Chapter 4

Oscillations in systems with many particles: standing waves
Introduction

Our discussion in the previous chapter shows that the calculation of normal modes in a vibrating system becomes more and more difficult as the number of masses increases. It appears that for real systems such as crystals, which have as many as $10^{23}$ atoms/cm$^3$, no supercomputer will ever be able to calculate the normal modes of vibration. However, we hinted at a possible solution of this problem for systems that have some type of symmetry. Crystals do display a very special type of symmetry: they have a pattern that repeats itself throughout the structure. These systems are called periodic. In this chapter, we will show that the calculation of normal modes in a periodic one-dimensional “crystal” is surprisingly simple. In fact, we will solve the problem once and forever: our solution will be valid for any number $N$ of masses. The periodicity of these systems is reflected in their normal mode solutions, which display themselves a periodicity: masses separated by a certain distance execute exactly the same motion. This distance is called the wavelength. Thus the fundamental concept of wavelength is linked to the periodic symmetry of the vibrating system.

The many-masses problem

Let us now consider a system of equal masses $m$ and equal springs of constant $K$, separated by a distance $a$, as depicted in Fig. 1

![Figure 1](image)

Figure 1 A linear chain of masses and springs.

Suppose that this system has a total of $N$ masses, where $N$ can be very large. In view of the discussion in the previous chapter, it appears that this problem would be computationally prohibitive. However, we notice that all masses satisfy the same equation:

$$m \frac{d^2x_n}{dt^2} = K(x_{n+1} - x_n) - K(x_n - x_{n-1})$$  \hspace{1cm} (1)

This equation reflects the fact that there are two forces acting on mass $n$: the force exerted by the spring that
connects it to mass $n-1$ and the force exerted by the spring that connects it to mass $n+1$. We obtain a similar equation if we stretch the chain -so that there is a tension $T$- and study its transverse (perpendicular to the chain) oscillations, as in a guitar string. In this case, the equation reads (see homework problem):

$$m \frac{d^2 y_n}{dt^2} = \frac{T}{a} (y_{n+1} - y_n) - \frac{T}{a} (y_n - y_{n-1})$$

where $a$ is the separation between masses and $y_n$ the transverse displacement of the mass. You might argue that the above equations of motion do not apply to the first and the last mass in the chain. However, you could imagine that you add two more masses at positions 0 and $N+1$. The first mass is glued to the wall on the left and doesn’t move. The other mass is glued to the wall on the right and doesn’t move either. It is clear that this does not change the physics of the problem. On the other hand, Eqs. (1) or (2) now apply to masses 1 and $N$, that is, to all real masses in our problem. Our solutions, however, must take into account the boundary conditions

$$x_0 = 0 \text{ (or } y_0 = 0) ; \quad x_{N+1} = 0 \text{ (or } y_{N+1} = 0)$$

Let us first try to find a solution to the equations of motion. We will worry later about the boundary conditions. For a normal mode of such a system, we must have, in analogy to our solution of the few-masses problem:

$$x_n = A_n \cos (\omega t + \alpha)$$

where the amplitude $A_n$ corresponds to what we called $A(1)$, $A(2)$, etc. in the previous chapter. Notice that for a normal mode all masses move with the same frequency and phase angle, as discussed earlier. Substituting into Eq. (1), we obtain

$$-m \omega^2 A_n = K(A_{n+1} - A_n) - K(A_n - A_{n-1})$$
which can be rewritten as

\[ A_{n+1} + A_{n-1} = A_n \left(2 - \frac{m\omega^2}{K}\right) \]  

(6)

A very similar equation is obtained for the transverse oscillations of the stretched spring, with \( K \) replaced by \( T/a \).

So far, we have succeeded in converting the set of \( N \) coupled differential equations of motion into a set of coupled algebraic equations for the coefficients \( A_n \). However, Eq. (6) looks very strange and we don’t know of any obvious solution. This problem could keep a mathematician busy for days, but a physicist has an extra resource: experiments!

Consider for example the transverse vibrations of a cord under tension. If you take a snapshot of the cord while it vibrates, you often see familiar patterns as in Fig. 2. These patterns look like sines and cosines. This suggests that we try

\[ A_n = A \sin kna \]  

(7)

where \( A \) is a constant and \( k \) a number that has units of inverse distance, so that it cancels the units of \( a \) inside the argument of the sine function. You should not confuse \( k \), called the \textbf{wavenumber}, with \( K \), the spring constant. Using Eq. (7), the coefficients in Eq. (6) can be written as:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Instantaneous displacement patterns for the transverse oscillations of a cord.}
\end{figure}
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\[
A_{n+1} = A \sin k(n+1)a = A \sin (kna + ka) = A [\sin (kna) \cos (ka) + \sin (ka) \cos (kna)]
\]

(8)

and

\[
A_{n-1} = A \sin k(n-1)a = A \sin (kna - ka) = A [\sin (kna) \cos (ka) - \sin (ka) \cos (kna)]
\]

(9)

Substituting in Eq. (6), we obtain

\[
2A \sin kna \cos ka = A \sin kna \left( 2 - \frac{m\omega^2}{K} \right)
\]

(10)

Eliminating the common factors, we obtain finally an expression for \(\omega^2\):

\[
\omega^2 = \frac{2K}{m} (1 - \cos ka)
\]

(11)

Using the identity \(\cos^2 \alpha = \frac{1}{2}(\cos 2\alpha + 1)\) this can be expressed as

\[
\omega = \sqrt{\frac{4K}{m} \sin \frac{ka}{2}}
\]

(12)

This means that our guess \(A_n = A \sin kna\) is indeed a solution to our problem provided Eq. (12) is satisfied. This equation gives the frequency of the normal mode corresponding to \(k\). So far, we have not restricted the possible values of \(k\) at all: any real number would do. But this would imply that there is an infinite number of normal modes, contradicting our expectation of \(N\) normal modes for \(N\) masses. As we shall see below, however, there are only \(N\) possible values of \(k\), so that the number of different normal modes is exactly \(N\), as expected.

**Standing waves**

The motion of mass \(n\) for the normal mode of frequency \(\omega\) is given by

\[
x_n = A \sin kna \cos (\omega t + \alpha)
\]

(13)

where \(k\) and \(\omega\) are related by Eq. (12). This is called a **standing wave**. If you look at a certain mass \(n\), the quantity
sin $kna$ is a constant, so that the mass has a simple harmonic motion given by $\cos (\omega t + \alpha)$, as expected for a normal mode. The motion repeats itself after a period $T = 2\pi/\omega$. On the other hand, if you freeze the time, as is the case when you take a snapshot of the system, $\cos (\omega t + \alpha)$ is a constant; if you look down the line of masses you see that their “frozen” displacements follow the trigonometric expression $\sin kna$.

![Figure 3](image_url)  
**Figure 3** The displacement pattern for the transverse oscillations of a chain of masses for one of its normal modes.

This is illustrated in Fig. 3 for the case of the transverse oscillations of a chain of masses. Notice that some masses do not move at all: these are the masses for which $\sin kna = 0$. These points are called **nodes**. Notice also a very special type of **spatial** periodicity: if, at a fixed time, you move a certain distance $\lambda$ down the line, you find another mass that is executing exactly the same motion as the mass from which you started. This distance $\lambda$ is called **wavelength**.

The wavelength $\lambda$ is related to the wavenumber $k$. Suppose that mass $n$ and mass $n + n_0$ execute the same motion. This means

$$\sin k(n + n_0)a = \sin kna$$  \hspace{1cm} (14)

This is satisfied for $kn_0a = 2\pi i$, with $i = 1, 2, 3, 4,...$ If $n_0a$ is the shortest distance between masses that execute the same motion, then by definition $\lambda = n_0a$ and $i = 1$, so that we obtain $k\lambda = 2\pi$ or

$$\lambda = \frac{2\pi}{k}$$  \hspace{1cm} (15)

Note the perfect analogy between the spacial period $\lambda$ and the temporal period $T$. While $T$ is given by $2\pi/\omega$, $\lambda$ is given by $2\pi/k$. So $T$ is to $\omega$ what $\lambda$ is to $k$. $T$ and $\omega$ refer to the...
temporal periodicity, whereas $k$ and $\lambda$ refer to the spacial periodicity. The wavelength and the period of a normal mode are not independent of each other. They are related by a so-called dispersion relation, which in the case of the chain is given by Eq. (12).

**Initial and boundary conditions**

So far we have written down the expression for a given normal mode of our chain. This expression contains 2 arbitrary constants: $A$ and $\alpha$. Since the system has $N$ masses, we need $2N$ arbitrary constants to accommodate the arbitrary initial position and velocity of our masses. In analogy with the two-mass case, we write the most general solution to the motion of our masses as a sum of normal modes

$$x_n(t) = A_1 \sin k_1 n a \cos (\omega_1 t + \alpha_1) + A_2 \sin k_2 n a \cos (\omega_2 t + \alpha_2) +$$
$$+ A_3 \sin k_3 n a \cos (\omega_3 t + \alpha_3) + \ldots$$

(16)

where we use numbers (1,2,3, ...) to label different normal modes. In the two-mass problem, we used letters ($A$ and $B$) to label the two normal modes, but the letter notation becomes cumbersome when we have to deal with many normal modes. If the number of normal modes is $N$, then we have a total of $2N$ constants $A$ and $\alpha$ exactly as needed to specify the initial position and velocity of each particle. So far, however, we have not placed any restriction on the possible values of $k$. It appears that we can find an infinite number of normal modes, which creates a big problem, because we would have too many arbitrary constants. However, not any value of $k$ is acceptable. This is because we must satisfy the boundary conditions. Let us look at the normal mode depicted in Fig. 3. Notice that the sine function which is the “envelope” of the displacements must be zero at the walls. In other words, we must have $y_0 = 0$ and $y_{N+1} = 0$ at all times. This means
The first condition is automatically satisfied because \( \sin 0 = 0 \); the second condition requires \( k(N+1)a = \pi i \), with \( i = 1, 2, 3, 4, ... \), so that the possible values of \( k \) are

\[
k_1 = \frac{\pi}{(N+1)a}, \quad k_2 = \frac{2\pi}{(N+1)a}, \quad k_3 = \frac{3\pi}{(N+1)a}, \ldots, \quad k_N = \frac{N\pi}{(N+1)a}
\]

(18)

This list gives the complete set of allowed values of \( k \), corresponding to the \( N \) normal modes of the system. It appears that one can find more normal modes by letting \( i = N + 1, N + 2, N + 3, \ldots \). However, one can show (see homework problem) that for values of \( i \) larger than \( N \) no new modes are obtained. The apparent “extra” modes are just a repetition of the ones we found for \( i \leq N \). Hence there are only \( N \) distinct normal modes of vibration for a linear chain with \( N \) masses, exactly as expected. Using \((N+1)a = L\), where \( L \) is the total length of the chain, and \( \lambda = 2\pi/k \), we can rewrite Eq. (18) in terms of the possible wavelengths of the normal modes:

\[
\lambda_1 = 2L, \quad \lambda_2 = L, \quad \lambda_3 = \frac{3}{2}L, \quad \lambda_i = \frac{2L}{i}
\]

(19)

Notice that the possible wavelengths are simple determined by the sine functions you can “fit” between 0 and \( L \) so that the sine function is zero at the two ends. You can easily convince yourself that the longest possible wavelength is \( 2L \). The displacement pattern corresponding to this mode is indicated in Fig. 2 (lower mode).

**The dispersion relation**

The relationship between wavenumber (or wavelength) and the frequency of oscillation, given by Eq. (12), is called the dispersion relation. The concept of dispersion relation will play an important role in the next few chapters. Notice that in the few-masses problems we obtained the frequencies of the different normal modes as the solutions of a
determinantal equation. In the case of the linear chain, the procedure is the following: from the boundary conditions, we figure out the possible values of \( k \) (or \( \lambda \)). Once we know the wavenumber of our normal mode, we use Eq. (12) to obtain the frequency. Notice that Eq. (12) is valid for all chains, no matter what the value of \( N \) is. If \( N \) changes, the allowed values of \( k \) will change, but Eq. (12) itself remains the same.

Another important feature of Eq. (12), plotted in Fig. 4, is its behavior for small values of \( k \). \((ka \ll 1, \text{ or } \lambda \gg a)\). In this case, by expanding the sine function to first order \((\sin x \approx x \text{ for } x \text{ small})\) we obtain a linear relationship.

**Figure 4** The dispersion relation for a linear chain of masses and springs. Solid curve: exact result. Dashed curve: linear extrapolation valid only for long wavelengths (small \( k \)).
where the factor $c = \sqrt{\frac{K}{m} a}$ has units of velocity. Eq. (20) will have important implications when we discuss traveling waves. The factor $c$ will turn out to be the velocity of propagation of the traveling wave. Notice that Eq. (20) implies that the frequency of the mode doubles if the wavelength is halved. This relationship, however, is only valid if the wavelength is sufficiently long, so that the expansion of the sine in Eq. (12) is justified.
Problems

1. Derive Eq.(2) for the transverse oscillations of a chain of equal masses connected by equal springs. Use the displacements $y_n$ perpendicular to the chain as your dynamic variables. Make the small angle approximation and neglect all terms of order higher than linear in the $y_n$’s.

2. The normal modes of a system of $N$ masses are given by $x_n = A \sin kna \cos (\omega t + \alpha)$. Show that when this expression is applied to the two-mass system and three-mass systems you obtain the normal modes derived in the previous chapter.

3. Sketch the displacement patterns for the five normal modes of a 5-beaded string attached walls at the two ends.

4. Consider a system of $N$ pendulums of length $l$ with bobs of mass $m$ coupled by springs of constant $K$ as in the figure.

\[ a) \text{ What would be the dispersion relation of this system if } g \text{ were zero?} \]

\[ b) \text{ What is the extra restoring force per unit displacement per unit mass acting on each mass if } g \text{ is not zero?} \]

\[ c) \text{ On the basis of } a) \text{ and } b) \text{ justify (or derive, if you feel like doing some extra work) the dispersion relation } \]

\[ \omega^2(k) = \frac{g}{l} + \frac{4K}{m} \sin^2 \frac{ka}{2} \]

for the system of pendulums and springs. Graph this expression.

\[ d) \text{ Why is it that the lowest possible frequency is zero for a system of only masses and springs but nonzero for the above system of pendulums and springs?} \]

5. How many masses should a system of masses and springs have so that the frequency of the second mode differs from twice the frequency of the lowest mode by less than 1%?

6. The wavenumber $k$ for the possible normal modes of an oscillating linear chain of $N$ masses is given by $k_i = \frac{\pi i}{(N + 1)a}$. We claimed in the lecture that all normal modes are obtained with $1 < i \leq N$.

\[ a) \text{ Show that for } i = N+1 \text{ the solution obtained is not a normal mode.} \]

\[ b) \text{ Show that for } i = N+2 \text{ the solution obtained is identical to the one corresponding to } i = N. \]

\[ c) \text{ Show that for } i = N+j, \text{ with } j \geq 2, \text{ the solution obtained is equivalent to the normal mode with } i = N-j+2. \]