The concept of ecological sustainability requires that at least some stocks of environmental assets be prevented from rising above or falling below certain threshold levels. This paper considers the implications of this requirement in the context of a general model of environmental control, paying particular attention to the informational and structural conditions that are necessary and sufficient to satisfy the requirement. These conditions turn out to be particularly stringent, and will not ordinarily be satisfied. The paper then considers the scope for achieving the goal of ecological sustainability when the system is not controllable through economic incentives, implying that prices are inadequate observers of environmental processes. It argues the merits of a policy of environmental stabilization based on direct observation of the physical system, in which ecological sustainability is assured through the imposition of physical restrictions on economic activity so as to avoid breaching the critical thresholds. Since the price assumptions of the model are very weak, the results are robust with respect to a range of economic theories of price.

1. INTRODUCTION

It has become natural to think of environmental management as a problem in ‘the control of resources’—to borrow the title of Dasgupta’s (1982) text. Formally, the optimal control problem involves the maximization (or minimization) of some index of performance as a function of a set of state variables and control inputs, subject to the constraint posed by the natural dynamics of those state variables. There is an obvious correspondence between this and an economic problem involving the maximization of some measure of welfare (or the minimization of some measure of social cost) through the appropriation of environmental goods and services, subject to the natural dynamics of the environment. Indeed, since the important contributions of Clark (1976) and Smith (1977) the control approach to the management of renewable environmental resources has been used with both increasing frequency and increasing effectiveness. It is now possible to point to a substantial control literature on the management of a wide range of biophysical stocks based on a well recognized set of bioeconomic models. The optimal rate of harvest of fish species or other forms of wildlife, optimal stocking densities on rangeland, optimal felling or rotation of forests, the optimal depletion or drawoff of groundwater, optimal soil conservation, and the

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2 An earlier version of this paper was prepared while the author was a visiting fellow at the Centre for Resource and Environmental Studies, Australian National University.
optimal provision of wildlife reserves have, for example, all been analysed in these terms.\textsuperscript{3} It is not, therefore, surprising that optimal control theory should provide a useful framework within which to address recent concern over the sustainability of natural resource utilization. The control approach not only enables the identification of an optimal resource management policy where the economy–environment system is controllable, it also establishes the necessary and sufficient conditions for the existence of such an optimal policy, and the scope for assuring the sustainability of resource use where those conditions are not satisfied. That is, the framework draws attention not only to what may be controlled, but also what may not be controlled, and an understanding of the limits of environmental control is a crucial part of the development of a policy for sustainable resource utilization.

This paper adapts the control framework to an analysis of the issues raised by the notion that an ecologically sustainable development strategy requires the preservation of at least some components of the stock of environmental assets—sometimes referred to as the stock of natural capital. After exploring the basis of this notion, the paper considers the structural and informational conditions required to satisfy a sustainability condition or constraint of this sort. These are the necessary and sufficient conditions for the optimal control of the system under the constraint. Since they turn out to be very stringent conditions which will not ordinarily be satisfied in real environmental control problems, the paper then considers the potential for preservation of the stock of environmental assets where system controllability cannot be guaranteed. As little attention has been paid to the nature of control solutions in economy–environment systems that are only partly controllable or partly observable, this is a neglected aspect of environmental control. The paper shows that while the preservation of the stock of environmental assets through the use of economic incentives is dependent on the controllability and (stochastic) observability of the global system, it is possible to identify a control policy based on physical restrictions which will protect against the decline of current environmental stocks irrespective of the controllability of the environment—providing only that the uncontrolled processes of the environment are stable. That is, it is possible to identify what may be termed an environmental stabilization policy.

The paper is divided into seven sections. Section 2 offers a general formulation of an economic-incentive based environmental control problem in a market economy. The formulation does not depend on any particular theory of price formation, being concerned only with the informational and structural conditions necessary and sufficient for system control through the use of economic incentives—given some initial set of prices. The basis of the price set is irrelevant. For the same reason, the formulation ignores strategic behaviour. That is, it does not consider the sequence of proactive and reactive decisions by the control

\textsuperscript{3} For significant developments in the application of the control theoretic framework to environmental management problems see Dasgupta (1982). For more recent applications to range, wildlife, and soil management, see Barrett (1989a–c). For a review of control models in frequent use, see Conrad and Clark (1987).
authority and all other economic agents that would underlie the formation of an
economic incentive policy in any real world case. Since the conclusions reached
for the case considered in the paper hold a fortiori for the case where the control
policy is determined in the context of strategic behaviour, this is not unduly
restrictive. Section 3 then reviews the relation between the concept of sus-
tainability and the requirement for the preservation of at least some environmen-
tal assets, and describes the conditions that this requirement imposes on the
control problem. Section 4 considers the conditions for the optimization of a
control problem of this form in the absence of uncertainty, paying special
attention to the (sufficient) structural conditions for the controllability of the
global system through a set of price incentives. Section 5 addresses the question
of uncertainty, and indicates the informational requirements of optimal control in
systems subject to uncertainty. Section 6 considers how the preservation of
environmental assets may be addressed in an economy–environment system that
is neither controllable nor (stochastically) observable. Section 7 offers some
concluding remarks.

2. FORMULATING THE ENVIRONMENTAL CONTROL PROBLEM

We are looking for a general formulation of the environmental control problem.
Let us first define an n-dimensional state vector, \( \mathbf{x}(t) \), which describes the set of
all physical resources available to the system at time \( t \). That is, \( \mathbf{x}(t) \) includes both
natural or environmental resources and manufactured or produced resources. The
initial state of the system is denoted \( \mathbf{x}(0) \) and its evolution over the interval \( [0, T] \)
is denoted by the state history, \( [\mathbf{x}(t), 0 \leq t \leq T] \). The state history is accordingly a
function of both natural and economic processes.

To begin with, we will ignore the problem of uncertainty. The dynamics of the
state variables in the absence of uncertainty may be defined by the ordinary
vector differential equation

\[
\dot{\mathbf{x}}(t) = f[\mathbf{x}(t), \mathbf{u}(t), t] \quad 0 \leq t \leq T, \tag{1}
\]

in which \( \mathbf{x}(t) \) has already been defined, and \( \mathbf{u}(t) \), an m-dimensional ‘control’
vector, denotes the allocation of resources by the economic agents in the system.
As the economic resources recorded in \( \mathbf{u}(t) \) are a subset of the total resource
base, \( m \) is strictly less than \( n \). In the absence of uncertainty, \( \mathbf{x}(t) \) and \( \mathbf{u}(t) \) are
known at each moment.

Equation (1) defines the ‘equations of motion’ of the system. For given initial
conditions,

\[
\mathbf{x}(0) = \mathbf{x}_0, \tag{2}
\]

it implies the state trajectory:

\[
\mathbf{x}(t) = \mathbf{x}_0 + \int_0^T f[\mathbf{x}(t), \mathbf{u}(t), t] \, dt \quad 0 \leq t \leq T.
\]
The general problem of optimal environmental control is to find that 'control' history, \([u(t), 0 \leq t \leq T]\), which maximizes some index of social welfare (to be defined later) over the interval \([0, T]\), subject to constraints designed to assure the sustainability of the system. To clarify the meaning of 'control' in this context, note that while it is assumed that there exists an environmental authority, and while the environmental authority is defined as the control authority, it is not assumed that the environmental authority directs the allocative decisions of resource users. In a market economy \(u(t)\) summarizes the independent behaviour of all economic agents in the system. Nevertheless, the environmental authority is assumed to have the capacity to intervene in the decision-making processes of individual resource users in two ways: through the use of economic instruments which influence the current price of resource use (such as taxes, subsidies or charges), and through the use of regulatory instruments which restrict the current aggregate level of resource use (such as emission/extraction quotas, whether these are directly allocated or subject to tradeable permits). It is beyond the scope of this paper to discuss the more subtle differences between these two classes of control instrument. As a recent review of the US literature in the area (Cropper and Oates, 1989) has argued, there is enormous diversity in both economic incentives and regulatory instruments, and the dividing line between them is not always clear. The way in which each instrument appears in the control problem in this paper will accordingly imply a cleaner distinction than may be warranted in practice. The important point is, though, that given uncertainty in the system the two classes of instrument are not symmetrical.

To understand the role of economic incentives in this formulation of the control problem, notice that it is assumed that the allocation of economic resources is some function of the set of prices confronting resource users at any given moment. Hence, the allocative decisions of resource users may be influenced by changes in the price set. To capture this, we assume that there exists a control function,

\[ u(t) = u[k, p(t), t] \quad 0 \leq t \leq T, \tag{3} \]

in which the arguments include both a non-zero \(m\)-dimensional time-dependent vector of prices, \(p(t)\), which are beyond the direct reach of the environmental authority, and a \(k\)-dimensional constant vector of economic incentives or control parameters, \(k\), which are not. It is supposed that \(k \leq m\). The control parameters might include, for example, standard Pigovian taxes or subsidies, user charges, licence fees, fines, and other levies. What is important is that the vector \(k\) is not restricted as to sign, and has the effect of modifying the price set confronting resource users. That is, it changes the structure of incentives. Given this, we can

\[ \text{Sometimes styled command-and-control measures.} \]

\[ \text{It is well known that under certain conditions price and quantity regulation are symmetrical, but these conditions include perfect information. In the absence of perfect information price and quantity controls are not symmetrical (cf. Weitzman, 1974). Since the environmental control problem is quintessentially a problem of control under uncertainty and ignorance as to the future environmental effects of economic activity, there is good reason to treat price and quantity controls separately.} \]
be more precise about the nature of the control problem. To the extent that environmental control is exercised through economic incentives it implies a parametric optimization problem. Social welfare is maximized by choice of \( k \), not by choice of \( u(t) \) directly.

It is worth emphasizing that this specification is very general with respect to the theoretical basis of \( p(t) \). It is, for example, consistent with both the classical price model discussed in Perrings (1987) and the Walrasian model that underpins much of the more recent environmental control literature. That is, the sequence \( \{p(t)\} \) may as readily be interpreted as a classical equilibrium price path (derived for a given distribution of the product), as a neo-classical intertemporally efficient price path (satisfying a generalized Hotelling rule, under which the equilibrium price of undepleted/undepreciated/undegraded resources rises at a rate equal to the marginal productivity of capital). Property rights, preferences, production technology, and the distribution of the product are subsumed in a function \( p[\cdot] \), the arguments of which are the set of all real resources in the system. That is

\[
p(t) = p[x(t), t], \quad 0 \leq t \leq T.
\]

The main implication of this is that the weights governing the allocation of economic resources in the absence of environmental control is a function of the general state of nature, \( x(t) \). Indeed, an immediate corollary is that a constant state of nature is sufficient to ensure constant relative prices. That is, since \( \dot{p}(t) = p_x(t), \dot{p}(t) = 0 \) if \( \dot{x}(t) = 0 \). Given the dependence of \( p(t) \) on \( x(t) \) we may characterize the environmental control problem as a closed loop, or feedback, parametric optimization problem.

The index of performance in this closed-loop parametric optimization problem is assumed to be a measure of welfare. Specifically it is assumed that the problem is to maximize:

\[
J = W[x(T), T]e^{-\delta T} + \int_0^T Y[x(t), u(t), t]e^{-\delta t} dt,
\]

in which \( W[\cdot] \) and \( Y[\cdot] \) are strictly concave continuous functions defining social preference orderings over the set of all feasible \( x(t) \). \( e^{-\delta t} \) is a discount factor beyond the reach of the control authority. The vectors \( x(t) \) and \( u(t) \) have already been defined.

Equation (5) indicates that we are dealing with a problem of Bolza, since welfare is assumed to be an algebraic function of the terminal value of the state variables, and an integral function of the state and the control variables over the whole period. The first function, \( W[\cdot] \), represents the discounted 'bequest' value of both produced and natural assets left at the end of the period. Although the term is misleading in this context, it is what is sometimes referred to as the 'scrap' value of the global asset base. The second function, \( Y[\cdot] \), represents the discounted flow of benefits deriving from the use of those assets in economic activity over the interval \([0, T]\). The use of the Bolza form of the problem does not involve any loss of generality, since it can be shown that a benefit function of the Bolza type can be transformed into a function with only a terminal benefit (of
the Mayer type), or one with only an integral term (of the Lagrange type). It is useful primarily because it is intuitive, and fits well with the way in which the environmental problem is currently perceived. In particular, it clarifies the trade-offs involved in the preservation of the asset base. The less significant is $W[-]$ relative to $Y[-]$, the more society weights the welfare gains from current income relative to the preservation of the asset base.

3. SUSTAINABILITY AND THE PRESERVATION OF ENVIRONMENTAL ASSETS

To understand the importance given to the preservation of environmental assets in the literature on sustainable resource utilization, it is useful to begin with the concept of income due to Hicks (1946): the maximum amount which may be spent on consumption in one period without reducing real consumption expenditure in future periods. This is maximum sustainable consumption expenditure, and may be defined as the highest level of consumption expenditure consistent with the preservation of the value of the asset base (Solow, 1986; Mäler, 1990). Following the debates on the limits to growth in the early 1970s, a series of papers considered whether consumption expenditure could in fact be sustained if the resources that were essential to economic activity were exhaustible. Solow (1974), Hartwick (1977, 1978a,b) and Dixit et al. (1980) all showed that in certain circumstances the answer was yes. However, the results reported in these papers were all critically dependent on the interpretation given to the term 'essential', and on the substitution properties of the aggregate production function employed.

Solow (1974) and Hartwick (1977, 1978a) defined an input to a process to be 'essential' if its absence implied that there could be no output from the process. At first blush this looks like a reasonable definition. It implies that in addition to any other properties of the production function, if the $i$th input is essential $f[x(t)] = 0$ where $x_i = 0$. It turns out, though, that whether or not the assumption that inputs are essential in this sense implies any real constraint on output depends on the remaining properties of the production function, including the substitution possibilities assumed to exist between inputs. All the authors cited assumed that production could be described in terms of constant elasticity of substitution functions, in which the elasticity of substitution was greater than or equal to one. But this assumption is not at all weak. The Cobb–Douglas function (CES with an elasticity of substitution equal to unity) adopted in Solow (1974) and Hartwick (1977, 1978a) has the convenient property that the average product of an essential resource has no upper bound. As the resource tends to zero, its average product may tend to infinity implying that its exhaustion does not constrain aggregate output in any meaningful way. The CES function with an elasticity of substitution greater than unity adopted in Hartwick (1978b) has the even more convenient property that no resource can be essential in the first place.

In fact, the technological assumptions of these papers bear little relation to the dynamic properties of real physical systems. The assumption that there exists effectively unlimited potential for substitution between natural and produced
resources contradicts everything that is currently understood about the evolution of thermodynamic systems, about the complementarity of resources in system structure, and about the importance of diversity in system resilience (Georgescu-Roegen, 1971; Holling, 1973, 1986; Daly, 1977, 1990; Kay, 1990). Since it is assumed by Solow and Hartwick that technology is stationary and given, it is a fortiori assumed that the potential exists now for such substitutability. None would deny that there is some scope for substitution between produced and environmental assets, nor that the potential for substitution is likely to increase over time, but to assert that the potential for substitution is currently unlimited is nothing short of absurd.

The link between the sustainability of income and the preservation of environmental assets derives from the limits on the potential for substitution between man-made and environmental assets. It is now well understood that ecosystems tend to be locally stable over defined ranges of the biophysical stocks that comprise their component parts, and they become unstable if the biophysical stocks rise above or fall below some critical threshold. Such biophysical stocks may be said to be complementary with the remaining stocks in the system. If this complementarity is ignored, the potential costs include the collapse of the system itself. In thermodynamic terms, for any ecosystem having evolved a particular self-organization there exist combinations of the biophysical stocks involved which are ‘thermodynamically possible’ but which are also incompatible with that self-organization. Whenever the resources of such an ecosystem are driven past certain threshold values, the system will switch from one ‘thermodynamic path’ to another, from one self-organization to another. Threshold values exist, for example, for the diversity of species in an ecosystem. While there may be a range of population sizes for the different species in an ecosystem over which the system remains stable, if any one population in an ecosystem falls below its critical threshold level, the self-organization of the ecosystem as a whole will be radically and irreversibly altered (cf. Pielou, 1975). Threshold values also exist for overall regressive succession; standing crop biomass; energy flows to grazing and decomposer food chains; mineral micro-nutrient stocks and so on (cf. Schaeffer et al., 1988). These thresholds imply that there exist levels of environmental assets at which it may be desirable to prevent any further depletion.

As a result, economists working within the same intellectual tradition as Solow and Hartwick et al. have argued that sustainability implies the need to preserve at least components of the stock of environmental assets (cf. Pearce, 1987, 1988; Pearce et al., 1989; Pearce and Turner, 1990; Barbier et al., 1990). The argument rests not only on the complementarity between produced and environmental assets, but also on the high degree of uncertainty about the role of many resources in the ecosystem, and the potential for inflicting irreversible damage through over exploitation of those resources. On these grounds, Pearce and Turner (1990) argue that it is prudent to conserve existing stocks ‘until we have a clearer understanding of what the optimal stock is’.

This paper considers two rather different formulations of the preservation condition coming out of these concerns: one to maintain current stocks of
environmental assets within the threshold levels of ecosystem stability as these are presently understood; the other to maintain current stocks of environmental assets at the same level. The second requirement is the more straightforward. It implies a terminal boundary condition on a subset of the resources in the global system. Specifically, it requires that a \( q \)-dimensional vector function of the terminal state,

\[
\mathbf{g}(\mathbf{x}(T), T) = 0
\]

should have prescribed values, \( q < n \). Notice that it is not implied that the terminal boundary condition implies an inviolate restriction on current economic activity. That is to say, it is neither a regulatory instrument governing current aggregate levels of resource use, nor an economic instrument affecting current resource prices. It may be interpreted as a target set of resource values judged by the environmental authority to be a necessity if future generations are to have the same opportunities as the present generation. If the particular control instruments used by the environmental authority are effective, then it will be possible to satisfy (6). This will be referred to as an environmental boundary condition.

The requirement that current stocks of environmental assets be maintained within the threshold levels of ecosystem stability does imply that the environmental authority should constrain current levels of economic activity. Specifically, it implies control over the level of extraction of resources in order to maintain the stability of the ecosystems to which they belong. Formally, such regulation implies an \( m \)-dimensional vector inequality constraint on the state and control histories:

\[
\mathbf{h}(\mathbf{x}(t), t) \leq 0 \quad 0 \leq t \leq T
\]

Notice that this allows for either one-sided or two-sided bounds on the variable in question, depending on the specification of \( \mathbf{h} \). So, for example, \( (u_i - u_{\text{max}}) \leq 0 \) defines an upper bound on allocation of the \( i \)-th resource; \( (u_{\text{min}} - u_i) \leq 0 \) defines a lower bound; and \( (u_i - u_{\text{max}})(u_i - u_{\text{min}}) \leq 0 \) defines both upper and lower bounds. In general, the restriction on the state variables would take the form of a lower bound: \( (x_{\text{min}} - x_i) \leq 0 \). Equation (7) will be referred to as an ecological sustainability constraint. This completes the description of the main elements of the control problem in the absence of uncertainty.

4. THE STRUCTURAL LIMITS TO ENVIRONMENTAL CONTROL

The general objective in the environmental control problem in market economies is to find that set of control parameters (or incentives) which will maximize the index of social welfare, subject to the system dynamics, as well as the environmental boundary condition and the ecological sustainability constraint. The elements of a general control problem of this form are summarized in (8.1) to (8.7)

\[
\text{maximize}_{\mathbf{u}} \quad J = W[\mathbf{x}(T), T]e^{-\delta T} + \int_0^T Y[\mathbf{x}(t), \mathbf{u}(t), t]e^{-\delta t} \, dt
\]

(8.1)
subject to

\[ \dot{x}(t) = f[x(t), u(t), t], \quad 0 \leq t \leq T \]  
(8.2)

\[ g[x(T), T] = 0 \]  
(8.3)

\[ h[x(t), u(t), t] \leq 0, \quad 0 \leq t \leq T \]  
(8.4)

\[ x(0) = x_0, \]  
(8.5)

with

\[ u(t) = u[k, p(t), t], \quad 0 \leq t \leq T \]  
(8.6)

\[ p(t) = p[x(t), t], \quad 0 \leq t \leq T. \]  
(8.7)

The necessary conditions for the maximization of welfare are familiar, and may be stated briefly. Abstracting both from the environmental boundary condition, (8.3), and the ecological sustainability constraint, (8.4), we may define the Hamiltonian

\[ H_0[x(t), u(t), \lambda_0(t), t] = Y[x(t), u(t), t]e^{-\delta t} + \lambda_0(t)f[x(t), u(t), t] \]  
(9)

where \( \lambda_0(t) \) is an \( n \)-dimensional vector of Lagrangian multipliers. The subscript '0' indicates that the problem is unconstrained. The necessary conditions are given by the corresponding Euler-Lagrange equations, and include: \(^6\)

\[ \dot{\lambda}_0(t) = -H_0[\cdot] = -Y_x(t)e^{-\delta t} - \lambda_0(t)F_x(t) \]  
(10.1)

\[ \lambda_0(T) = W_x(t)e^{-\delta t} \]  
(10.2)

\[ H_0[\cdot] = Y_k e^{-\delta t} + \lambda_0(t)F_k(t) = 0 \]  
(10.3)

The first point to make is that both the maximum condition (10.3) and the adjoint vector (10.1) are concerned with the impact of perturbations of the state variables and the control parameters respectively evaluated along an optimal trajectory. That is, they are estimated at \( x(t) = x^*(t) \) and \( k = k^* \). This implies that both are local, not global conditions. Satisfaction of (10.3) is required if the Hamiltonian is to be stationary with respect to variation in the control parameters along an optimal path, but it implies nothing about the existence of other optimal paths. Put another way, the fact that there may be a set of incentives which maximizes social welfare whilst satisfying the equations of motion along some optimal development path does not mean that there are no other possibly dominant optimal paths. A second and more important point to make is that since the Euler–Lagrange equations are necessary, not sufficient, they say nothing about whether or not a locally optimal trajectory is in fact attainable. In the context of the particular problem discussed in this paper, satisfaction of these equations says nothing about the ability of a control authority to use a form of parametric control to guide the system.

To see this, take the environmental boundary condition, and consider the properties of the general system which will assure the attainability of an optimal development path that satisfies this condition. There exists a \( q \)-dimensional discounted vector function of terminal benefits associated with the environmental boundary conditions

\[ J_1 = g[x(T), T]e^{-\delta T}. \]  
(11)

\(^6\) The necessary conditions are derived (for the particular problem discussed here) in Appendix 1.
The Hamiltonian corresponding to each component of this vector is itself a component of the \(q\)-dimensional column vector
\[
H_t[x(t), u(t), \Lambda_t(t), t] = \Lambda_t(t) f^2[t],
\] (12)
\(\Lambda_t\) being a \(q \times n\) matrix formed from \(n\) \(q\)-vectors of Lagrangian multipliers attached to each of the terminal conditions in (8.3). If an optimal development path satisfying the environmental boundary condition exists, the first variation of the (augmented) terminal benefit function with respect to the control parameters will be zero. Formally, this implies that
\[
\Delta J^* = \kappa \int_0^T \{\Lambda_t(t) F_k(t) [Y_k e^{-\delta t} + F_k(t)' \lambda_0(t)] \} dt + \kappa \int_0^T \{\Lambda_t(t) F_k(t) F_k(t)' \Lambda_t(t) \mu \} dt
\] (13)
will be zero, where the variation \(\Delta k\) has been selected such that \(\Delta k = \kappa H_k[\cdot]' = \kappa[Y_k e^{-\delta t} + F_k(t) \lambda_0(t) + F_k(t) \Lambda_t(t) \mu]\), \(\kappa\) being a small positive constant, and where \(\mu\) is the \(q\)-vector of Lagrange multipliers corresponding to the terminal boundary conditions. Attainability of an optimal path will be assured providing that the vector \(\mu\) does exist. This requires that the vector equation
\[
\mu = -Q^{-1} r
\] (14)
has a solution, \(Q\) being the matrix \(\int_0^T \{\Lambda_t(t) F_k(t) F_k(t)' \Lambda_t(t) \} dt\) and \(r\) being the vector \(\int_0^T \{\Lambda_t(t) F_k(t) [Y_k e^{-\delta t} + F_k(t) \lambda_0(t)] \} dt\). It follows that a sufficient condition for the maximization of welfare subject to an environmental boundary condition is the invertibility of \(Q\).\(^7\)

Consider the rank of \(Q\). This depends upon the rank of the constituent matrices: \(\Lambda_t(t), \Lambda_t(t), F_k(t), \) and \(F_k(t)\). The first two of these matrices are \(q \times n\) and \(n \times q\) respectively, while the last two are \(n \times k\) and \(k \times n\). Hence the dimension of \(Q\) is \(q \times q\). If the constituent matrices are of full rank, this implies that the maximum rank of \(Q\) is the lesser of \(q\) and \(k\), and \((q > k)\). This implies that \(Q^{-1}\) exists only if \(k \geq q\): the number of control parameters is greater than or equal to the number of environmental assets to be preserved. It is only if \(k > q\) that a set of optimizing control parameters can be chosen to satisfy the environmental boundary condition.

The existence of the inverse of \(Q\) is a sufficient condition for the optimality of the problem summarized in (8.1)-(8.3) and (8.5)-(8.7).\(^8\) If \(Q^{-1}\) exists, the

\(^7\) This sufficient condition is derived in Appendix 2.

\(^8\) In addition to the normality condition, satisfaction of two other conditions are sufficient to ensure optimality of a development path traced by the state history, \(x(t), 0 \leq t \leq T\). The first of these is the Legendre-Clebsch condition which requires that the Hessian matrices composed of the second derivatives of the Hamiltonians with respect to the control parameters are negative definite throughout the interval \([0, T]\). That is
\[
\begin{bmatrix}
\partial^2 H_t[\cdot] \\
\partial k^2
\end{bmatrix} < 0
\] (*)
where \(H_t[\cdot] = H_t[x^*(t), u^*(t), \Lambda^*_t(t), t]\). If the control problem is normal, and if the necessary conditions have been met, then (*) is sufficient to guarantee optimality. However, it is only locally sufficient, applying in the neighbourhood of an optimal development path. In general, if the dynamic system is non-linear, there is no reason to believe that (*) would hold for significant deviations from an optimal path. The second sufficiency condition is the Jacobi condition. This requires that there be no conjugate points along an optimal path, which ensures the uniqueness of that path.
problem is said to be 'normal'. If not, it is said to be 'abnormal'. It is this normality condition which is crucial if the environmental boundary condition is to be satisfied. What the normality condition guarantees is that the system is structurally controllable through the vector of parameters, $k$. It is possible, through those parameters, to force the system to meet condition (8.3). The fact that $Q$ is full rank in a normal problem ensures at least one control parameter. In terms of the environmental control problem, it is possible to influence the system through the set of price incentives so as to preserve all of those environmental resources that are held to be crucial to global sustainability. If $Q^{-1}$ does not exist then manipulation of economic incentives will leave one or more such environmental resources untouched, and therefore unprotected from degradation due to economic activity.

It is difficult to overemphasize the importance of the normality condition in the environmental control problem, yet the condition has been almost completely ignored in the literature on environmental management. In general, attention has been focused on a limited set of necessary conditions (given by the Euler–Lagrange equations). Where sufficiency has been considered, it has been limited to the convexity condition. In general, however, the literature has simply assumed controllability. This is somewhat ironical given the weight placed on controllability questions in the earlier literature on macroeconomic stabilization where an assumption of normality, though still extremely powerful, was rather more credible than in the case of economy–environment systems (cf. Aoki, 1976). One of the central issues in the economics of environmental management has been the role of economic incentives in assuring the sustainable use of environmental resources. Yet it is by no means clear that the environmental control problem is 'normal'. Indeed, both the weight of evidence and the logic of economy–environment systems suggests otherwise (cf. Perrings, 1987). However, if it is not safe to assume that the problem is normal—that the dynamic system is controllable—then it becomes important to identify the limits to environmental control and the scope for achieving alternative environmental objectives.

5. UNCERTAINTY IN THE ENVIRONMENTAL CONTROL PROBLEM

We shall return to the problem of assuring the sustainability of non-controllable systems momentarily. At this point, however, we turn to a second source of difficulty in the environmental control problem: the uncertainty associated with the system dynamics. There are two problems here. The first is that in a feedback control problem optimality requires the continuous measurement of state variables, and available measures may be subject to error. The second is that the system dynamics may themselves be known imperfectly, implying that feedback control will necessarily be misdirected.

If we take the difficulty created by measurement error first, it is clear that not all measurement error is fatal from a control perspective. In many cases measures may be subject only to white noise (Gaussian error). A variety of filters exist to enable the propagation of state estimates with minimum error in such cases, and such filters tend to operate reasonably well if the system under analysis is linear.
Non-linearities in the system complicate matters by the effect they have on error transmission through the system. The probability density functions for stochastic effects may, for example, change as those effects are transmitted through the system. However, providing that the effects are small and additive, they may be partially filtered out of non-linear systems (Stengel, 1986). The problem is altogether more difficult if errors are large and non-Gaussian, and if they are not additive.

The difficulties created by misspecification of the system dynamics are much less tractable. The misspecification of the system increases the probability that both the measures sought and the controls applied will be inconsistent with the control objectives. It is increasingly being argued that system uncertainty in the environmental control problem extends well beyond Gaussian measurement errors, and includes basic misspecifications of system dynamics. Indeed, the technological assumptions that characterize the Solow/Hartwick models are simply incompatible with any reasonable approximation of the dynamics of the global system. The result is a very large element of fundamental uncertainty about both the measures and dynamics of the system, implying that the control authority is simply unable to form realistic expectations about the distribution of future outcomes.

The main source of uncertainty appears to be the evolutionary nature of the global system. Not all evolution creates the same amount of difficulty. Faber and Proops 1990, for example, make the point that while genotypic evolution (evolution of genetic potential) which changes the development potential of interdependent species or ecosystems is in principle unpredictable, phenotypic evolution (evolution within a genotype in response to environmental change) is not. Failure to predict phenotypic evolutionary trends may be due to the product of ignorance about the functional structure of ecosystems, but failure to predict genotypic evolutionary trends is inherent in the nature of the changes involved. Genotypic evolution is, accordingly, the least tractable source of system uncertainty. From a thermodynamic perspective, such genotypic evolution may be interpreted in terms of the potential for self-organizing far-from-equilibrium thermodynamic systems to undergo essentially unforseeable and 'catastrophic' switches from one thermodynamic branch to another (Kay, 1989, 1990).

The implication of this is that the environmental control problem involves very significant error that is likely to be beyond the reach of available filter techniques. It involves the potential for novel and entirely surprising outcomes. This is not to say that knowledge of the system, or at least of parts of the system, cannot be improved over time. Estimates of the distribution of possible environmental outcomes of economic activity may well be improved through, for example, a passive Bayesian learning process. However, it does imply that there is likely to remain a very large measure of fundamental ignorance about the future effects of current actions.

To get a sense of the knowledge requirements in the optimal environmental control problem, it is useful to consider a formulation of the problem suitable for the case where there exists no unique optimal trajectory. This is the case both for
stochastic systems and, a fortiori, for systems characterized by fundamental uncertainty. The approach used in such circumstances is dynamic programming. The benefit function adopted is identical to (8.1) save that what is optimized is some expectation of terminal and integral benefits conditional on the available state of knowledge or information set. At time $T$ this may be written:

$$J = E\{(W[x(T), T]e^{-\delta T} + \int_0^T Y[x(t), u(t), t]e^{-\delta t} dt) \mid S[0, T]\}$$

in which $S[0, T]$ is an information set at time $T$ that summarizes realized indirect measures of the real resource base, $[p(t), 0 \leq t \leq T]$; and realized allocations of real resources, $[u(t), 0 \leq t \leq T]$. That is

$$S[0, T] = \{p[0, T], u[0, T]\}.$$ 

We first define a value function, $V^*$, associated with (15) which is equal to the discounted benefit derived from the optimal parametric control of the system for the period to go to the terminal time, $T$. So at a time, $t_1$, during the interval $[0, T]$, $V^*$ is defined by

$$V^*(t_1) = E\{(W[x^*(T), T]e^{-\delta T} + \int_{t_1}^T Y[x^*(t), u^*(t), t]e^{-\delta t} dt) \mid S[0, t_1]\}$$

in which $u^*(t) = u[k^*, p(t), t]$. The starting point for the maximization of (16) is accordingly the terminal value

$$E\{(V^*[x^*(T), T]) = E\{W[x^*(T)e^{-\delta T}] \mid S[0, T]\}$$

That is, it is the optimal value of the resource base at the terminal time, $x^*(T)$, conditioned on the information set available at the terminal time, $S[0, T]$.

The difficulty here is obvious. While the optimal value of $x^*(T)$ (in a 'normal' problem) will satisfy the terminal boundary or economic sustainability condition, its calculation requires the information set $S[0, T]$. In other words, it requires a state of knowledge that can only be available at time $T$. The knowledge that is available to the control authority at any time before $T$ is less than the knowledge required to optimize the problem. Formally, for any time $t$, $0 \leq t < T$, there exists

$$E\{(W[x^*(T), T]e^{-\delta T} + \int_{t_1}^T Y[x^*(t), u^*(t), t]e^{-\delta t} dt) \mid S[0, t_1]\}$$

To identify the conditions in which $V^*$ is at a maximum at $t_1$, we may first expand the time derivative of $V^*(t_1)$ in series. Suppressing time indices, we have:

$$\frac{dV^*}{dt} \Delta t = E\left\{(\frac{\partial V^*}{\partial t} \Delta t + \frac{\partial V^*}{\partial x} \Delta x + \frac{1}{2} \left[ \frac{\partial^2 V^*}{\partial x^2} \Delta x^2 \right] \Delta t + \ldots \right\} \mid S[0, t_1]\}.$$  

Dividing through by $\Delta t$, and equating to the derivative of $V^*(t_1)$ with respect to time at $t_1$ yields

$$0 = \max_k E\left\{ (Y[x^*]e^{-\delta t} + \frac{\partial V^*}{\partial t} \Delta t + \frac{\partial V^*}{\partial x} \Delta x + \frac{1}{2} \left[ \frac{\partial^2 V^*}{\partial x^2} \Delta x^2 \right] \Delta t + \ldots ) \mid S[0, t_1]\}.$$ 

For an optimum trajectory, this expectation should be maximized for the interval $[T, t_1]$ through choice of the parametric control vector $k$, conditioned on the available information set available at time $t_1$. 

Equation (16) may be written in the form

$$V^*(t_1) = E\{(W[x^*(T), T]e^{-\delta T} + \int_{t_1}^T Y[x^*(t), u^*(t), t]e^{-\delta t} dt) \mid S[0, t_1]\}$$
an information set $S[t + T]$, where $t+$ is a time just greater than $t$, which is required for the optimization of the trajectory of $x(t)$, but which is not available to the control authority. There are two possibilities to consider here. If the conditioning effect of this information is fully predictable (and if structural conditions for the controllability of the system exist) then the stochastic control of the system may still be optimal, and the economic sustainability condition may be satisfied. If the conditioning effect of $S[t+, T]$ is not fully predictable and can only be approximated, then even if the system is controllable, the control will be suboptimal and sustainability cannot be guaranteed.

Given that the environmental system is characterized by fundamental uncertainty, the available information set does not include a sufficient profile of the statistical properties of the unavailable information set to predict its conditioning effect on the future behaviour of the system. This is partly because of the existence of novel developments whose implications for the time-behaviour of the system are unclear. In a far-from-equilibrium evolutionary system, the distribution of outcomes associated with such developments cannot be inferred from the history of the system both because of the paucity of relevant observations, and because of the effect of novelty on the system parameters. However, it is also because of the indirect, noisy, and incomplete character of the system observers in a market economy: the price set.

6. ECOLOGICAL SUSTAINABILITY IN THE CONTROL PROBLEM

This brings us to the nub of the question. If it is not possible to satisfy an environmental boundary condition through the use of economic incentives, either because of the uncontrollability of the system or because of the existence of fundamental uncertainty, what is to be done? The cause of ecological sustainability is not lost, but it does require a different approach to the problem. To see what this might be, let us first observe that where a uniquely optimal trajectory is unattainable because the system is uncontrollable, it may still be possible to achieve a degree of system stability providing that the uncontrolled (and unobserved) processes of the environment are themselves stable. A system may be said to be stable if a bounded input generates a bounded response. Thus an ecosystem may be said to be stable if the responses to bounded exactions on or insertions into that ecosystem are themselves bounded: that is, the system does not undergo some catastrophic transformation or collapse as a result. System stability is accordingly very close to the meaning of 'resilience' in systems ecology, and so to the concept of ecological sustainability. System stability and system sustainability are much the same thing. Indeed, if it is possible to assure system stability, it will in general be possible to assure system sustainability. The main problem from an environmental management perspective is that ecosystems tend

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10 This begs the question of what sort of system response we are interested in. The most appropriate response would seem to be in the distribution of the system parameters. This approach to the problem of ecological sustainability has been explored in Common and Perrings (1991).
to be stable over certain parameter ranges only. If an ecosystem is not controllable, its sustainability may be assured only if depletion or pollution of the populations or stocks it comprises does not drive them beyond the region within which they are stable. The role of the ecological sustainability constraint, equation (8.4), is to impose direct restrictions on resource-using economic activities in order to maintain ecological populations/stocks within stable and therefore sustainable bounds.

A separate problem is that if the system is not controllable through the use of economic incentives, environmental processes will not be observable or will be only imperfectly observable through the price system. The price system will not, therefore, provide an adequate means of monitoring the environment. Instead, it will be necessary to rely on direct observation. The crucial indicators here are the threshold values for the populations or stocks comprising an ecosystem, beyond which ecosystem stability is lost. This implies the use of physical indicators both of stocks and of change in ecosystem resilience. This would include the number of species in an ecosystem and the relative size and distribution of the population in each species relative to its critical threshold level (cf. Pielou, 1975); changes in overall regressive succession; standing crop biomass; relative energy flows to grazing and decomposer food chains; mineral micro-nutrient stocks; and in the mechanisms of and capacity for damping oscillations (Di Castri, 1987; Schaeffer et al., 1988). Formally, the addition of direct observations to the information set implies that the ecological sustainability constraint vector $h[x(t), u(t), t] = 0$ would be conditioned on an information set

$$S_D[0, t] = \{\dot{x}[0, t], p[0, t], u[0, t]\},$$

which includes a direct estimate of (components of) the resources-base $\dot{x}(t)$.

To see the role of the ecological sustainability constraint in the environmental control problem it is convenient to return to a simple stochastic version of the problem discussed in Section 4. Since we do not assume controllability of the system, it is convenient to disregard the environmental boundary condition. This is the problem indexed '0' in Section 4. If we now add the ecological sustainability constraint defined by equation (8.4) we can identify an augmented benefit function of the form:

$$J_0 = E\{W[\cdot]e^{-\delta T} + \int_0^T [Y[\cdot]e^{-\delta t} + \lambda_0(t)f[\cdot] - \dot{x}(t)] ~dt \mid S_D[0, t]\}$$

(18)

with

$$\eta_0(t) = 0 \quad \text{if} \quad h[\cdot] < 0$$

$$\eta_0(t) = 0 \quad \text{if} \quad h[\cdot] = 0$$

$\eta_0(t)$ being a vector of Lagrange multipliers adjoining the ecological sustainability constraint to the benefit function $J_0$. The Hamiltonian for the problem is

$$H_0[\cdot] = Y[\cdot]e^{-\delta t} + \lambda_0(t)f[\cdot] + \eta_0(t)h[\cdot]$$
and the adjoint vector and transversality conditions take the form:

\[ \dot{\lambda}_0(t) = \begin{cases} -Y_x(t)e^{-\delta t} - \lambda'_0(t)F_x(t), & h[.] < 0 \\ -Y_x(t)e^{-\delta t} - \lambda'_0(t)F_x(t) - \eta'_0(t)H_x(t), & h[.] = 0 \end{cases} \quad (19.1) \]

\[ \lambda_0(T) = W_x(t)e^{-\delta T} \quad (19.2) \]

while the maximum condition may be written

\[ \begin{cases} Y_k(t)e^{-\delta t} + \lambda'_0(t)F_k(t) & h[.] < 0 \\ Y_k(t)e^{-\delta t} + \lambda'_0(t)F_k(t) + \eta'_0(t)H_k(t) & h[.] = 0 \end{cases} \quad (19.3) \]

Equations (19) form the Euler–Lagrange equations for the revised problem, giving the necessary conditions on \( \lambda_0(t) \), \( \eta_0(t) \), \( H_{0k}[.] \), and \( H_{0x}[.] \) for optimization of the problem. Both the adjoint equation and the maximum condition now have two forms depending on whether the sustainability constraint is or is not effective along an optimal trajectory. The vector \( \eta_0(t) \) measures the sensitivity of the benefit function to the sustainability constraint. If a control is ineffective, which may be thought of as implying that a resource is non-scarce or that emission levels are within some well defined threshold, the corresponding multiplier has a zero value. If a control is effective, the corresponding multiplier defines the marginal benefit of the relaxation of the constraint. In policy terms an ineffective control translates as free access to the resource in question. An effective control translates as restricted access to the resource, whether secured by formal physical limits (quota, permits, safe minimum standards) or the use of economic instruments to meet physical limits (user fees, charges, fines, etc.).

Now consider what the sustainability constraint is doing in the context of the control problem. It is well understood that the imposition of boundaries of this sort on the control instruments fully determines the optimizing control in special cases. This is true, for example, of benefit functions that do not have a well defined maximum (or minimum). It is not, however, true in the case of the environmental control problem under discussion here. The boundaries imposed under a sustainability constraint do not necessarily determine the optimal allocation of economic resources. Nor do they necessarily affect the role of the market in securing the allocation of resources. Indeed, this depends on the degree of system controllability.

It is reasonable, from what has been said, to suppose that given available information on the global economy–environment system the structural conditions do not exist for a natural capital boundary condition to be satisfied through a parametric control programme. Put another way, there is no reason to believe that the parametric optimization of the expected value of the benefit function will generate a trajectory which both respects the maximum principle and satisfies the environmental boundary condition. The system is neither perfectly controlled nor perfectly understood. This does not mean that all resource use may not be influenced through the price mechanism, but it does mean that some resource use is either unaffected by prices, or is influenced in an indeterminate way. The role of the sustainability constraint is to regulate the pressure on environmental
resources in such a way that the parametric optimization of the system is consistent with its local stability.

7. CONCLUDING REMARKS

Ecological sustainability constraints of this form can be thought of as precautionary constraints. The limits they impose on economic activity will depend (1) on the (local) stability of the ecosystems involved, (2) on the conjectured losses if those ecosystems should become unstable due to the effects of economic activity, and (3) on the degree of system controllability. If the structural conditions for the controllability of the global economy-environment system are not satisfied, and if the system is imperfectly observed, there exists the potential for unanticipated and 'catastrophic' effects at points far removed from the original source of damage. It is these effects that are the object of an ecological sustainability constraint. What all ecological sustainability constraints share is that they bound the current level of pressure imposed on particular populations, and that the bounds reflect perceptions about the potential future losses in terms of the wider system of population extinction. In terms of a simple population indicator, for example, if a species population growth function has the property that it is critically depensatory at some more or less well defined population level (that the population will collapse with some probability if it falls below that level), then the ecological sustainability constraint would be based on the critical depensatory point. Where it would be placed relative to the critical depensatory point would, however, depend on the perceived significance of the future welfare effects of the collapse of the population. The implication of a control approach to the management of the economy-environment system is the same as the implication of a systems approach to ecology. It is the dynamic interdependence of populations that determines the magnitude of future losses. Hence even if constraints are imposed on species populations or resource stocks, the value of those constraints should reflect not the species/stocks own-growth function, but its role in the ecosystems whose stability is to be protected.

Sustainability constraints may accordingly be thought of as analogous to the safe minimum standards that have long been a feature of engineering design. But whether they are directly enforced as a condition of investment (cf. Goodland and Ledec, 1987) or secured through some influence function involving the price mechanism depends on the reach of the latter. That is, whether the inequalities of (8.4) are satisfied through direct regulation or through some form of incentive-based parametric control naturally depends critically on the controllability of the system. If the resource stocks to be maintained are within the reach of the price system, then the sustainability constraint may be satisfied through some sort of economic incentive-based parametric control. If those stocks are beyond the reach of the price system direct regulation is necessary. Indeed, it may well be that the best evidence for the uncontrollability of the global system is the prevalence of instruments of direct regulation. Transferable or non-transferable quotas in fisheries, game licences, open and close seasons on the predation of
‘game’ species are all indicators not so much of the failure of markets, as of their extraordinarily limited scope as system observers and system controllers.

REFERENCES

APPENDIX 1

In the unconstrained problem, adjoin (8.2) to the benefit function by the Lagrangian vector \( \lambda_0(t) \), yielding

\[
J_0^* = W[x(T), T]e^{-\sigma T} + \int_0^T \left[ Y[x(t), u(t), t]e^{-\sigma t} + \lambda'_0(t)\{f[x(t), u(t), t] - \dot{x}(t)\} \right] dt. 
\]  
(A1.1)

Given the Hamiltonian

\[
H_0[x(t), u(t), \lambda_0(t), t] = Y[x(t), u(t), t]e^{-\sigma t} + \lambda'_0(t)f[x(t), u(t), t], 
\]  
(A1.2)

\( J_0^* \) may be written, after integrating the last term on the right-hand side by parts

\[
J_0^* = W[x(T), T]e^{-\sigma T} + \int_0^T \{ H_0[x, u, \lambda_0(t), t] + \dot{\lambda}_0(t)\} dt - \left[ Y[x(0), u(0), T] - \lambda'_0(0)x(0) \right]. 
\]  
(A1.3)

The first variation of this function with respect to the control parameters is given by

\[
\Delta J_0^* = \{ W(t)e^{-\sigma t} - \lambda'_0(T) \} \Delta x(\Delta k) + \int_0^T \{ H_{0x}[\cdot]\Delta k + [H_{0u}[\cdot] + \dot{\lambda}_0(t)]\Delta x(\Delta k) \} dt 
\]  
(A1.4)

where \( H_{0x} = \{ \partial H_0[\cdot]/\partial k \} \), and \( H_{0u} = \{ \partial H_0[\cdot]/\partial x \} \). Selecting values of \( \lambda_0(T) \) that will ensure that \( W(t)e^{-\sigma t} - \lambda'_0(T) \) and \( [H_{0x}[\cdot] + \dot{\lambda}_0(t)] \) are equal to zero, yields the first two necessary conditions for a maximum of \( J_0^* \):

\[
\dot{\lambda}_0(t) = -H_{0x}[\cdot] = -Y_k(t)e^{-\sigma t} - \dot{\lambda}_0(t)F_k(t) 
\]  
(A1.5)

\[
\lambda_0(T) = W_k(T)e^{-\sigma t} 
\]  
(A1.6)

Substituting these values into (A1.4) reduces the first variation to

\[
\Delta J_0^* = \int_0^T \{ H_{0x}[\cdot] \Delta k \} dt 
\]  
(A1.7)

from which it is immediate that the benefit function will be at a maximum only if

\[
H_{0x}[\cdot] = Y_k e^{-\sigma t} + \dot{\lambda}_0(t)F_k(t) = 0 
\]  
(A1.8)
where

\[ Y_k = \begin{bmatrix} \frac{\partial Y}{\partial u} \\ \frac{\partial Y}{\partial p} \end{bmatrix}, \quad F_k = \begin{bmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial f}{\partial p} \end{bmatrix}, \quad Y_s = \begin{bmatrix} \frac{\partial Y}{\partial u} + \frac{\partial Y}{\partial p} \end{bmatrix}, \quad F_s = \begin{bmatrix} \frac{\partial f}{\partial u} + \frac{\partial f}{\partial p} \end{bmatrix}, \quad W_s = \begin{bmatrix} \frac{\partial W}{\partial u} \end{bmatrix} \]

the partial derivatives of the vectors \( f, u, \) and \( p \) being conformably dimensioned Jacobian matrices, and the partial derivatives of \( Y \) and \( W \) being conformably dimensioned row vectors. Equations (A1.5), (A1.6) and (A1.8) give the necessary conditions on \( \lambda_0(t), H_{0k}[\cdot], \) and \( H_{0k}[\cdot] \) if all parts of the first variation are to be equal to zero in the neighbourhood of an optimal trajectory.

**APPENDIX 2**

In the problem subject to the constraint (8.3), adjoin (8.2) to the vector of terminal benefits associated with the environmental boundary conditions, \( J_1 = g[x(T), T]e^{-\delta T} \), by \( \Lambda'_i, \) a \( q \times n \) matrix of \( n \) \( q \)-vectors of Lagrangian multipliers, yielding

\[ J_1^* = g[x(T), T]e^{-\delta T} + \int_0^T \Lambda'_i(t) \{f[x(t), u(t), t] - \dot{x}(t)\} \, dt \quad (A2.1) \]

The corresponding Hamiltonian is the \( q \)-dimensional column vector

\[ H_1[x(t), u(t), \Lambda_i(t), t] = \Lambda'_i(t)f'[\cdot]. \]

Hence \( J_1^* \) may be written in the form

\[ J_1^* = g[x(T), T]e^{-\delta T} + \int_0^T \{H_1[\cdot] + \dot{\Lambda}_i(t)x(t)\} \, dt - [\Lambda'_i(T)x(T) - \Lambda'_i(0)x(0)] \quad (A2.2) \]

The first variation of (A2.2) with respect to the control parameters is

\[ \Delta J_1^* = [g_1 e^{-\delta T} - \Lambda'_i(T)]\Delta x(\Delta k) + \int_0^T \{H_{1k}[\cdot]\Delta k + [H_{1k}[\cdot] + \dot{\Lambda}_i(t)]\Delta x(\Delta k)\} \, dt \quad (A2.3) \]

Selecting values for \( \Lambda_i \) that will force the coefficients of \( \Delta x \) to zero implies that

\[ \dot{\Lambda}_i(t) = -H_{1k}[\cdot] \quad (A2.4) \]

and

\[ g_1 e^{-\delta T} = \Lambda'_i(T) \quad (A2.5) \]

Reducing (A2.3) to

\[ \Delta J_1^* = \int_0^T \{H_{1k}[\cdot]\Delta k\} \, dt \quad (A2.6) \]
Adjoin (A2.6) to the first variation of the benefit function, $J_0$, with a vector of Lagrange multipliers, $\mathbf{\mu}'$, and write the first variation of the general benefit function, $J$, as

$$\Delta J = \Delta J_0^+ + \mathbf{\mu}' \Delta J_1^+ = \int_0^T \{ H_0(\mathbf{k}) \Delta \mathbf{k} + \mathbf{\mu}' H_1(\mathbf{k}) \Delta \mathbf{k} \} \, dt, \quad (A2.7)$$

implying that on the optimal trajectory

$$H_1(\mathbf{k}) = \mathbf{Y}^e - \mathbf{\lambda}_0(\mathbf{F}_k(t)) + \mathbf{\mu}' \mathbf{\Lambda}_i(t) \mathbf{F}_k(t) = 0. \quad (A2.8)$$

Now select $\Delta \mathbf{k} = \kappa H_1(\mathbf{k})$ and substitute this into (A2.6).

$$\Delta J_1^+ = \kappa \int_0^T \{ \mathbf{\Lambda}_i(t) \mathbf{F}_k(t)[\mathbf{Y}^e - \mathbf{\Lambda}_i(t)] \} \, dt + \kappa \int_0^T \{ \mathbf{\Lambda}_i(t) \mathbf{F}_k(t) \mathbf{F}_k(t) \mathbf{\Lambda}_i(t) \mathbf{\mu} \} \, dt. \quad (A2.9)$$

Defining $\mathbf{Q}$ and $\mathbf{r}$ as in the text, this may be solved for the vector of Lagrange multipliers corresponding to the terminal boundary conditions.