Please note that there is more than one way to answer most of these questions. The following only represents a sample solution.

Problem 1: Linz 4.1.13 and 4.2.5

4.1.13: If $L$ is a regular language, prove that $L_1 = \{uv : u \in L, |v| = 2\}$ is also regular.

Let $L_{|2|} = \{v : |v| = 2\}$. Then $L_{|2|}$ is regular since it can be denoted by the regular expression $\Sigma \Sigma$. Here if the alphabet is $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, then the regular expression $\Sigma$ is used as shorthand to denote $(\sigma_1 + \sigma_2 + \cdots + \sigma_n)$.

Now note that $L_1 = LL_{|2|}$, since $w \in L_1$ iff $w = uv$ with $u \in L$ and $|v| = 2$ iff $w = uv$ with $u \in L$ and $v \in L_{|2|}$ iff $w \in LL_{|2|}$. Since both $L$ and $L_{|2|}$ are regular and by Theorem 4.1 regular languages are closed under concatenation, we conclude the $L_1$ is regular.

4.2.5: A language is said to be a palindrome language if $L = L^R$. Find an algorithm for determining if a given regular language is a palindrome language.

On input $L$ where $L$ is a regular language:

1. Represent $L$ as a DFA, $D$.
2. Following Theorem 4.2 we construct an NFA, $\hat{N}$ that accepts $L^R$.
3. Convert $\hat{N}$ into an equivalent DFA, $\hat{D}$. So now $L(\hat{D}) = L^R$.
4. Now we can follow the algorithm in Theorem 4.7 to determine if $L = L^R$, which will give us the result of whether $L$ is a palindrome language.

Note the above algorithm will return that $L$ is a palindrome language iff $L = L^R$, which was the condition we had to check.

Problem 2: Linz 4.3.6 and 4.3.7

4.3.6: Determine whether or not the following languages are regular.

(a) $L = \{a^n b^n : n \geq 1\} \cup \{a^m b^n : n \geq 1, m \geq 1\}$.

Let $L_1 = \{a^n b^n : n \geq 1\}$ and $L_2 = \{a^n b^m : n \geq 1, m \geq 1\}$. Then $L_1 \subseteq L_2$ since $w = a^n b^n \in L_1$ implies $w \in L_2$, just taking $m = n$. Since $L_1 \subseteq L_2$, we get $L_1 \cup L_2 = L_2$. Thus, this problem is
equivalent to determining if $L_2$ is regular.

$L_2$ is regular since it can be denoted by the regular expression $aa^*bb^*$, that is at least one $a$ followed by at least one $b$. Thus, we conclude that $L = L_1 \cup L_2 = L_2$ is regular.

(b) $L = \{a^n b^n : n \geq 1\} \cup \{a^n b^{n+2} : n \geq 1\}$.

We will show $L$ is not regular by using the pumping lemma.

- For a contradiction, assume $L$ is regular.
- Since $L$ is a regular there exists a pumping length $m > 0$.
- Take $w = a^m b^m$, then $w \in L$ and $|w| > m$.
- Then from the pumping lemma there exists $x, y, z \in \Sigma^*$ such that $w = xyz$ with (1) $|xy| \leq m$ and (2) $|y| \geq 1$.
- From (1) and (2) above, $y = a^k$ for some $1 \leq k \leq m$.
- Take $i = 2$, then $xy^i z = a^{m+k} b^m \notin L$ because the number of $a$'s is greater than the number of $b$'s, so $xy^i z$ cannot be in any of the three parts of the union.
- This is a contradiction of the pumping lemma. Thus, $L$ cannot be regular.

4.3.7: Show that the language

$$L = \{a^n b^n : n \geq 0\} \cup \{a^n b^{n+1} : n \geq 0\} \cup \{a^n b^{n+2} : n \geq 0\}$$

is not regular.

We will show $L$ is not regular by using the pumping lemma.

- For a contradiction, assume $L$ is regular.
- Since $L$ is a regular there exists a pumping length $m > 0$.
- Take $w = a^m b^m$, then $w \in L$ and $|w| > m$.
- Then from the pumping lemma there exists $x, y, z \in \Sigma^*$ such that $w = xyz$ with (1) $|xy| \leq m$ and (2) $|y| \geq 1$.
- From (1) and (2) above, $y = a^k$ for some $1 \leq k \leq m$.
- Take $i = 2$, then $xy^i z = a^{m+k} b^m \notin L$ because the number of $a$'s is greater than the number of $b$'s, so $xy^i z$ cannot be in any of the three parts of the union.
- This is a contradiction of the pumping lemma. Thus, $L$ cannot be regular.
Problem 3: Linz 4.1.16

Show that the statement “If \( L_1 \) is regular and \( L_1 \cup L_2 \) is also regular, then \( L_2 \) must be regular” were true for all \( L_1 \) and \( L_2 \), then all languages would be regular.

Let \( L_1 = \Sigma^* \), then \( L_1 \) is regular (denoted by the regular expression \( \Sigma^* \), where the regular expression \( \Sigma \) is as defined in the solution of Problem 1, 4.1.13 above). Let \( L_2 \) be any language. By the definition of a language, \( L_2 \subseteq L_1 \). Therefore, \( L_1 \cup L_2 = L_1 \), and hence is regular. Thus, if the statement in the problem description were true, then \( L_2 \) must be regular. Since \( L_2 \) was any arbitrary language, we would have that all languages would be regular.

The statement in this problem is clearly false though. Let \( L_1 = \Sigma^* \), \( L_2 = \{a^n b^n : n \geq 0\} \). Then \( L_1 \) is regular, and \( L_1 \cup L_2 = L_1 \) is regular, but we know \( L_2 \) is not regular, disproving the statement.

Problem 4: Linz 4.2.14

Find an algorithm for determining whether a regular language \( L \) contains an infinite number of even-length strings.

On input \( L \) where \( L \) is a regular language:

1. Represent \( L \) as a DFA, \( D \).
2. Construct a DFA for the language \( L' = \{w : |w| \text{ is even}\} \) (\( L' \) is regular since it can be represented by the RE \( (\Sigma \Sigma)^* \)).
3. Construct a DFA for \( L'' = L \cap L' \), which is regular from Theorem 4.1 . (This is the language of all even length strings in \( L \)).
4. Now we can follow the algorithm in Theorem 4.6 to determine if \( L'' \) is infinite, which will give us the result of whether \( L \) has an infinite number of even-length strings.

Note the above algorithm will return that \( L \) is contains an infinite number of even-length strings iff \( L'' \) is infinite. This is true since \( L'' \) contains all the even-length strings in \( L \), because it is the intersection of \( L \) and the language of all even-length strings, \( L' \).

Problem 5: Linz 4.3.15 (c)-(g)

Consider the languages below. For each, make a conjecture whether or not it is regular. Then prove your conjecture.

(c) \( L = \{a^n b^l : n/l \text{ is an integer}\} \).

We will show \( L \) is not regular by using the pumping lemma.

- For a contradiction, assume \( L \) is regular.
- Since \( L \) is a regular there exists a pumping length \( m > 0 \).
- Take \( w = a^m b^{m+1} \), then \( w \in L \) \((\frac{m+1}{m+1} = 1)\) and \(|w| > m \).
– Then from the pumping lemma there exists \(x, y, z \in \Sigma^*\) such that \(w = xyz\) with (1) \(|xy| \leq m\) and (2) \(|y| \geq 1\).

– From (1) and (2) above, \(y = a^k\) for some \(1 \leq k \leq m\).

– Take \(i = 0\), then \(xy^iz = a^{m+1-k}b^{m+1} \notin L\) because \(0 < \frac{m+1-k}{m+1} < 1\), since \(0 < m+1-k < m+1\).

– This is a contradiction of the pumping lemma. Thus, \(L\) cannot be regular.

(d) \(L = \{a^nb^l : n + l \text{ is a prime number}\}\).

We will show \(L\) is not regular by using a closure property and then the pumping lemma.

– For a contradiction, assume \(L\) is regular.

– Then from Theorem 4.3 \(h(L)\) is regular for any homomorphism \(h\).

– Let \(h\) be the homomorphism defined as \(h(a) = a\) and \(h(b) = a\).

– Then \(h(L) = \{a^na^l : n + l \text{ is a prime number}\} = \{a^{n+l} : n + l \text{ is a prime number}\} = \{a^p : p \text{ is a prime number}\}\).

– We will now show \(h(L)\) cannot be regular causing the contradiction.

– Since \(h(L)\) is a regular there exists a pumping length \(m > 0\).

– Take \(w = a^p\), where \(p\) is a prime number larger than \(m\) (Such a \(p\) exists since there is an infinite number of prime numbers). Then \(w \in L\) and \(|w| > m\).

– Then from the pumping lemma there exists \(x, y, z \in \Sigma^*\) such that \(w = xyz\) with (1) \(|xy| \leq m\) and (2) \(|y| \geq 1\).

– From (1) and (2) above, \(y = a^k\) for some \(1 \leq k \leq m\).

– Take \(i = p + 1\), then \(xy^iz = a^{p+k}b^{p(k+1)} \notin L\) because there are \(p \cdot (k + 1)\) \(a\)'s, where \(p, k + 1 > 1\). Thus, the number of \(a\)'s cannot be prime.

– This is a contradiction of the pumping lemma. Thus \(h(L)\) cannot be regular, which contradicts Theorem 4.3. Thus, we conclude \(L\) cannot be regular.

(e) \(L = \{a^nb^l : n \leq l \leq 2n\}\).

We will show \(L\) is not regular by using the pumping lemma.

– For a contradiction, assume \(L\) is regular.

– Since \(L\) is a regular there exists a pumping length \(m > 0\).

– Take \(w = a^mb^m\), then \(w \in L\) and \(|w| > m\).

– Then from the pumping lemma there exists \(x, y, z \in \Sigma^*\) such that \(w = xyz\) with (1) \(|xy| \leq m\) and (2) \(|y| \geq 1\).
– From (1) and (2) above, \( y = a^k \) for some \( 1 \leq k \leq m \).

– Take \( i = 2 \), then \( xy^i z = a^{m+k} b^m \notin L \) because the number of \( a \)'s \((m+k)\) is not less than or equal to the number of \( b \)'s \((m)\) in \( xy^i z \).

– This is a contradiction of the pumping lemma. Thus, \( L \) cannot be regular.

(f) \( L = \{a^nb^l : n \geq 100, l \leq 100\} \).

\( L \) is regular. To see this, note \( L = L_1L_2 \), where \( L_1 = \{a^n : n \geq 100\} \) and \( L_2 = \{b^l : l \leq 100\} \). We will show that \( L_1 \) and \( L_2 \) are regular.

Define an NFA \( N_1 = (\{q_0, q_1, \ldots, q_{100}\}, \{a, b\}, \delta_1, q_0, \{q_{100}\}) \), where \( \delta_1(q_i, a) = \{q_{i+1}\} \) for \( 0 \leq i \leq 99 \), \( \delta_1(q_{100}, a) = \{q_{100}\} \) and \( \delta_1(q_i, b) = \{\} \) for \( 0 \leq i \leq 100 \). Then, \( L(N_1) = L_1 \) since a string is accepted by \( N_1 \) iff it has at least 100 \( a \)'s and no \( b \)'s. Since there is an NFA that accepts \( L_1 \), \( L_1 \) is regular.

Define an NFA \( N_2 = (\{q_0, q_1, \ldots, q_{100}\}, \{a, b\}, \delta_2, q_0, \{q_0, q_1, \ldots, q_{100}\}) \), where \( \delta_2(q_i, b) = \{q_{i+1}\} \) for \( 0 \leq i \leq 99 \), \( \delta_2(q_{100}, b) = \{\} \) and \( \delta_2(q_i, a) = \{\} \) for \( 0 \leq i \leq 100 \). Then, \( L(N_2) = L_2 \) since a string is accepted by \( N_2 \) iff it has at most 100 \( b \)'s and no \( a \)'s. Since there is an NFA that accepts \( L_2 \), \( L_2 \) is regular.

Since \( L_1 \) and \( L_2 \) are regular and regular languages are closed under concatenation, then \( L = L_1L_2 \) is regular.

(g) \( L = \{a^n b^l : |n - l| = 2\} \).

We will show \( L \) is not regular by using the pumping lemma.

– For a contradiction, assume \( L \) is regular.

– Since \( L \) is a regular there exists a pumping length \( m > 0 \).

– Take \( w = a^m b^{m+2} \), then \( w \in L \) and \( |w| > m \).

– Then from the pumping lemma there exists \( x, y, z \in \Sigma^* \) such that \( w = xyz \) with (1) \( |xy| \leq m \) and (2) \( |y| \geq 1 \).

– From (1) and (2) above, \( y = a^k \) for some \( 1 \leq k \leq m \).

– Take \( i = 0 \), then \( xy^i z = a^{m-k} b^{m+2} \notin L \) because \( |m - k - (m + 2)| = | - k - 2 | = k + 2 \neq 2 \).

– This is a contradiction of the pumping lemma. Thus, \( L \) cannot be regular.