. Write $R^m = \times_{i=1}^I R_i^m$ (resp. $R = \times_{i=1}^I R_i$) for the set of $m$-rationalizable (resp. rationalizable) choice-type pairs for the Bayesian game $(\Gamma, \mathcal{T})$, and write $R^{m,*} = \times_{i=1}^I R_i^{m,*}$ (resp. $R^* = \times_{i=1}^I R_i^*$) for the set of $m$-rationalizable (resp. rationalizable) choice-type pairs for the Bayesian game $(\Gamma, \mathcal{T}^*)$.

**Definition F3** *Let $\mathcal{T}$ and $\mathcal{T}^*$ be two $\Theta$-based interactive type structures, so that $\mathcal{T}$ can be embedded into $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$. Then, the pair $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the **Rationalizable Extension Property for the $\Theta$-based game** $\Gamma$ if the following holds: If $(c_1, t_1, \ldots, c_I, t_I) \in R$ then $(c_1, h_1(t_1), \ldots, c_I, h_I(t_I)) \in R^*$. Say the pair $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the **Rationalizable Extension Property** if it satisfies the Rationalizable Extension Property for each $\Theta$-based game $\Gamma$.*

[1] The rationale for this statement is analogous to the rationale for self-admissible sets in Brandenburger-Friedenberg-Keisler [11, 2008].

**Definition F4** *Let $\mathcal{T}$ and $\mathcal{T}^*$ be two $\Theta$-based interactive type structures, so that $\mathcal{T}$ can be embedded into $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$. Then, the pair $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the* ***Rationalizable Pull-Back Property for the $\Theta$-based game*** *$\Gamma$ if the following holds: If $(c_1, h_1(t_1), \ldots, c_I, h_I(t_I)) \in R^*$, then $(c_1, t_1, \ldots, c_I, t_I) \in R$. Say the pair $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the* ***Rationalizable Pull-Back Property*** *if it satisfies the Rationalizable Pull-Back Property for each $\Theta$-based game $\Gamma$.*

**Proposition F1** *Fix $\Theta$-based structures $\mathcal{T}$ and $\mathcal{T}^*$, so that $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$. Fix, also, a $\Theta$-based game $\Gamma$. If, for each $m$, $R_1^m, \ldots, R_I^m$ and $R_1^{m,*}, \ldots, R_I^{m,*}$ are measurable, then $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the Rationalizable Extension and Pull-Back Properties for $\Gamma$.*

The following Corollary is an immediate consequence of Proposition F1.

**Corollary F1** *Fix $\Theta$-based structures $\mathcal{T}$ and $\mathcal{T}^*$, so that $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$. If, for each $\Theta$-based game $\Gamma$ and each $m$, $R_1^m, \ldots, R_I^m$ and $R_1^{m,*}, \ldots, R_I^{m,*}$ are measurable. Then, $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the Rationalizable Extension and Pull-Back Properties.*

The proof of Proposition F1 will follow from the next Lemma.

**Lemma F2** *Fix a $\Theta$-based game $\Gamma$ and $\Theta$-based structures $\mathcal{T}$ and $\mathcal{T}^*$, where $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$. Suppose, for each $m$, $R_1^m, \ldots, R_I^m$ and $R_1^{m,*}, \ldots, R_I^{m,*}$ are measurable. Then, for each $i = 1, \ldots, I$,*

*(i)* $(c_i, t_i) \in R_i^m$ *implies* $(c_i, h_i(t_i)) \in R_i^{m,*}$;

*(ii)* $(c_i, t_i) \in [C_i \times T_i] \backslash R_i^m$ *implies* $(c_i, h_i(t_i)) \in [C_i \times T_i^*] \backslash R_i^{m,*}$.

**Proof.** The proof is by induction on $m$. For $m = 0$, the result is immediate. Assume the result holds for $m$. We will show that it also holds for $m + 1$.

Begin with part (i): Fix $(c_i, t_i) \in R_i^{m+1}$. Then, we can find a measure $\mu \in \Delta(\Theta \times C_{-i} \times T_{-i})$ satisfying (i)-(iii) of Definition F1. Extend $h_{-i}$ to $\overrightarrow{h}_{-i} : \Theta \times C_{-i} \times T_{-i} \to \Theta \times C_{-i} \times T_{-i}^*$, with $\overrightarrow{h}_{-i}(\theta, c_{-i}, t_{-i}) = (\theta, c_{-i}, h_{-i}(t_{-i}))$ for each $(\theta, c_{-i}, t_{-i}) \in \Theta \times C_{-i} \times T_{-i}$. Let $\mu^*$ be the image measure of $\mu$ under $\overrightarrow{h}_{-i}$. We will show that $\mu^*$ satisfies analogs of (i)-(iii), relative to the structure $\mathcal{T}^*$, i.e.,

*(i*)* for each $d_i \in C_i$, $\int_{\Theta \times C_{-i}} \pi_i(\theta, c_i, c_{-i})\, d \operatorname{marg}_{\Theta \times C_{-i}} \mu^* \geq \int_{\Theta \times C_{-i}} \pi_i(\theta, d_i, c_{-i})\, d \operatorname{marg}_{\Theta \times C_{-i}} \mu^*$;
*(ii*)* $\mu^*\left(\Theta \times R_{-i}^{m,*}\right) = 1$; and
*(iii*)* $\beta_i^*(h_i(t_i)) = \operatorname{marg}_{\Theta \times T_{-i}^*} \mu^*$.
This will establish that $(c_i, h_i(t_i)) \in R_i^{m+1,*}$.

Condition *(i*)* follows from (i) and the fact that $\operatorname{marg}_{\Theta \times C_{-i}} \mu = \operatorname{marg}_{\Theta \times C_{-i}} \mu^*$. For part *(ii*)*, note that

$$\begin{aligned}
\mu^*\left(\Theta \times R_{-i}^{m,*}\right) &= \mu((\overrightarrow{h}_{-i})^{-1}\left(\Theta \times R_{-i}^{m,*}\right)) \\
&= \mu(\Theta \times R_{-i}^m) \\
&= 1,
\end{aligned}$$

where the first line uses the fact that $R_{-i}^{m,*}$ is measurable, the second line follows from parts (i)-(ii) of the induction hypothesis, and the third line follows from (ii) of Definition F1. For part *(iii*)*, fix some event $E^* \subseteq \Theta \times T_{-i}^*$ and note that

$$\begin{aligned}
\beta_i^*(h_i(t_i))(E^*) &= \beta_i(t_i)((\mathrm{id} \times h_{-i})^{-1}(E^*)) \\
&= \mathrm{marg}_{\Theta \times T_{-i}} \mu((\mathrm{id} \times h_{-i})^{-1}(E^*)) \\
&= \mu(C_{-i} \times (\mathrm{id} \times h_{-i})^{-1}(E^*)) \\
&= \mu((\overrightarrow{h}_{-i})^{-1}(C_{-i} \times E^*)) \\
&= \mu^*(C_{-i} \times E^*) \\
&= \mathrm{marg}_{\Theta \times T_{-i}^*} \mu^*(E^*),
\end{aligned}$$

where the first line follows from the definition of a type morphism, the second line follows from condition (iii), and the fourth and fifth lines follow from construction.

Now we turn to part (ii). Suppose $(c_i, h_i(t_i)) \in R_i^{m+1,*}$. Then, we can find a measure $\mu^*$ satisfying conditions *(i*)*-*(iii*)* above. Let $\overrightarrow{g}_{-i} : \Theta \times C_{-i} \times h_{-i}(T_{-i}) \to \Theta \times C_{-i} \times T_{-i}$ be a map, with $\overrightarrow{g}_{-i}(\theta, c_{-i}, h_{-i}(t_{-i})) = (\theta, c_{-i}, t_{-i})$. Recall that $h_{-i}$ is an embedding. As such, it is injective and so $\overrightarrow{g}_{-i}$ is well-defined. Likewise, $h_{-i}$ is bimeasurable, and so $\overrightarrow{g}_{-i}$ is the product of measurable maps and so measurable. Using *(ii*)*, $\mu^*(\Theta \times C_{-i} \times h_{-i}(T_{-i})) = 1$. So, the image measure of $\mu^*$ under $\overrightarrow{g}_{-i}$ is well-defined. Write $\mu$ for this measure. We need to show that $\mu$ satisfies conditions (i)-(iii) of Definition F1.

Conditions (i)-(ii) are shown by repeating the arguments for *(i*)* and *(ii*)* above. We focus on condition (iii). Fix some event $E$ in $\Theta \times T_{-i}$ and recall that

$$\begin{aligned}
\mu(C_{-i} \times E) &= \mu^*((\overrightarrow{g}_{-i})^{-1}(C_{-i} \times E)) \\
&= \mu^*(C_{-i} \times (\mathrm{id} \times h_{-i})(E)) \\
&= \mathrm{marg}_{\Theta \times T_{-i}^*} \mu^*((\mathrm{id} \times h_{-i})(E)) \\
&= \beta_i^*(h_i(t_i))((\mathrm{id} \times h_{-i})(E)) \\
&= \beta_i(t_i)((\mathrm{id} \times h_{-i})^{-1}((\mathrm{id} \times h_{-i})(E))) \\
&= \beta_i(t_i)(E),
\end{aligned}$$

where the first line follows from the fact that $\overrightarrow{g}_{-i}$ is measurable, the fourth line follows from condition *(iii*)*, the fifth line follows from the definition of a type morphism, and the last line uses the fact that $h_{-i}$ is injective. This establishes (iii). ■

As a corollary of Lemma F1 and Proposition F1, we have:

**Corollary F2** *Fix $\Theta$-based structures $\mathcal{T}$ and $\mathcal{T}^*$, so that $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$. If, for a $\Theta$-based game $\Gamma$, $\times_{i=1}^I R_i$ and $\times_{i=1}^I R_i^*$ are both non-empty, then $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the Rationalizable Extension and Pull-Back Properties for $\Gamma$.*

## References


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