Using Trajectory Measurements to Estimate the Region of Attraction of Nonlinear Systems

Brendon K. Colbert[1] and Matthew M. Peet[2]

***Abstract*— We propose a method to estimate the region of attraction of a nonlinear ODE based only on measurements of the trajectory - implying that the nonlinear vector field need not be known a priori. This method is based on using trajectory data to determine values of a form of converse Lyapunov function at a finite number of points in the state-space. Least absolute deviations is then used to fit this data to a Sum-of-Squares polynomial whose level sets then become estimates for the region of attraction. This learned Lyapunov function can then be used to predict whether newly generated initial conditions lie in the region of attraction. Extensive numerical testing is used to show that the method correctly predicts whether a new initial condition is within the region of attraction of the nonlinear ODE on over 95% of a generated set of test data.**

I. INTRODUCTION

Finding the region of attraction of nonlinear Ordinary Differential Equations (ODEs) is well-studied and important problem. For instance, estimates of the region of attraction have been used in cases such as verifying and validating flight control, [6], and analyzing cancer dormancy equilibria as in [16]. Unfortunately, in many real-world cases the vector field defining the ODE may not be known a priori. In such cases, there are no currently available methods for estimating the region of attraction.

In this paper then we consider systems of the form

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is the vector field and $x_0 \in \mathbb{R}^n$ is the initial condition. We assume the vector field is unknown, but trajectory data is available. Specifically, define $g(x_0, t)$ to be the solution map of Eq. (1), where $g(x, 0) = x(0)$ and $\frac{d}{dt}g(x,t) = f(g(x,t))$ for all $x \in \mathbb{R}^n$ and $t \geq 0$. Then we assume trajectory data is available in the form of $g(x_i, k\Delta t)$ for $k = 1, \ldots, m$ and $i = 1, \ldots, m$. The question is then how to use this data to estimate the region of attraction.

Current methods for estimating the region of attraction can be separated as either Lyapunov based or non-Lyapunov based. Many of the non-Lyapunov based methods involve numerically integrating the vector field of the ODE (and therefore require a priori knowledge of the function $f(x)$). One such numerical method, [7], involves identifying the equilibrium points whose unstable manifolds contains initial conditions that approach the equilibrium point of interest, which is usually performed by integrating the vector field. The union of these manifolds are then within the region of attraction. Other methods involve numerically integrating the vector field in the forward and reverse direction and observing the trajectories of a set of initial conditions as in [9].

Lyapunov based methods estimate the region of attraction on a compact set $X \in \mathbb{R}^n$ by searching for a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$, and a maximum scalar c to generate the set $D := \{\ x \mid V(x) \leq c,\ x \neq 0\}$ such that the time derivative of the Lyapunov function $\dot{V}(x) < 0$ and $V(x) > 0$ for all $x \in D$. There are no constraints on the function $V(x)$ except that it must be positive on the set D, for instance we see logarithmic Lyapunov functions in [1].

If we consider polynomial Lyapunov functions then the search for Lyapunov proofs of stability can be cast as a semi-definite programming problem using the Sum-of-Squares (SOS) technique. The toolbox SOSTools [18], provides a general purpose sum-of-squares programming solver that solves these types of problems using Semi-Definite Programming (SDP) solvers such as SeDuMi [19]. Note, however, that even when the ODE of a system is known, $\dot{x} = f(x)$, and the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is polynomial, methods for finding the region of attraction involve a bilinear SOS optimization problem that can be, at least approximately, solved using bisection [13] or genetic algorithms [11]. SDP techniques have also been used to find Lyapunov functions as a certificate of stability such as in [5] and [12] for switched and hybrid systems.

Perhaps most relevant to this paper, in the case where an ODE and Lyapunov function are known, but the ODE does not capture all the dynamics of the system, [4] establishes a data-based approach to estimate these unmodeled dynamics and return an estimate of the region of attraction for the system.

Regarding the problem at hand, we need to determine a method that depends on given trajectories as opposed to the nonlinear ODE of a system. To do so we rely on the set of converse Lyapunov theorems [10], which guarantee the existence of Lyapunov functions for establishing stability on the region of attraction. In this paper we consider such a converse Lyapunov function,

$$V(x) = \int_0^\infty \|g(x,t)\|^2 dt. \tag{2}$$

Since we do not have the solution map, $g(x, t)$, for the given system we can not determine $V(x)$ analytically. However, by observing and integrating trajectories $g(x_i, t)$ over possibly multiple initial conditions x_i, we may determine $y_i = V(x_i)$

*This work was not supported by any organization

[1]Brendon K. Colbert is with the Department of Mechanical Engineering, Arizona State University, Tempe, AZ, 85298 US brendon.colbert@asu.edu

[2] Matthew M. Peet is with Faculty of Mechanical Engineering, Arizona State University Tempe, AZ, 85298 US mpeet@asu.edu

for each of the initial conditions and then search for a function, $V^*(x)$ that minimizes the sum of the errors $|V^*(x_i) - y_i|$. Specifically this is the least absolute deviations problem

$$\min_{h\in H} \quad \sum_{i=1}^{m} |h(x_i) - y_i|\,, \tag{3}$$

where we will define H to be the convex cone of sum-of-squares polynomials, $x_i \in \mathbb{R}^n$ from $i = 1, ..., m$ to be the given initial conditions and $y_i = V(x_i)$. We briefly note that there has been some work on using trajectory data to fit Lyapunov functions for purposes other that estimating the region of attraction [15], [17]. However, these results provide no labeling (i.e. the true or estimated value of $V(x_i)$ is unknown) and hence cannot be used to estimate stability regions.

Upon obtaining the optimal Lyapunov function $V^* = h \in H$ through solution of optimization problem (3), we estimate the region of attraction as a maximal level set of the Lyapunov function. That is, if we define

$$V^*_\gamma = \{\ x \mid \quad V^*(x) \le \gamma\}.$$

and $\gamma^* = \max_i V^*(x_i)$, then $V^*_{\gamma^*}$ becomes our estimate for the region of attraction.

The paper is organized as follows. Notation is introduced in Section II, and Lyapunov theory is presented in Section III where we further detail the converse Lyapunov function and how we approximate the region of attraction using the level sets. In Section IV we consider how to solve optimization problem (3), and in Section V we discuss the method for computing an estimate of the region of attraction. Finally Section VI presents numerical results of the algorithms on two different dynamical systems, including the Van Der Pol oscillator.

II. NOTATION

We use $\mathbb{S}^n$ and $\mathbb{S}^{n+}$ to denote the symmetric matrices and cone of positive semi-definite matrices of size $n \times n$ respectively. Furthermore let the function $Z_d : \mathbb{R}^n \to \mathbb{R}^q$ denote the vector of monomials of degree d or less, where $q = \binom{n+d}{n}$. Finally, we denote the ring of multivariate polynomials with real coefficients as $\mathbb{R}[x]$, and the cone of SOS polynomials as Σ_s. We say a function V is positive definite if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$.

III. LYAPUNOV THEORY

First recall that we are considering nonlinear ordinary differential equations of the form,

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$. We assume, for simplicity, that a solution map, $g(x, t)$, exists where $\partial_t g(x, t) = f(g(x, t))$ and $g(x, 0) = x$.

We will assume without loss of generality that $f(0) = 0$, and thus $x = 0$ is an equilibrium point of the nonlinear ODE.

Definition 1: Given a nonlinear ODE, $\dot{x} = f(x)$, the point $x = 0$ is asymptotically stable on the set X if,

$$\lim_{t\to\infty} g(x, t) = 0, \quad \forall x \in X \tag{4}$$

where $\partial_t g(x, t) = f(g(x, t))$ and $g(x, 0) = x$.

The purpose of this paper then is to use trajectory data to determine the largest set on which a given nonlinear ODE is asymptotically stable. We defining this set, S, as the region of attraction via

$$S := \{x \in \mathbb{R}^n \quad | \lim_{t\to\infty} g(x, t) = 0\}.$$

To find estimates of S we use Lyapunov functions. Lyapunov functions are globally nonnegative functions, $V : \mathbb{R}^n \to \mathbb{R}$ that can be used to prove asymptotic stability on a set X.

Specifically, if $X \subset \mathbb{R}^n$ is a compact set, and $V : X \to \mathbb{R}$ is a continuously differentiable function such that

$$\begin{aligned} V(0) =&0 \\ V(x) >&0 \quad \text{for } x \in X, x \neq 0 \\ \nabla V(x)^T f(x) <&0 \quad \text{for } x \in X, x \neq 0 \end{aligned}$$

then the nonlinear ODE $f(x)$ is asymptotically stable on any sublevel set of $V(x) \subset X$, where a sublevel set of $V(x)$ is defined as

$$V_\gamma = \{\ x \in \mathbb{R}^n \mid \quad V(x) \le \gamma\}.$$

To show that Lyapunov functions may be used to precisely estimate the region of attraction, S, we quote the following converse Lyapunov result [14].

Theorem 2: For a nonlinear ODE $\dot{x} = f(x)$, where $f : X \to \mathbb{R}^n$ is locally Lipschitz and $X \subset S$, where S is the region of attraction, then there is a continuous positive definite function $W(x)$ such that

$$\begin{aligned} V(x) = \int_0^\infty \|g(x,t)\|^2 dt \ge 0, \qquad V(0) = 0 \\ \nabla V f(x) \le -W(x), \quad \forall\ x \in S \end{aligned}$$

where $\partial_t g(x, t) = f(g(x, t))$ and $g(x, 0) = x$ and for any $c > 0$, the set $\{V(x) \le c\}$ is a compact subset of S.

This Lyapunov function has the important property that $S = \lim_{c\to\infty} V_c$.

In practice, of course, we do not have a closed form solution of this converse Lyapunov function. However, because the value of the converse Lyapunov function at point, x, is defined as the integral of the forward-time trajectory from initial condition x, we may estimate the value of this converse function at a point x_i by observing the trajectory with initial condition x_i and integrating $g(x_i, t)$ to obtain $V(x_i)$. If the trajectory is stable, that is $\lim_{t\to\infty} g(x_i, t) = 0$, then $V(x_i)$ will be finite and in this way, it is possible to use almost any converse Lyapunov form to create labeled data from stable trajectories for the purpose of estimating the converse function.

IV. FITTING DATA TO SUM-OF-SQUARES POLYNOMIALS USING LEAST ABSOLUTE DEVIATIONS

We first assume that we have a set of inputs $x_i \in \mathbb{R}^n$ for $i = 1, ..., m$ as well as corresponding values, $y_i \in \mathbb{R}^+$.

The problem we are interested in solving then is how to find an optimal Lyapunov function, say $V : \mathbb{R}^n \to \mathbb{R}^+$ that is constrained to be globally positive and that maps from the inputs x_i to y_i most accurately.

In this section, however, we will consider the more general problem of how we may fit a function, say $h : \mathbb{R}^n \to \mathbb{R}^+$ that is guaranteed to be globally positive and that maps from a set of inputs x_i to y_i with minimal error. Perhaps the simplest method of function fitting is to minimize some variation on $\sum_i \|h(x_i) - y_i\|$. Methods of this form include least squares where $\|h(x_i) - y_i\| = (h(x_i) - y_i)^2$. However, because the objective in least squares is nonlinear, we select a computationally cheaper approach and consider the closely related problem of least absolute deviations, defined as

$$\min_{h \in H} \sum_{i=1}^{m} |h(x_i) - y_i|, \tag{5}$$

where H is a set of functions that we are searching over. While the objective function is still not linear, we may define a dummy variable $\gamma \in \mathbb{R}^m$ as well as $2m$ constraints to obtain the following optimization problem,

$$\begin{aligned} \min_{h \in H, \gamma \in \mathbb{R}^m} \quad & \sum_{i=1}^{m} \gamma_i \\ \text{such that:} \quad & h(x_i) - y_i \geq \gamma_i \\ & y_i - h(x_i) \geq \gamma_i, \end{aligned} \tag{6}$$

which is equivalent to optimization problem (5).

Now that the objective of the optimization problem is linear we must decide on the set of admissible functions, H. Because we are interested in finding Lyapunov functions, and because Lyapunov functions must be globally non-negative, we will choose H to be the set of SOS polynomials.

Formally we may define the SOS polynomials as the set,

$$\Sigma_S := \{\ f \mid f(x) = \sum_{i=1}^{n} p_i(x)^2,\ p_i(x) \in \mathbb{R}[x],\ n \in \mathbb{N}\}$$

which are clearly nonnegative for all $x \in \mathbb{R}^n$. A given function, of degree $2d$, is $p \in \Sigma_S$ if and only if there exists a matrix $P \in \mathbb{S}^{q+}$ such that

$$p(x) = Z_d(x)^T P Z_d(x),$$

where $Z_d(x) \in \mathbb{R}^q$ is the vector of monomials of degree d or less and $q = \binom{n+d}{n}$.

The constraints that $P \in \mathbb{S}^{q+}$ and $p(x) = Z_d(x)^T P Z_d(x)$ can be enforced using semi-definite programming (SDP). If we substitute the parametrization $h(x) = Z_d^T(x_i) P Z_d(x_i)$ in optimization problem (6), we have

$$\begin{aligned} \min_{P \in \mathbb{R}^{q \times q}, \gamma \in \mathbb{R}^m} \quad & \sum_{i=1}^{m} \gamma_i \\ \text{such that:} \quad & Z_d^T(x_i) P Z_d(x_i) - y_i \geq \gamma_i \\ & y_i - Z_d^T(x_i) P Z_d(x_i) \geq \gamma_i, P \succeq 0. \end{aligned} \tag{7}$$

Since the value of $Z_d^T(x_i) P Z_d(x_i)$ is linear with respect to the elements of P, this optimization problem then has $2m$ linear constraints, one semi-definite constraint of size $q \times q$ and a linear objective function. This problem can then be efficiently solved as a semi-definite program using interior point methods such as those in [2] and a suitable solver such as mosek [3] or SeDuMi [19].

We will define the solution to optimization problem (7) as P^* and can recover the optimal function as, $h^*(x) = Z_d(x)^T P^* Z_d(x)$. We will next define a method to use this optimal function to determine an estimate of the region of attraction.

V. OPTIMAL APPROXIMATIONS OF A CONVERSE LYAPUNOV FUNCTION

In the previous section we defined an optimization problem which determines the optimal map from a set of inputs x_i to a set of outputs y_i with the constraint that the image of the map be nonnegative. In this section, we show how to use trajectory data to associate the inputs x_i with labels y_i, where $y_i = V(x)$, where V is the converse Lyapunov form obtained in Equation (2). Furthermore, given the solution to the resulting Optimization Problem (7), we show that the result can be combined with the trajectory data to obtain estimates of the region of attraction.

A. Determining the Value of a Converse Lyapunov Function

In this subsection, we use trajectory data to define inputs of the form x_i and associated labels of the form $y_i = \log(1 + V(x_i))$ where V if the converse Lyapunov function defined in Equation (2). If we let $g(x,t)$ be the solution map to the nonlinear ODE, then we define our trajectory data to be of the form of vectors $a(i,j) = g(x_i, j\Delta t)$ for $j = 1, ..., K$ where Δt is the measurement time-step and x_i are the initial conditions used to generate the trajectories. Then

$$V(x_i) = \int_0^{K\Delta t} \|g(x,t)\|^2 dt + V(a(i,K)).$$

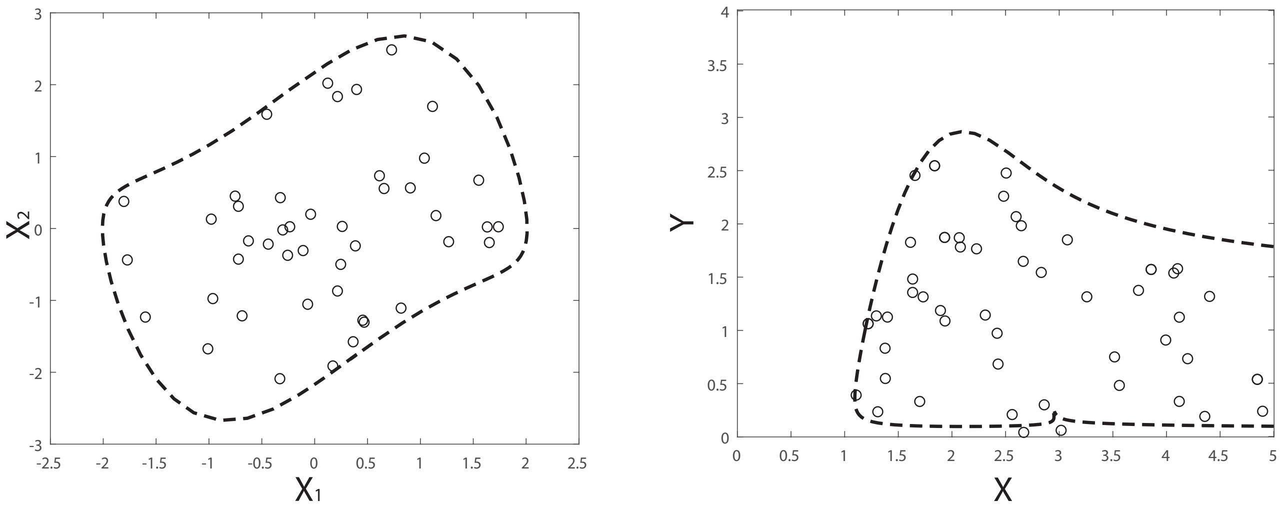
If we take K sufficiently large, we may assume $a(i,K) \approx 0$. If we likewise assume Δt is small, then we make the approximation

$$V(x_i) = \int_0^{K\Delta t} \|g(x,t)\|^2 dt \approx \sum_{j=0}^{K} \|a(i,j)\|^2 \Delta t. \tag{8}$$

In practice, of course, we use a trapezoidal approximation of this integral. Ideally Δt is constant, however, the trapezoidal approximation of the integral does not require that the trajectory be measured on a specific time step. Thus we may still approximate the converse Lyapunov function in cases where measurements are taken in irregular time steps.

We conclude that a given set of trajectories at K time instants associated with L initial conditions gives us KL values of $V(x)$.

As mentioned previously in this subsection, however, we do not use $V(x_i)$ as our label. This is because $V(x)$ grows very quickly as x approaches the edge of the region of attraction and decreases quickly near the origin, resulting in several orders of magnitude variation. This variation causes performance issues when solving Optimization Problem (7)



(a) The initial condition data (black circles) used in optimization problem 7 for the Van Der Pol Oscillator. The area contained between the black dotted line is within the region of attraction of the system.

(b) The initial condition data (black circles) used in optimization problem 7 for the Predator-Prey model. The area contained under the black dotted line is within the region of attraction of the system.

Fig. 1: Subfigures (a) and (b) show the initial conditions used to generate the data for optimization problem (7).

- points of a smaller magnitude have less influence on the value of the optimal function $V^*(x)$. To resolve this issue we instead use data labels $y_i = log(1 + V(x_i))$ as data for the optimization algorithm where the values of $V(x_i)$ are obtained from Equation (8). This means, however, that the actual output from the optimization solver is

$$h^*(x) = Z_d^T(x) P^* Z_d(x) \cong \log(1 + V(x))$$

from whence we may obtain our estimate of the converse function as

$$V^*(x) = 10^{h^*(x)} - 1.$$

Note that when $h^*(x) > 0$ if and only if $V(x) > 0$ and $h^*(x) = 0$ if and only if $V(x) = 0$. Furthermore, $\dot{V}(x(t)) \leq 0$ implies $\dot{h}^*(x(t)) \leq 0$ and hence $h^*(x)$ is fitting to a valid Lyapunov function for the system - albeit not the original converse from Section 2.

B. Estimating the Region of Attraction

Having shown how trajectory data can be used to provide training data for finding a Lyapunov function, we now discuss how to use that function to estimate the region of attraction. We denote this fitted Lyapunov function as $V^*(x) = 10^{Z_d^T(x) P^* Z_d(x)} - 1$.

Recall that in section III we discussed that the level set of the converse Lyapunov function,

$$V_\gamma := \{x \in \mathbb{R}^n \quad | \quad V(x) \leq \gamma\}$$

can be used as an estimate of the region of attraction, and that as the value of γ increases, this estimate becomes more accurate. Therefore, if one had the converse Lyapunov function (2), then by choosing a suitably large γ one could estimate the region of attraction arbitrarily well. In this case however we have an estimate, $V^*(x)$, of the Lyapunov function that is optimal with respect to the trajectory data we are given. In areas where we do not have trajectory information $V^*(x)$ is not likely to be accurate, so we must consider a metric with which to select γ.

TABLE I: Test set accuracy of the SOS optimal function on the Van Der Pol Oscillator and the Predator-Prey model data. Accuracy of the Lyapunov function is defined as the sum of the absolute error of the function for each test point divided by the total number of test points.

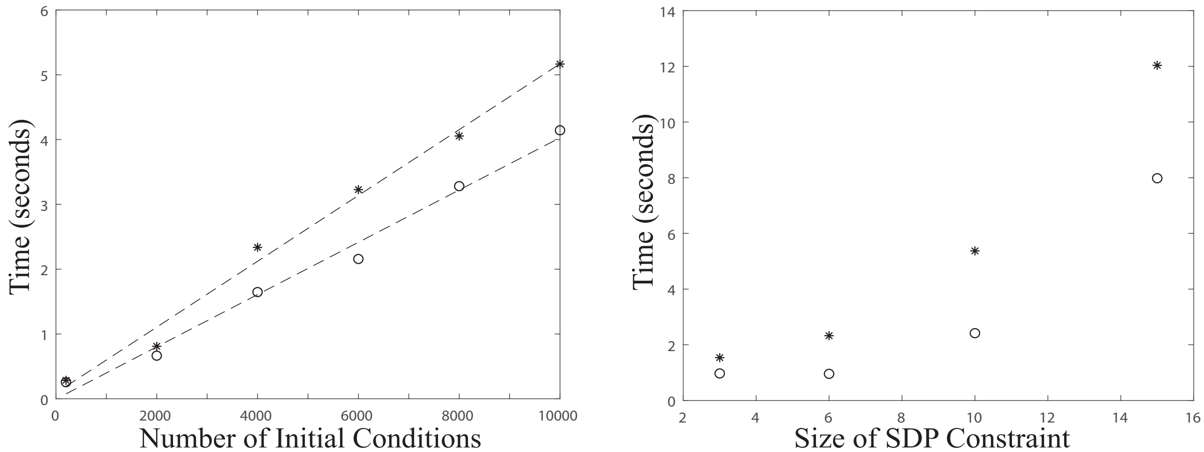
ODE	d = 2	d = 4	d = 6
Van Der Pol	0.3828	0.3942	0.1708
Predator-Prey	0.6043	0.2850	0.1546

Consider the largest value of the converse Lyapunov function (2) of all the trajectories used to find the optimal $V^*(x)$, which we will denote as $\gamma^* = \max_j \{V^*(x_j)\}$, where $j = 1, ..., m$ and x_j are the initial conditions of the given trajectories. Then we will only consider the level set of our optimal function, $V^*(x)$, that is less than or equal to γ^* since this is the smallest value of the actual converse Lyapunov function (2) that should contain all of the trajectory measurements.

However, it is possible that γ^* may be too large of a value and the estimate of the region of attraction may contain points outside of the true region of attraction S. We will then consider a factor of safety, $0 < \eta \leq 1$, to define a smaller estimate of the region of attraction. We then have that our estimate for the region of attraction is

$$E_{\eta\gamma^*} := \{\ x\ |V^*(x) \leq \eta\gamma^*\},$$

where values of η that are closer to zero result in a smaller, more conservative estimate of the region of attraction when compared to values of η that are closer to one.



(a) The computation time of Optimization Problem (7) versus the number of data points. The black circles indicate a 2nd order polynomial was used and the black stars are 4th order polynomials. The black dashed line is the line of best fit through these points.

(b) The computation time of Optimization Problem (7) versus the size of the semi-definite matrix, $q = \binom{n+d}{n}$. The black circles are indicate the problem was optimized with 2000 data points and the black stars indicate the problem was optimized with 4000 data points.

Fig. 2: Subfigures (a) and (b) plot the computation time of Optimization Problem (7) with respect to the number of data points (a) and the size of the semi-definite matrix (b).

VI. NUMERICAL TESTS

Here we have results from a number of numerical tests. In all cases, our data set consists of trajectories generated from $L = 50$ different initial conditions taken from within the region of attraction, between a radius of 1 and 4 from the equilibrium point $x = 0$ and simulate the trajectory of the nonlinear ODE for 10 seconds with $\Delta t = .1$ and $K = 100$, although we only use the $j = 1$ through $j = 4$ time-steps per trajectory for data generation, resulting in 200 data points of the form x_i, y_i. In addition, we add normally distributed noise scaled as 10^{-2} where the labels are typically in the interval $y_i \in [3, 13]$. Fig. 1 shows the initial conditions used to generate the data for optimization problem (7).

A. Numerical Results of the Optimization Problem

We first consider how increasing the degree of the SOS polynomial from 2 to 4 to 6 improves the fit between $V^*(x)$ and $\log(1 + V(x))$ for two nonlinear ODEs. To evaluate the accuracy of the fit, we created a second test set of trajectory data with 500 initial conditions x_i evenly spread in the region of attraction and calculated the value of the converse Lyapunov function at each point.

Our first test system is the Van der Pol oscillator in reverse time, defined as

$$\begin{aligned} \dot{x_1} &= -x_2 \\ \dot{x_2} &= x_1 + x_2(x_1^2 - 1). \end{aligned} \tag{9}$$

The second test system is a biological model of predator-prey dynamics as described in [8],

$$\begin{aligned} \dot{x} &= x(-(x-\alpha)(x-\beta) - \gamma y) \\ \ddot{y} &= y(-c + x), \end{aligned} \tag{10}$$

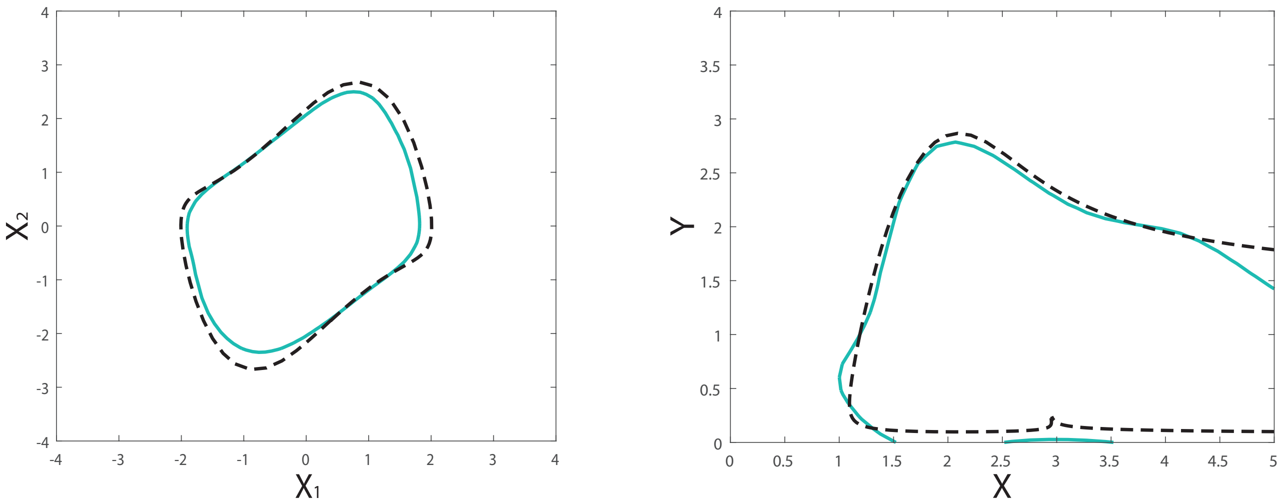
where $\alpha = 1, \beta = 3, \gamma = 0.5$ and $c = 2.1$. Here α represents the minimum density for successful mating, and β represents the asymptotic carrying capacity. We are interested then in the region of attraction of the point $x = 2.1$, and $y = 1.98$, which is a locally asymptotically stable equilibrium point. The region of attraction of this point then is the region over which the predator-prey system will asymptotically converge to a non-zero, desirable, equilibrium point.

For both examples, we calculate the average of $|V^*(x_i) - V(x_i)|$ for the 500 x_i and these averages can be found in Table I for degrees 2, 4, and 6. We note that the accuracy increases with respect to the test set as the degree of the polynomial increases. Since the polynomial function was not optimized with respect to the data generated for this test set, the accuracy on the test set is a measure of how well the polynomial function approximates values of the converse Lyapunov function that were not used in the optimization problem.

In Fig. 2 we plot the computation time for Optimization Problem (7) versus: the number of data points in Subfigure (a); and versus the length of the semi-definite matrix $P \in \mathbb{R}^{q \times q}$ where $q = \binom{n+d}{n}$ in Subfigure (b). The best linear least squares fit to the complexity data in Subfigure (a) had a slope of .004 and .005 for the degree two polynomial and the degree four polynomial respectively. Based on this numerical data the complexity of optimization problem (7) seems to scale linearly with respect to m, the number of data points used in the optimization problem.

B. Numerical Results for Estimating the Region of Attraction

We will again consider the same two ODE's discussed in the prior section, and the same optimal function $V^*(x)$. Given a new set of initial conditions, which we will call the test set of data, we wish to determine the number of initial conditions correctly identified as being in the region of attraction, falsely labeled as being in the region of attraction (false positives)



(a) The area contained within the black dotted line is within the region of attraction of the Van Der Pol Oscillator and the area contained within the blue line is the estimate of the region of attraction defined as $E_{\eta\gamma^*}$ where $\eta = 1$.

(b) The area contained within the black dotted line is within the region of attraction of the predator-prey model and the area contained within the blue line is the estimate of the region of attraction defined as $E_{\eta\gamma^*}$ where $\eta = 1$.

Fig. 3: Subfigures (a) and (b) show the estimated region of attraction E_{γ^*} versus the true region of attraction identified by observing the trajectories of the system in reverse.

TABLE II: The percentage of initial conditions that were correctly determined to be within the region of attraction, falsely reported to be within the region of attraction and falsely reported to be outside of the region of attraction by the optimal Lyapunov function obtained from Optimization Problem (7).

ODE	Degree	Correct	False Pos.	False Neg.
Van Der Pol	2	96.16 %	0.00 %	3.84 %
	4	95.92 %	0.00 %	4.08 %
	6	97.50 %	0.02 %	2.48 %
Predator-Prey	2	61.22 %	35.22 %	3.56 %
	4	93.89 %	1.89 %	4.22 %
	6	96.11 %	2.11 %	1.78 %

and those that are falsely labeled as not being in the region of attraction (false negatives). In Table II we examine the effect of the polynomial degree on the accuracy of the region of attraction estimate and in Table III we examine the effect of changing the selected value of γ.

First consider again the reverse time Van Der Pol oscillator (9). Using the 6th degree sum-of-squares polynomial function, $V^*(x)$ we will obtain an estimate of the region of attraction for the Van der Pol oscillator. In Fig. 3 (a) we see a graphical representation of the true region of attraction of the system S and the estimated region of attraction for the 6th degree polynomial, $E_{\eta\gamma^*}$ for $\eta = 1$.

In Table II we have that increasing the degree of the polynomial function increases the accuracy of the estimated region of attraction. The 6th degree polynomial correctly identified 97.50% of the test set as being within the region of attraction with a 2.48% false negative rate and a 0.02% false positive rate, outperforming the lower degree polynomials in all categories except the false positive rate.

Next we determined the accuracy of the method as we decrease the value of η to return a smaller estimate of the region of attraction. In Table III we see that decreasing η decreases the percent of test data correctly categorized, but it decreases the number of false positives. We see then that decreasing the value of γ shrinks the estimate of the region of attraction, making it less accurate overall, but reducing the chances of reporting a false positive result.

We next consider again the biological model of predator-prey dynamics as described in [8], Problem (10). We again use the 6th degree optimal SOS function $V^*(x)$ optimized on the 50 initial conditions in subsection (a). In Fig. 3 (b) we see a graphical representation of the true region of attraction of the system S and the estimated region of attraction denoted $E_{\eta\gamma^*}$ for $\eta = 1$.

In Table II we have have that increasing the degree of the polynomial function increases the accuracy of the estimated region of attraction. Since the predator-prey model has a more complicated region of attraction the 2nd degree polynomial is insufficient to capture this region. In fact we see an increase in accuracy of over 30% from the 2nd degree to the 4th degree sum-of-squares polynomial. The 6th degree sum-of-squares polynomial correctly identified 96.11% of the test set as being within the region of attraction with a 2.1% false positive rate and a 1.78% false negative rate.

Next we determined the accuracy of the method as we decrease the value of η for determining the level set $E_{\eta\gamma^*}$. In Table III we see that decreasing η again decreases the percent of test data correctly categorized, but in this case it causes a significant decrease in the number of false positives. In fact when decreasing η from 1 to 0.8 we have that the false positive rate drops from 2.11% to 0.00%.

In cases where test data is available, one may find the percentage of false positive and false negative values and use these as a metric for selecting η. For instance if a more

TABLE III: The percentage of initial conditions that were correctly determined to be within the region of attraction, falsely reported to be within the region of attraction and falsely reported to be outside of the region of attraction by the optimal Lyapunov function obtained from Optimization Problem (7).

ODE	η	Correct	False Pos.	False Neg.
Van Der Pol	0.6	94.64 %	0.00 %	5.36 %
	0.8	96.28 %	0.00 %	3.72 %
	1	97.50 %	0.02 %	2.48 %
Predator-Prey	0.8	92.89 %	0.00 %	7.11 %
	0.9	95.89 %	0.67 %	3.44 %
	1	96.11 %	2.11 %	1.78 %

conservative estimate of the region of attraction is needed, the value of η can be selected by choosing the largest value of η which has no false positive results on the test set.

VII. CONCLUSIONS

In this paper we have proposed a method for estimating the region of attraction of a nonlinear ODE given only data on the trajectory of the system over a finite set of initial conditions. This method is therefore independent of any knowledge of the vector field, and the region of attraction can therefore be predicted without requiring the system dynamics to be identified. We ran numerical tests on systems with 2 states, but note that this approach can be easily applied to systems with more states. However, due to the complexity of semi-definite programming, the computational expense for systems with greater than 10 states will likely be large.

Our approach is based on using the trajectory data to estimate the value of a particular form of converse Lyapunov function $V(x_i)$ at discrete points x_i in the state-space. The points x_i are then used as inputs and associated with labels $y_i = V(x_i)$ which are then used to fit a function V^* which represents an estimate of this converse function. The learned function is constrained to be an SOS polynomial and the resulting optimization problem is an SDP. The level sets of this learned function are then used as estimates of the region of attraction of the system. We show that by increasing the degree of the polynomial, we obtain increasingly accurate estimates of the region of attraction as measured by numerical validation on an auxiliary set of testing data.

In cases where the nonlinear ODE governing a system is not known, there are few, if any, methods for estimating the region of attraction. This paper presents an argument, the argument being that by mapping trajectory data to a converse Lyapunov form we may obtain labels, labels which can then be used to fit a Lyapunov function, a Lyapunov function which can then be used to generate highly accurate estimates of the region of attraction. Naturally, this method can be extended to other forms of converse Lyapunov function, many of which are defined explicitly in terms of the solution map.

REFERENCES

[1] A.A. Ahmadi, M. Krstic, and P.A. Parrilo. A globally asymptotically stable polynomial vector field with no polynomial lyapunov function. In *Decision and Control and European Control Conference*, pages 7579–7580, 2011.

[2] F. Alizadeh, J.A. Haeberly, and M.L. Overton. Primal-dual interior-point methods for semidefinite programming: convergence rates, stability and numerical results. *SIAM Journal on Optimization*, 8(3):746–768, 1998.

[3] MOSEK ApS. *The MOSEK optimization toolbox for MATLAB manual. Version 7.1 (Revision 28).*, 2015.

[4] F. Berkenkamp, R. Moriconi, A.P. Schoellig, and A. Krause. Safe learning of regions of attraction for uncertain, nonlinear systems with gaussian processes. In *Proceedings on Decision and Control*, 2016.

[5] M.S. Branicky. Multiple lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):475–482, 1998.

[6] A. Chakraborty, P. Seiler, and G.J. Balas. Nonlinear region of attraction analysis for flight control verification and validation. *Control Engineering Practice*, 19(4):335–345, 2011.

[7] H-D Chiang, M.W. Hirsch, and F.F. Wu. Stability regions of nonlinear autonomous dynamical systems. *IEEE Transactions on Automatic Control*, 33(1):16–27, 1988.

[8] M. Gatto and S. Rinaldi. Stability analysis of predator-prey models via the liapunov method. *Bulletin of mathematical biology*, 39(3):339–347, 1977.

[9] R. Genesio, M. Tartaglia, and A. Vicino. On the estimation of asymptotic stability regions: State of the art and new proposals. *IEEE Transactions on Automatic Control*, 30(8):747–755, 1985.

[10] W. Hahn. The converse of the stability theorems. In *Stability of Motion*, pages 225–256. 1967.

[11] F Hamidi, H Jerbi, W Aggoune, M Djemai, and M Naceur Abdkrim. Enlarging region of attraction via LMI-based approach and Genetic Algorithm. In *Communications, Computing and Control Applications*, pages 1–6, 2011.

[12] M. Johansson and A. Rantzer. Computation of piecewise quadratic lyapunov functions for hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):555–559, 1998.

[13] M. Jones, H. Mohammadi, and M.M. Peet. Estimating the region of attraction using polynomial optimization: A converse Lyapunov result. In *Proceedings of the IEEE Conference on Decision and Control*, 2017.

[14] H.K. Khalil. Noninear systems. 2, 1996.

[15] S.M. Khansari-Zadeh and A. Billard. Learning control Lyapunov function to ensure stability of dynamical system-based robot reaching motions. *Robotics and Autonomous Systems*, 62(6):752–765, 2014.

[16] A. Merola, C. Cosentino, and F. Amato. An insight into tumor dormancy equilibrium via the analysis of its domain of attraction. *Biomedical Signal Processing and Control*, 3(3):212–219, 2008.

[17] N. Noroozi, P. Karimaghaee, F. Safaei, and H. Javadi. Generation of lyapunov functions by neural networks. In *Proceedings of the World Congress on Engineering*, volume 2008, 2008.

[18] S. Prajna, P. Antonis, and P.A. Pablo. Introducing sostools: A general purpose sum of squares programming solver. In *Proceedings of the IEEE Conference on Decision and Control*, 2002.

[19] J.F. Sturm. Using sedumi 1.02, a matlab toolbox for optimization over symmetric cones. *Optimization methods and software*, 11(1-4):625–653, 1999.