Online Supplement: When Do Type Structures Contain All Hierarchies of Beliefs?

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This note provides certain technical steps omitted from the main text. The reader is asked to consult the main text for notation.

Lemma S1 *For each m:*

(i) $Z^c_{m+1} = Z^c_1 \times \prod_{n=1}^m \mathcal{P}\left(Z^d_n\right)$;

(ii) $\rho^c_{m+1}\left(x^c, t^d\right) = (x^c, \delta^d_1(t^d), \ldots, \delta^d_m(t^d))$.

Proof. For $m = 1$, both parts are immediate. Assume that the result holds for $m \geq 2$. Then

$$\begin{aligned} Z^c_{m+2} &= Z^c_{m+1} \times \mathcal{P}\left(Z^d_{m+1}\right) \\ &= Z^c_1 \times \textstyle\prod_{n=1}^m \mathcal{P}\left(Z^d_n\right) \times \mathcal{P}\left(Z^d_{m+1}\right). \end{aligned}$$

Also,

$$\begin{aligned} \rho^c_{m+2}\left(x^c, t^d\right) &= (\rho^c_{m+1}(x^c, t^d), \delta^d_{m+1}(t^d)) \\ &= (x^c, \delta^d_1(t^d), \ldots, \delta^d_m(t^d), \delta^d_{m+1}(t^d)), \end{aligned}$$

establishing the result. ■

Lemma S2 *Let Ω be metrizable and Φ be Polish. If $f : \Omega \to \Phi$ is measurable (resp. continuous), then $\underline{f} : \mathcal{P}(\Omega) \to \mathcal{P}(\Phi)$ is measurable (resp. continuous).*

Proof. The case where f is continuous is found in Aliprantis-Border [1, 1999; Theorem 14.14]. We treat the case where f is measurable.

First, note that, since Φ is Polish, $\mathcal{B}(\mathcal{P}(\Phi))$ is generated by all the sets of the form $\{\nu \in \mathcal{P}(\Phi) : \nu(E) \in K\}$, ranging over E Borel in Φ and K measurable in $[0, 1]$ (see Kechris [2, 1995; Theorem 17.24]). It suffices to show that each set $\underline{f}^{-1}(\{\nu \in \mathcal{P}(\Phi) : \nu(E) \in K\})$ is in $\mathcal{B}(\mathcal{P}(\Omega))$. If so, then $\underline{f}$ is measurable. (See Aliprantis-Border [1, 1999; Corollary 4.23].)

Note that

$$\begin{aligned}\underline{f}^{-1}(\{\nu \in \mathcal{P}(\Phi) : \nu(E) \in K\}) &= \left\{\mu \in \mathcal{P}(\Omega) : \underline{f}(\mu)(E) \in K\right\} \\ &= \left\{\mu \in \mathcal{P}(\Omega) : \mu\left(f^{-1}(E)\right) \in K\right\}.\end{aligned}$$

Since f is measurable, $f^{-1}(E)$ is Borel in Ω. Now, by Aliprantis-Border [1, 1999; Lemma 14.16], $\underline{f}^{-1}(\{\nu \in \mathcal{P}(\Phi) : \nu(E) \in K\})$ is in $\mathcal{B}(\mathcal{P}(\Omega))$. ■

Lemma S3 *Let I be (at most) a countable collection of integers $1, 2, \ldots$. Let Ω be a topological space and, for each $i \in I$, let Φ_i be a Polish space. Fix measurable (resp. continuous) maps $f_i : \Omega \to \Phi_i$ and define $f : \Omega \to \prod_i \Phi_i$ so that*

$$f(\omega) = (f_1(\omega), f_2(\omega), \ldots)$$

for all $\omega \in \Omega$. Then, f is measurable (resp. continuous).

Proof. Theorem 8.5 in Chapter 2 of Munkres [3, 1975] establishes the case where each f_i is continuous. Suppose each f_i is measurable. Let E be a Borel set on $\prod_i \Phi_i$. Then, $f^{-1}(E) = \bigcap_i f_i^{-1}(E)$. Each $f_i^{-1}(E)$ is measurable, so that $f^{-1}(E)$ is measurable. ■

Lemma S4 *The maps ρ_m^c and δ_m^c are measurable. Moreover, if the structure is continuous, the maps ρ_m^c and δ_m^c are continuous.*

Proof. First, note that $\rho_1^c = \mathrm{proj}_{X^c}$, and so is certainly continuous. So, by Lemma S2, $\underline{\rho}_1^c$ is continuous. Also using this Lemma, we have that δ_1^c is measurable (and continuous if β^c is continuous).

Now, assume that the result holds for m. By the induction hypothesis and Lemma S3, ρ_{m+1}^c is measurable. (When ρ_m^c, δ_m^d are continuous, Lemma S3 says that ρ_{m+1}^c is continuous.) Now, by Lemma S2, $\underline{\rho}_{m+1}^c$ is measurable (resp. continuous when ρ_{m+1}^c is continuous). So, each δ_{m+1}^c is measurable (and continuous when β^c is continuous). ■

Lemma S5 *The map δ^c is measurable. Moreover, δ^c is continuous if the structure is continuous.*

Proof. Immediate from Lemmas S3 and S4. ■

Proof of Lemma 3.1. By induction on m.

$m = 1$: Since η_1^c is the identity map, $\eta_1^c(\mu_1^c) = \mu_1^c$, as required. Now note that that since ζ_1^d is the identity map, so is $\underline{\zeta}_1^d$, and so $\underline{\zeta}_1^d = \eta_1^d$. This establishes that $\zeta_2^c\left(x_1^c, \mu_1^d\right) = \left(\zeta_1^c(x_1^c), \underline{\zeta}_1^d\left(\mu_1^d\right)\right)$, as required.

$m \geq 2$: Assume that the result holds for m. It is immediate from the induction hypothesis that

$\eta_{m+1}^{c}\left(\mu_{1}^{c},\ldots,\mu_{m+1}^{c}\right)=\left(\mu_{1}^{c},\underline{\zeta}_{2}^{c}\left(\mu_{2}^{c}\right),\ldots,\underline{\zeta}_{m+1}^{c}\left(\mu_{m+1}^{c}\right)\right)$. Now,

$$\begin{aligned}\zeta_{m+2}^{c}\left(x_{1}^{c},\mu_{1}^{d},\ldots,\mu_{m}^{d},\mu_{m+1}^{d}\right) &= \left(\zeta_{1}^{c}\left(x_{1}^{c}\right),\eta_{m+1}^{d}\left(\mu_{1}^{d},\ldots,\mu_{m}^{d},\mu_{m+1}^{d}\right)\right)\\ &= \left(\zeta_{1}^{c}\left(x_{1}^{c}\right),\eta_{m}^{d}\left(\mu_{1}^{d},\ldots,\mu_{m}^{d}\right),\underline{\zeta}_{m+1}^{d}\left(\mu_{m+1}^{d}\right)\right)\\ &= \left(\zeta_{m+1}^{c}\left(x_{1}^{c},\mu_{1}^{d},\ldots,\mu_{m}^{d}\right),\underline{\zeta}_{m+1}^{d}\left(\mu_{m+1}^{d}\right)\right),\end{aligned}$$

where the second line follows from the fact that $\eta_{m+1}^{c}\left(\mu_{1}^{c},\ldots,\mu_{m+1}^{c}\right)=\left(\mu_{1}^{c},\underline{\zeta}_{2}^{c}\left(\mu_{2}^{c}\right),\ldots,\underline{\zeta}_{m+1}^{c}\left(\mu_{m+1}^{c}\right)\right)$, already established. ■

Lemma S6 *Let $I=\{1,2,\ldots\}$ be (at most) a countable set of integers. For each i, let $f_i:\Omega_i\to\Phi_i$ be an embedding. Define $f:\prod_{i\in I}\Omega_i\to\prod_{i\in I}\Phi_i$ so that $f\left(\omega_1,\omega_2,\ldots\right)=\left(f_1\left(\omega_1\right),f_2\left(\omega_2\right)\ldots\right)$. Then, f is also an embedding.*

Proof. Let $g_i:\Omega_i\to f_i\left(\Omega_i\right)$ be such that $g_i\left(\omega_i\right)=f_i\left(\omega_i\right)$. We have that each g_i is a homeomorphism. Define f as in the statement of the Lemma and $g:\prod_i\Omega_i\to f\left(\prod_i\Phi_i\right)$ so that $g\left(\omega\right)=f\left(\omega\right)$ for all $\omega\in\prod_i\Omega_i$. We will show that g is also a homeomorphism. The injectivity of g is immediate from the injectivity of each of the maps g_i. The surjectivity of g is immediate.

To show that g is continuous, fix closed sets C_i in $f_i\left(\Omega_i\right)$ where $C_i=f_i\left(\Omega_i\right)$ for all but finitely many i. Under the product and relative topologies, these sets form a basis for $\prod_{i\in I}f_i\left(\Omega_i\right)$. So, it suffices to show that $g^{-1}\left(\prod_{i\in I}C_i\right)$ is closed. Let J be the set of i with $C_i\neq f_i\left(\Omega_i\right)$. Then,

$$\begin{aligned}g^{-1}\left(\textstyle\prod_{i\in I}C_i\right) &= \textstyle\prod_{i\in J}\left[g_i^{-1}\left(C_i\right)\right]\times\prod_{i\in I\setminus J}\left[g_i^{-1}\left(C_i\right)\right]\\ &= \textstyle\prod_{i\in J}\left[g_i^{-1}\left(C_i\right)\right]\times\prod_{i\in I\setminus J}\Omega_i.\end{aligned}$$

Since each g_i is continuous, then each $g_i^{-1}\left(C_i\right)$ is closed. It follows that $g^{-1}\left(\prod_{i\in I}C_i\right)$ is indeed closed, as required.

To show that g is closed, fix closed sets F_i in Ω_i, where $F_i=\Omega_i$ for all but finitely many i. Again, these sets form a basis for $\prod_{i\in I}\Omega_i$ in the product topology, so that it suffices to show that $g\left(\prod_{i\in I}F_i\right)$ is closed. Let J be the subset of i with $F_i\neq\Omega_i$. Then

$$\begin{aligned}g\left(\textstyle\prod_{i\in I}F_i\right) &= \textstyle\prod_{i\in J}g_i\left(F_i\right)\times\prod_{i\in I\setminus J}g_i\left(F_i\right)\\ &= \textstyle\prod_{i\in J}g_i\left(F_i\right)\times\prod_{i\in I\setminus J}f_i\left(\Omega_i\right),\end{aligned}$$

where the last line follows from the fact that each g_i is surjective. Since each $g_i\left(F_i\right)$ is closed, it follows that $g\left(\prod_{i\in I}F_i\right)$ is closed. ■

Proof of Lemma 3.2. By induction on m. For $m=1$, the result is immediate since ζ_1^c and η_1^c are the identity maps. Assume that the result holds for m. We will show that it also holds for $m+1$. By the induction hypothesis, ζ_1^c and η_m^d are embeddings. Since ζ_{m+1}^c is the product of ζ_1^c and η_m^d, Lemma S6 in Appendix A gives that ζ_{m+1}^c is an embedding. Similarly, η_{m+1}^c is the product of η^c_m and $\underline{\zeta}^c_{m+1}$. The former is an embedding by the induction hypothesis. Moreover, the induction hypothesis gives that ζ^c_{m+1} is an embedding so that $\underline{\zeta}^c_{m+1}$ is an embedding (Kechris [2, 1995; Exercise 17.28]). Again, by Lemma S6, η^c_{m+1} is an embedding. ■

Proof of Lemma 3.3. First, note that η^c is injective since each of the maps η^c_m is injective: Fixing $(\mu_1, \mu_2, \ldots) \neq (\varpi_1, \varpi_2, \ldots)$ in H^c, we can find some initial segment of these sequences that are distinct, i.e., some m with $(\mu_1, \ldots, \mu_m) \neq (\varpi_1, \ldots, \varpi_m)$. Then, by Lemma 3.2, $\eta^c_m(\mu_1, \ldots, \mu_m) \neq \eta^c_m(\varpi_1, \ldots, \varpi_m)$ so that $\eta^c(\mu_1, \mu_2, \ldots) \neq \eta^c(\varpi_1, \varpi_2, \ldots)$.

Now, we turn to showing the continuity of η^c. For each m, fix closed sets C_m in $\mathcal{P}(Z^c_m)$ with $C_m = \mathcal{P}(Z^c_m)$ for all but finitely many m. Since these sets form a basis for the product topology, it suffices to show that $(\eta^c)^{-1}(\prod_m C_m)$ is closed. Note, there is some M such that

$$(\eta^c)^{-1}\left(\prod_{m=1}^{\infty} C_m\right) = \left[(\eta^c_M)^{-1}\left(\prod_{m=1}^{M} C_n\right) \times \prod_{m=M+1}^{\infty} \mathcal{P}(Z^c_m)\right] \cap H^c.$$

By Lemma 3.2, $(\eta^c_m)^{-1}(\prod_{n=1}^{M} C_n)$ is closed, and by Lemma A3, H^c is closed. It follows that $(\eta^c)^{-1}(\prod_{m=1}^{\infty} C_m)$ is closed.

Next, we show that η^c is a closed map. For this, fix closed sets F_m in $\mathcal{P}(X^c_m)$ with $F_m = \mathcal{P}(X^c_m)$ for all but finitely many m. Since these sets form the basis for the product topology for $\prod_{m=1}^{\infty} \mathcal{P}(X^c_m)$, it suffices to show that $\eta^c(H^c \cap (\prod_{m=1}^{\infty} F_m))$ is closed. Note that there exists some M with

$$\eta^c\left(H^c \cap \prod_{m=1}^{\infty} F_m\right) = \left[\eta^c_M\left(H^c_M \cap \prod_{m=1}^{M} F_m\right) \times \prod_{m=M+1}^{\infty} \mathcal{P}(X^c_m)\right] \cap H^c.$$

By Lemma A2, H^c_M is closed. So, again using Lemmas 3.2 and A3, $\eta^c_M(H^c \cap \prod_{m=1}^{\infty} F_m)$ must be closed, as required. ■

Proof of Lemma 3.5. It suffices to show that, for each m, $\delta^c_{m+1}(t^c) = \delta^c_m(t^c)$. If so, a standard inductive argument completes the proof. Fix an event E in Z^c_m and notice that

$$\begin{aligned} \left(\rho^c_{m+1}\right)^{-1}\left(E \times \mathcal{P}\left(Z^d_m\right)\right) &= [(\rho^c_m)^{-1}(E)] \cap [X^c \times (\delta^d_m)^{-1}\left(\mathcal{P}\left(Z^d_m\right)\right)] \\ &= (\rho^c_m)^{-1}(E). \end{aligned}$$

From this,

$$\begin{aligned} \delta^c_{m+1}(t^c)\left(E \times \mathcal{P}\left(Z^d_m\right)\right) &= \beta^c(t^c)(\left(\rho^c_{m+1}\right)^{-1}(E \times \mathcal{P}(Z^d_m))) \\ &= \beta^c(t^c)((\rho^c_m)^{-1}(E)) \\ &= \delta^c_m(t^c)(E), \end{aligned}$$

as required. ■

Proof of Lemma 3.8. Fix some $n \leq m$ and some event G in Z^c_n. Then $\underline{\zeta}^c_n(\mu_n)(G) = \mu_n((\zeta^c_n)^{-1}(G))$. We know that $\text{marg}_{X^c_n} \mu_{m+1} = \mu_n$, so that

$$\begin{aligned}
\mu_n((\zeta_n^c)^{-1}(G)) &= \mu_{m+1}(\{(\nu_1,\ldots,\nu_{m+1}) \in H_{m+1}^c : (\nu_1,\ldots,\nu_n) \in (\zeta_n^c)^{-1}(G)\}) \\
&= \mu_{m+1}((\zeta_{m+1}^c)^{-1}(G \times \textstyle\prod_{k=n+1}^{m+1} \mathcal{P}(Z_k^d))) \\
&= \underline{\zeta}_{m+1}^c(\mu_{m+1})(G \times \textstyle\prod_{k=n+1}^{m+1} \mathcal{P}(Z_k^d)) \\
&= \operatorname{marg}_{Z_n^c} \underline{\zeta}_{m+1}^c \left(\mu_{m+1}\right)(G)\,,
\end{aligned}$$

as required. ■

References

[1] Aliprantis, C., and K. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, Springer, 1999.

[2] Kechris, A., *Classical Descriptive Set Theory*, Springer-Verlag, 1995.

[3] Munkres, J., *Topology: A First Course*, Prentice-Hall, 1975.