A DETAILED DESCRIPTION OF THE DISCREPANCY IN FORMULAS FOR THE STANDARD ERROR OF THE DIFFERENCE BETWEEN A RAW AND PARTIAL CORRELATION: A TYPOGRAPHICAL ERROR IN OLKIN AND FINN (1995)

by

Chondra M. Lockwood and David P. MacKinnon

Contact:
David MacKinnon
480-727-6120
David.MacKinnon@asu.edu

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INTRODUCTION

This report documents the derivation of the variance of the difference between a raw (or simple) and partial correlation. This function is used as a measure of the mediated effect in the manuscript entitled "A Comparison of Methods to Test the Significance," by David P. MacKinnon, Chondra M. Lockwood, Jeanne M. Hoffman, Virgil Sheets, and Stephen G. West. A discussion follows of the discrepancies between derivations conducted by the authors of this manuscript and Olkin and Finn (1995) from which this method was drawn.

Olkin and Finn (1995)

Olkin and Finn (1995) present a solution for the variance of the difference between a simple correlation and the same correlation partialled for a third variable. This approach can provide a test of mediation. To the extent that the relationship between an independent variable (X) and a dependent variable (Y) is carried through a mediator (M), the correlation between X and Y will be reduced when partialled for M. This function is shown in Equation 1, where r_{xy} is the correlation between X and Y, r_{my} is the correlation between M and Y, and r_{xm} is the correlation between X and M.

$$olkin = r_{xy} - r_{xy.m} = r_{xy} - \frac{r_{xy} - r_{xm} r_{my}}{(1 - r_{my}^2)^{1/2} (1 - r_{xm}^2)^{1/2}} \tag{1}$$

THE MULTIVARIATE DELTA METHOD

The multivariate delta method for deriving the variance of a function requires a covariance matrix of the elements in the function as well as a vector of partial derivatives of the function with respect to each element. The variance estimate is the covariance matrix pre- and post-multiplied by the vector of derivatives.

In the cases described in this report, we have three elements: r_{xy}, r_{my}, and r_{xm}. For any function *f* of these elements, the multivariate delta variance formula is in Equation 2, where **Φ** is the variance-covariance matrix and **a** is the vector of partial derivatives.

$$\sigma_f^2 = a\Phi a' = \begin{bmatrix} \frac{\partial f}{\partial r_{xy}} & \frac{\partial f}{\partial r_{my}} & \frac{\partial f}{\partial r_{xm}} \end{bmatrix} \begin{bmatrix} \sigma_{r_{xy}}^2 & \sigma_{r_{xy} r_{my}} & \sigma_{r_{xy} r_{xm}} \\ \sigma_{r_{xy} r_{my}} & \sigma_{r_{my}}^2 & \sigma_{r_{my} r_{xm}} \\ \sigma_{r_{xy} r_{xm}} & \sigma_{r_{my} r_{xm}} & \sigma_{r_{xm}}^2 \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial r_{xy}} \\ \frac{\partial f}{\partial r_{my}} \\ \frac{\partial f}{\partial r_{xm}} \end{bmatrix} \tag{2}$$

Variances and Covariances among Correlations

Olkin and Siotani (1976) presented the formulas for asymptotic variances of and covariances among correlations. The formulas to complete the covariance matrix **Φ** from Equation 2 are given in Equations 3 through 8.

$$\sigma^2_{r_{xy}} = \frac{\left(1 - r^2_{xy}\right)^2}{N} \tag{3}$$

$$\sigma^2_{r_{my}} = \frac{\left(1 - r^2_{my}\right)^2}{N} \tag{4}$$

$$\sigma^2_{r_{xm}} = \frac{\left(1 - r^2_{xm}\right)^2}{N} \tag{5}$$

$$\sigma^2_{r_{xy}r_{my}} = \frac{\tfrac{1}{2}\left(2r_{xm} - r_{xy}r_{my}\right)\left(1 - r^2_{xm} - r^2_{xy} - r^2_{my}\right) + r^3_{xm}}{N} \tag{6}$$

$$\sigma^2_{r_{xy}r_{xm}} = \frac{\tfrac{1}{2}\left(2r_{my} - r_{xm}r_{xy}\right)\left(1 - r^2_{my} - r^2_{xm} - r^2_{xy}\right) + r^3_{my}}{N} \tag{7}$$

$$\sigma^2_{r_{my}r_{xm}} = \frac{\tfrac{1}{2}\left(2r_{xy} - r_{xm}r_{my}\right)\left(1 - r^2_{xm} - r^2_{xy} - r^2_{my}\right) + r^3_{xy}}{N} \tag{8}$$

Partial Derivatives - Olkin and Finn

The partial derivatives of the Olkin and Finn function (Equation 1) are listed in Equations 9 through 11. These will be the elements of a vector of partial derivatives called $\mathbf{a}_{\mathbf{olkin}}$. See Appendix A for more detailed explanation of the derivations.

$$a1_{olkin}\,\frac{\partial olkin}{\partial r_{xy}} = 1 - \frac{1}{(1 - r^2_{my})^{1/2}(1 - r^2_{xm})^{1/2}} \tag{9}$$

$$a2_{olkin=}\,\frac{\partial olkin}{\partial r_{my}} = \frac{r_{xm} - r_{xy}r_{my}}{(1 - r^2_{xm})^{1/2}(1 - r^2_{my})^{3/2}} \tag{10}$$

$$a3_{olkin} = \frac{\partial olkin}{\partial r_{xm}} = \frac{r_{my} - r_{xm}r_{xy}}{(1 - r^2_{my})^{1/2}(1 - r^2_{xm})^{3/2}} \tag{11}$$

Mathematica Derivations

As a check on the derivations, the derivations presented in Equations 9 through 11 were also conducted using Mathematica (Wolfram Research, 1996). The results of this program are shown in Appendix B and are identical to Equations 9 through 11.

TYPOGRAPHICAL ERROR IN THE OLKIN AND FINN DERIVATIONS

In Olkin and Finn (1995), the variance formula is presented in a simplified manner, rather than the multivariate delta presentation described above. The formula in their text is reproduced in Equation 12. A photocopy of the formula from their text (p. 160) is presented in Appendix C. Note that in Equation 12, the notation has been changed to be consistent with this report, i.e. in terms of r_{xy}, r_{my}, and r_{xm} rather than ρ_{01}, ρ_{02}, and ρ_{12}. Olkin and Finn (1995) use 0, 1 and 2 to refer to the variables Y, X and M, respectively.

$$olkin\ text\text{:}\operatorname{var}(r_{xy}-r_{xy.m})=\frac{a\Phi a'}{\left(1-r_{my}^2\right)\left(1-r_{xm}^2\right)} \tag{12}$$

The denominator of Equation 12 was removed from the partial derivatives in vector **a** presented in the text of Olkin and Finn (1995). The vector presented in the Olkin and Finn article is reproduced in Equation 13 and in Appendix C, where page 160 from the article is shown.

$$olkin\ text\text{:}a=\left[1-\sqrt{\left(1-r_{my}^2\right)\left(1-r_{xm}^2\right)},\quad \frac{r_{xy}r_{my}-r_{xm}}{1-r_{my}^2},\quad \frac{r_{xy}r_{my}-r_{my}}{1-r_{my}^2}\right] \tag{13}$$

Note that the numerator of Olkin and Finn's (1995) variance formula (Equation 12) looks equivalent to Equation 2. Because **Φ** is the same in the two equations, the difference is in the vector **a**. It appears that each element in **a** in the Olkin and Finn text has been divided by the quantity $\sqrt{\left(1-r_{my}^2\right)\left(1-r_{xm}^2\right)}$. However, after each partial derivative from Equation 13 was divided by this quantity, the results do not correspond exactly to the to the multivariate delta method presented earlier. The partial derivatives divided by this quantity are presented in Equations 14 through 16.

$$\frac{a1_{text}}{\sqrt{\left(1-r_{my}^2\right)\left(1-r_{xm}^2\right)}}=\frac{1}{\left(1-r_{my}^2\right)^{1/2}\left(1-r_{xm}^2\right)^{1/2}}-1 \tag{14}$$

$$\frac{a2_{text}}{\sqrt{\left(1-r_{my}^2\right)\left(1-r_{xm}^2\right)}}=\frac{r_{xy}r_{my}-r_{xm}}{\left(1-r_{my}^2\right)^{3/2}\left(1-r_{my}^2\right)^{1/2}} \tag{15}$$

$$\frac{a3_{text}}{\sqrt{\left(1-r_{my}^2\right)\left(1-r_{xm}^2\right)}}=\frac{r_{xy}r_{my}-r_{my}}{\left(1-r_{my}^2\right)^{3/2}\left(1-r_{xm}^2\right)^{1/2}} \tag{16}$$

Equations 14 and 15 are equivalent to Equations 9 and 10, respectively, when these quantities are squared as they are in Equations 2. The discrepancy is for the third partial derivative. Equation 16 is not equivalent to Equation 11.

Appendix D contains a description of where these methods are equivalent and where they are not. In sum, the discrepancy stems from the third element of **a**. We propose that this discrepancy stems from a typographical error in the third element of **a** in Olkin and Finn's (1995) text on page 160 of the article.

METHOD OF FINITE DIFFERENCES APPROACH

The method of finite differences for the two possible partial derivative solutions was programmed in SAS (see Appendix E).The program clearly showed that the partial derivative in equation 11 is correct.

REFERENCES

MacKinnon, D. P., Lockwood, C. M., Hoffman, J. M., Sheets, V., & West, S. G. (2000). *A Comparison of Methods to Test the Significance of the Mediated Effect*. Manuscript submitted to *Psychological Methods*.

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APPENDIX A

DETAILED DERIVATIONS OF OLKIN AND FINN FUNCTION

$$f = r_{yx} - \frac{r_{yx} - r_{ym} r_{mx}}{\sqrt{(1 - r_{ym}{}^{2})(1 - r_{mx}{}^{2})}}$$

$$let\ a = r_{yx},\ b = r_{ym},\ c = r_{mx}$$

$$\therefore\ f = a - \frac{a - u}{xv}$$

$$let\ u = bc,\ let\ v = (1 - c^2)^{\frac{1}{2}},\ let\ x = (1 - b^2)^{\frac{1}{2}}$$

$$\frac{\partial f}{\partial a} = a' + \left(\frac{-a + u}{xv}\right)'$$

1. Summation rule

$$\left(\frac{-a + u}{xv}\right)' = \frac{xv(-a + u)' - (-a + u)(xv)'}{(xv)^2}$$

2. Quotient rule

$$\frac{\partial f}{\partial a} = a' + \frac{xv(-a + u)' - (-a + u)(xv)'}{(xv)^2}$$

3. Substitution of results step 2 into step 1.

$$\frac{\partial f}{\partial a} = 1 + \frac{-xv + xv(u)' - (-a + u)(xv)'}{(xv)^2}$$

4. Derivative of a with respect to itself equals 1

$$u' = (bc)' = 0$$

5. Derivative of a constant equals 0

$$(xv)' = 0$$

6. Derivative of a constant equals 0

$$\frac{\partial f}{\partial a} = 1 + \frac{-(1 - b^2)^{\frac{1}{2}}(1 - c^2)^{\frac{1}{2}}}{\left((1 - b^2)^{\frac{1}{2}}(1 - c^2)^{\frac{1}{2}}\right)^2}$$

7. Substitution of steps 5 and 6 into step 4

$$\frac{\partial f}{\partial a} = 1 - \frac{1}{(1 - b^2)^{\frac{1}{2}}(1 - c^2)^{\frac{1}{2}}}$$

8. Equation reduces algebraically

$$\frac{\partial f}{\partial r_{yx}} = 1 - \frac{1}{\left(1-r_{ym}^{2}\right)^{\frac{1}{2}}\left(1-r_{mx}^{2}\right)^{\frac{1}{2}}}$$

9. Replace original elements.

$$\frac{\partial f}{\partial b} = a' + \left(\frac{-a+u}{xv}\right)'$$

1. Summation rule

$$\left(\frac{-a+u}{xv}\right)' = \frac{xv(-a+u)' - (-a+u)(xv)'}{(xv)^{2}}$$

2. Quotient rule

$$\frac{\partial f}{\partial b} = a' + \frac{xv(-a+u)' - (-a+u)(xv)'}{(xv)^{2}}$$

3. Substitution of results step 2 into step 1.

$$\frac{\partial f}{\partial b} = 0 + \frac{xv(0+u)' - (-a+u)(xv)'}{(xv)^{2}}$$

4. Derivative of a constant equals 0

$$= \frac{xv(u)' - (-a+u)(xv)'}{(xv)^{2}}$$

Equation reduces algebraically

$$u' = (bc)' = c$$

5. Derivative of $xf^{i}=ixf^{i-1}$

$$(xv)' = \left(1-c^{2}\right)^{\frac{1}{2}}\left(\tfrac{1}{2}\right)(-2b)\left(1-b^{2}\right)^{-\frac{1}{2}}$$

$$= -b\left(1-c^{2}\right)^{\frac{1}{2}}\left(1-b^{2}\right)^{-\frac{1}{2}}$$

6. Chain rule

Equation reduces algebraically

$$\frac{\partial f}{\partial b} = \frac{(1-b^{2})^{\frac{1}{2}}(1-c^{2})^{\frac{1}{2}}(c) - (-a+bc)(-b)(1-c^{2})^{\frac{1}{2}}(1-b^{2})^{-\frac{1}{2}}}{\left((1-b^{2})^{\frac{1}{2}}(1-c^{2})^{\frac{1}{2}}\right)^{2}}$$

7. Substitution of steps 5 and 6 into step 4.

$$\frac{\partial f}{\partial b} = \frac{(1-b^2)^{1/2}(1-c^2)^{1/2}(c) - \left(ab - cb^2\right)(1-c^2)^{1/2}(1-b^2)^{-1/2}}{(1-b^2)(1-c^2)}$$

$$= \frac{(1-b^2)^{-1/2}(1-c^2)^{1/2}\left[c\left(1-b^2\right) - \left(ab - cb^2\right)\right]}{(1-b^2)(1-c^2)}$$

$$= \frac{c - ab}{(1-b^2)^{3/2}(1-c^2)^{1/2}}$$

8. Reduce equation algebraically

$$\frac{\partial f}{\partial r_{ym}} = \frac{r_{mx} - r_{yx} r_{ym}}{\left(1 - r_{ym}^2\right)^{3/2}\left(1 - r_{mx}^2\right)^{1/2}}$$

9. Replace original elements.

$$\frac{\partial f}{\partial c} = a' + \left(\frac{-a+u}{xv}\right)'$$

1. Summation rule

$$\left(\frac{-a+u}{xv}\right)' = \frac{xv(-a+u)' - (-a+u)(xv)'}{(xv)^2}$$

2. Quotient rule

$$\frac{\partial f}{\partial c} = a' + \frac{xv(-a+u)' - (-a+u)(xv)'}{(xv)^2}$$

3. Substitution of results step 2 into step 1.

$$\frac{\partial f}{\partial c} = 0 + \frac{xv(0+u)' - (-a+u)(xv)'}{(xv)^2}$$

4. Derivative of a constant equals 0

$$= \frac{xv(u)' - (-a+u)(xv)'}{(xv)^2}$$

Equation reduces algebraically

$$u' = (bc)' = b$$

5. Derivative of $xf^i = ixf^{i-1}$

$$(xv)' = \left(1-b^2\right)^{1/2}\left(\tfrac{1}{2}\right)(-2c)\left(1-c^2\right)^{-1/2}$$

6. Chain rule

$$= -c\left(1-b^2\right)^{1/2}\left(1-c^2\right)^{-1/2}$$

Equation reduces algebraically

$$\frac{\partial f}{\partial c} = \frac{(1-b^2)^{1/2}(1-c^2)^{1/2}(b) - (-a+bc)(-c)(1-b^2)^{1/2}(1-c^2)^{-1/2}}{\left((1-b^2)^{1/2}(1-c^2)^{1/2}\right)^2}$$

7. Substitution of steps 5 and 6 into step 4.

$$\frac{\partial f}{\partial c} = \frac{(1-b^2)^{1/2}(1-c^2)^{1/2}(b) - \left(ac-bc^2\right)(1-b^2)^{1/2}(1-c^2)^{-1/2}}{(1-b^2)(1-c^2)}$$

8. Reduce equation algebraically

$$= \frac{(1-b^2)^{1/2}(1-c^2)^{-1/2}\left[b\left(1-c^2\right)-\left(ac-bc^2\right)\right]}{(1-b^2)(1-c^2)}$$

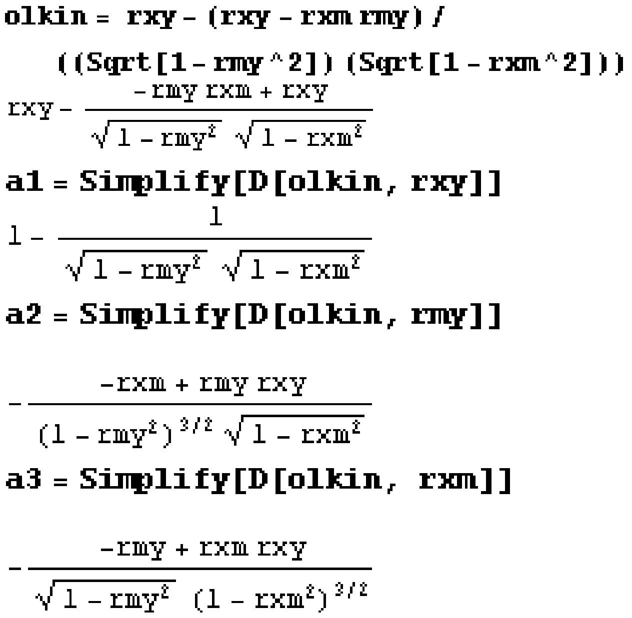
$$= \frac{b-ac}{(1-b^2)^{1/2}(1-c^2)^{3/2}}$$

$$\frac{\partial f}{\partial r_{mx}} = \frac{r_{ym} - r_{yx}r_{mx}}{\left(1-r_{ym}^2\right)^{1/2}\left(1-r_{mx}^2\right)^{3/2}}$$

9. Replace original elements.

APPENDIX B

MATHEMATICA DERIVATIONS



```
olkin = rxy - (rxy - rxm rmy) /
   ((Sqrt[1 - rmy ^ 2]) (Sqrt[1 - rxm ^ 2]))
```

$$rxy - \frac{-rmy\ rxm + rxy}{\sqrt{1 - rmy^2}\ \sqrt{1 - rxm^2}}$$

```
a1 = Simplify[D[olkin, rxy]]
```

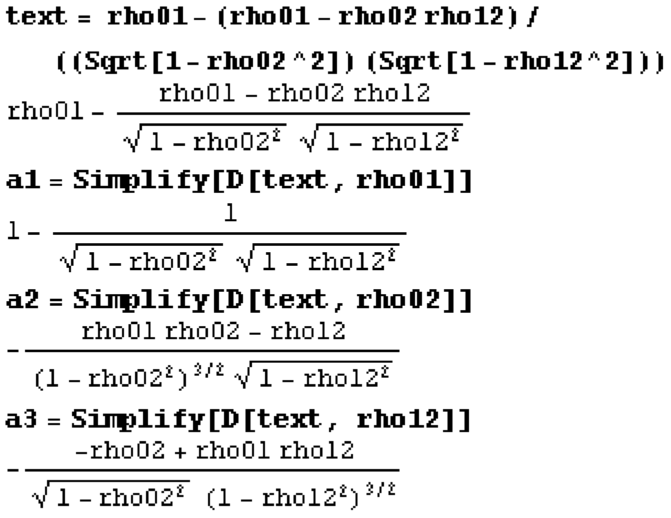
$$1 - \frac{1}{\sqrt{1 - rmy^2}\ \sqrt{1 - rxm^2}}$$

```
a2 = Simplify[D[olkin, rmy]]
```

$$-\frac{-rxm + rmy\ rxy}{(1 - rmy^2)^{3/2}\ \sqrt{1 - rxm^2}}$$

```
a3 = Simplify[D[olkin, rxm]]
```

$$-\frac{-rmy + rxm\ rxy}{\sqrt{1 - rmy^2}\ (1 - rxm^2)^{3/2}}$$



```
text = rho01 - (rho01 - rho02 rho12) /
   ((Sqrt[1 - rho02 ^ 2]) (Sqrt[1 - rho12 ^ 2]))
```

$$rho01 - \frac{rho01 - rho02\ rho12}{\sqrt{1 - rho02^2}\ \sqrt{1 - rho12^2}}$$

```
a1 = Simplify[D[text, rho01]]
```

$$1 - \frac{1}{\sqrt{1 - rho02^2}\ \sqrt{1 - rho12^2}}$$

```
a2 = Simplify[D[text, rho02]]
```

$$-\frac{rho01\ rho02 - rho12}{(1 - rho02^2)^{3/2}\ \sqrt{1 - rho12^2}}$$

```
a3 = Simplify[D[text, rho12]]
```

$$-\frac{-rho02 + rho01\ rho12}{\sqrt{1 - rho02^2}\ (1 - rho12^2)^{3/2}}$$

APPENDIX C

PAGE 160 FROM OLKIN AND FINN (1995)

in Equation 8: these are ϕ_{12}, ϕ_{14}, and ϕ_{24} in the present model. The remaining covariances, except for $\text{cov}(r_{02}, r_{13})$ and $\text{cov}(r_{03}, r_{12})$, have the same form. The two exceptions require Equation 5. Specifically,

$$\begin{aligned}\phi_{25} = \text{cov}(r_{02}, r_{13}) = [(\tfrac{1}{2})\rho_{02}\rho_{13}(\rho_{01}^2 + \rho_{03}^2 + \rho_{12}^2 + \rho_{23}^2) \\ + \rho_{01}\rho_{23} + \rho_{03}\rho_{12} - (\rho_{02}\rho_{01}\rho_{03} + \rho_{02}\rho_{12}\rho_{13} \\ + \rho_{01}\rho_{12}\rho_{13} + \rho_{03}\rho_{23}\rho_{13})]/n, \\ \phi_{34} = \text{cov}(r_{03}, r_{12}) = [(\tfrac{1}{2})\rho_{03}\rho_{12}(\rho_{01}^2 + \rho_{02}^2 + \rho_{13}^2 + \rho_{23}^2) \\ + \rho_{01}\rho_{23} + \rho_{02}\rho_{13} - (\rho_{03}\rho_{01}\rho_{02} + \rho_{03}\rho_{13}\rho_{23} \\ + \rho_{01}\rho_{13}\rho_{12} + \rho_{02}\rho_{23}\rho_{12})]/n. \quad (9)\end{aligned}$$

Substituting sample values in these expressions, the estimated variance–covariance matrix is

$$\hat{\Phi} = \frac{1}{1{,}415}\begin{pmatrix} .6598 & .1056 & .0870 & .1265 & .1302 \\ & .9226 & -.0169 & .3893 & .0052 \\ & & .9240 & .0066 & .3949 \\ & & & .9377 & -.0118 \\ & & & & .9532 \end{pmatrix}.$$

Consequently, $\hat{\mathbf{a}}\hat{\Phi}\hat{\mathbf{a}}' = .1461/1{,}415$ and $\hat{\sigma}_\infty = .0102$. The difference of the two squared multiple correlations is $-.002$, and a 95% confidence interval for $\rho_{0(12)}^2 - \rho_{0(13)}^2$ is $r_{0(12)}^2 - r_{0(13)}^2 \pm 1.96\hat{\sigma}_\infty = -.002 \pm (1.96)(.0102)$, which yields the interval $[-.022, .018]$. Because zero is contained in the interval, the hypothesis H: $\rho_{0(12)}^2 = \rho_{0(13)}^2$ would not be rejected by these data. Both friends' and parents' substance use, and friends' and classmates' substance use, predict eighth-grade substance use equally well. Both pairs of predictors account for approximately 20% of the variation in eighth-grade usage.

Model C: Determining the Effect of a Third Variable on the Association of Two Others

This model (and Model D) analyzes the extent to which the correlation of two variables x_0 and x_1 can be attributed to a third variable x_2. It is possible that an explanation for the correlation of x_0 with x_1 is provided by common features or behaviors reflected in x_2. If so, the partial correlation $\rho_{01\cdot 2}$ will be smaller than the zero-order correlation ρ_{01}. Model C can also be used to examine whether x_2 acts as a suppressor variable with respect to x_0 and x_1. To the extent that this is the case, the correlation after adjustment for x_2 (i.e., $\rho_{01\cdot 2}$) may be larger than the simple correlation ρ_{01}.

In this illustration, we explore the relationship of students' use of abusable substances in seventh grade with their families' use. Although the correlation is not particularly high ($r = .238$), we examine the extent to which it is attributable to SES common to parents and their children. The variables are x_0 = USE-7, x_1 = FAMILY-7, and x_2 = SES.

This procedure compares the simple correlation ρ_{01} with the partial correlation $\rho_{01\cdot 2}$, using estimates r_{01}, $r_{01\cdot 2}$, and $\hat{\sigma}_\infty^2 = \widehat{\text{var}}(r_{01} - r_{01\cdot 2})$. The variance is

$$\text{var}(r_{01} - r_{01\cdot 2}) = \mathbf{a}\Phi\mathbf{a}'/[(1 - \rho_{02}^2)(1 - \rho_{12}^2)],$$

where the vector

$$\mathbf{a} = \left(1 - \sqrt{(1 - \rho_{02}^2)(1 - \rho_{12}^2)}, \frac{\rho_{01}\rho_{02} - \rho_{12}}{1 - \rho_{02}^2}, \frac{\rho_{01}\rho_{02} - \rho_{02}}{1 - \rho_{02}^2}\right).$$

P is the symmetric matrix of correlations among x_0, x_1, and x_2, and Φ is the matrix of variances and covariances among the sample correlations. Both of these have the same form as in Model A, and the elements of Φ are given by Equations 7 and 8.

The sample correlation matrix for USE-7, FAMILY-7, and SES is

$$R = \hat{P} = \begin{pmatrix} 1.000 & 0.238 & 0.044 \\ & 1.000 & 0.089 \\ & & 1.000 \end{pmatrix}.$$

The estimate of ρ_{01} is $r_{01} = .238$. The estimate of $\rho_{01\cdot 2}$ is

$$\hat{\rho}_{01\cdot 2} = \frac{r_{01} - r_{02}r_{12}}{\sqrt{(1 - r_{02}^2)(1 - r_{12}^2)}} = .235.$$

Although the difference .003 is very small, we proceed with the analysis to illustrate the method.

Substituting sample values in the expressions for **a** and Φ,

$$\hat{\mathbf{a}} = (.00514, -.07725, -.03742)$$

and

$$\hat{\Phi} = \frac{1}{1{,}415}\begin{pmatrix} .8900 & .0780 & .0361 \\ & .9952 & .2334 \\ & & .9843 \end{pmatrix}.$$

Consequently, $\hat{\mathbf{a}}\hat{\Phi}\hat{\mathbf{a}} = .00861/1{,}415$ and $\hat{\sigma}_\infty = \sqrt{(.00861/1{,}415)/.9898} = .0025$.

The sample difference is $.238 - .235 = .003$, and a 95% confidence interval for the difference $\rho_{01} - \rho_{01\cdot 2}$ is $r_{01} - r_{01\cdot 2} \pm 1.96\hat{\sigma}_\infty = .003 \pm (1.96)(.0025)$, which yields the interval $[-.0017, .0080]$. We conclude that SES does not explain why a seventh grader's use of abusable substances is related to reported use by his or her family. The hypothesis H: $\rho_{01} = \rho_{01\cdot 2}$ is supported.

Model D: Deciding Which of Two Variables Has a Stronger Effect on the Association of Two Others

This model asks whether the correlation of two variables, x_0 and x_1, is explained better by the third variable x_2 or x_3. It involves a comparison of the partial correlation of x_0 and x_1 adjusted for the effects of x_2 with the partial correlation of x_0 and x_1 adjusted for x_3. In our illustration, we examine the relative stability of the use of abusable substances from seventh grade to eighth grade ($r = .599$). We ask whether this stability is explained better by the effect of family members as models during this period of time or by the possible influence of friends' usage. The variables are x_0 = USE-8, x_1 = USE-7, x_2 = FAMILY-7, and x_3 = FRIENDS-8.

This procedure compares the partial correlation $\rho_{01\cdot 2}$ with $\rho_{01\cdot 3}$, using estimates $r_{01\cdot 2}$, $r_{01\cdot 3}$, and $\hat{\sigma}_\infty^2 = \widehat{\text{var}}(r_{01\cdot 2} - r_{01\cdot 3})$. For this model, the variance is $\text{var}(r_{01\cdot 2} - r_{01\cdot 3}) = \mathbf{a}\Phi\mathbf{a}'$, where the vector $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$ has elements

APPENDIX D

TYPOGRAPHICAL ERROR IN OLKIN AND FINN (1995)

$f = r_{yx} \frac{r_{yx} - r_{ym}r_{mx}}{\sqrt{\left(1-r_{ym}^2\right)\left(1-r_{mx}^2\right)}}$ - difference between the raw and partial correlation

FIRST PARTIAL DERIVATIVE

$\frac{\partial f}{\partial r_{yx}} = 1 - \frac{1}{\left(1-r_{ym}^2\right)^{1/2}\left(1-r_{mx}^2\right)^{1/2}}$ Olkin and Finn $a1 = 1 - \sqrt{\left(1-r_{ym}^2\right)\left(1-r_{mx}^2\right)}$

How the quantities are related : $\frac{(a1)^2}{\left(1-r_{mx}^2\right)\left(1-r_{ym}^2\right)} = \left(\frac{\partial f}{\partial r_{yx}}\right)^2$

$$\frac{(a1)^2}{\left(1-r_{mx}^2\right)\left(1-r_{ym}^2\right)} = \frac{\left(1-\sqrt{\left(1-r_{ym}^2\right)\left(1-r_{mx}^2\right)}\right)^2}{\left(1-r_{mx}^2\right)\left(1-r_{ym}^2\right)}$$

$$= \frac{1-2\sqrt{\left(1-r_{ym}^2\right)\left(1-r_{mx}^2\right)} + \left(1-r_{ym}^2\right)\left(1-r_{mx}^2\right)}{\left(1-r_{mx}^2\right)\left(1-r_{ym}^2\right)}$$

$$\left(\frac{\partial f}{\partial r_{yx}}\right)^2 = \left(1 - \frac{1}{\left(1-r_{ym}^2\right)^{1/2}\left(1-r_{mx}^2\right)^{1/2}}\right)^2$$

$$= 1 - \frac{2}{\left(1-r_{ym}^2\right)^{1/2}\left(1-r_{mx}^2\right)^{1/2}} + \frac{1}{\left(1-r_{ym}^2\right)\left(1-r_{mx}^2\right)}$$

$$= \frac{\left(1-r_{ym}^2\right)\left(1-r_{mx}^2\right) - 2\left(1-r_{ym}^2\right)^{1/2}\left(1-r_{mx}^2\right)^{1/2} + 1}{\left(1-r_{ym}^2\right)\left(1-r_{mx}^2\right)}$$

$$= \frac{1 + -2\left(1-r_{ym}^2\right)^{1/2}\left(1-r_{mx}^2\right)^{1/2} + \left(1-r_{ym}^2\right)\left(1-r_{mx}^2\right)}{\left(1-r_{ym}^2\right)\left(1-r_{mx}^2\right)}$$

$f = r_{yx} \frac{r_{yx} - r_{ym} r_{mx}}{\sqrt{\left(1 - r_{ym}^2\right)\left(1 - r_{mx}^2\right)}}$ - difference between the raw and partial correlation

SECOND PARTIAL DERIVATIVE

$\frac{\partial f}{\partial r_{ym}} = \frac{r_{mx} - r_{yx} r_{ym}}{\left(1 - r_{mx}^2\right)^{1/2}\left(1 - r_{ym}^2\right)^{3/2}}$ Olkin and Finn $a2 = \frac{r_{yx} r_{ym} - r_{mx}}{1 - r_{ym}^2}$

How the quantities are related : $\frac{(a2)^2}{\left(1 - r_{mx}^2\right)\left(1 - r_{ym}^2\right)} = \left(\frac{\partial f}{\partial r_{ym}}\right)^2$

$$\frac{(a2)^2}{\left(1 - r_{mx}^2\right)\left(1 - r_{ym}^2\right)} = \frac{\left(\frac{r_{yx} r_{ym} - r_{mx}}{1 - r_{ym}^2}\right)^2}{\left(1 - r_{mx}^2\right)\left(1 - r_{ym}^2\right)}$$

$$= \frac{\left(r_{yx} r_{ym} - r_{mx}\right)^2}{\left(1 - r_{mx}^2\right)\left(1 - r_{ym}^2\right)^3}$$

$$\left(\frac{\partial f}{\partial r_{ym}}\right)^2 = \left(\frac{r_{mx} - r_{yx} r_{ym}}{\left(1 - r_{mx}^2\right)^{1/2}\left(1 - r_{ym}^2\right)^{3/2}}\right)^2$$

$$= \frac{\left(r_{mx} - r_{yx} r_{ym}\right)^2}{\left(1 - r_{mx}^2\right)\left(1 - r_{ym}^2\right)^3} = \frac{\left(r_{yx} r_{ym} - r_{mx}\right)^2}{\left(1 - r_{mx}^2\right)\left(1 - r_{ym}^2\right)^3}$$

$$f = r_{yx} \frac{r_{yx} - r_{ym} r_{mx}}{\sqrt{\left(1 - r_{ym}^2\right)\left(1 - r_{mx}^2\right)}}$$ - difference between the raw and partial correlation

THIRD PARTIAL DERIVATIVE

$$\frac{\partial f}{\partial r_{mx}} = \frac{r_{ym} - r_{mx} r_{yx}}{\left(1 - r_{mx}^2\right)^{1/2}\left(1 - r_{ym}^2\right)^{3/2}}$$

Olkin and Finn $a3 = \dfrac{r_{yx} r_{ym} - r_{ym}}{1 - r_{ym}^2}$

How the quantities are related : $\dfrac{(a3)^2}{\left(1 - r_{mx}^2\right)\left(1 - r_{ym}^2\right)} = \left(\dfrac{\partial f}{\partial r_{mx}}\right)^2$

$$\frac{(a3)^2}{\left(1 - r_{mx}^2\right)\left(1 - r_{ym}^2\right)} = \frac{\left(\dfrac{r_{yx} r_{ym} - r_{ym}}{1 - r_{ym}^2}\right)^2}{\left(1 - r_{mx}^2\right)\left(1 - r_{ym}^2\right)}$$

$$= \frac{\left(r_{yx} r_{ym} - r_{ym}\right)^2}{\left(1 - r_{mx}^2\right)\left(1 - r_{ym}^2\right)^3}$$

$$\left(\frac{\partial f}{\partial r_{mx}}\right)^2 = \left(\frac{r_{ym} - r_{mx} r_{yx}}{\left(1 - r_{mx}^2\right)^{3/2}\left(1 - r_{ym}^2\right)^{1/2}}\right)^2$$

$$= \frac{\left(r_{ym} - r_{mx} r_{yx}\right)^2}{\left(1 - r_{mx}^2\right)^3\left(1 - r_{ym}^2\right)} = \frac{\left(r_{mx} r_{yx} - r_{ym}\right)^2}{\left(1 - r_{mx}^2\right)^3\left(1 - r_{ym}^2\right)}$$

$$f = r_{yx} \frac{r_{yx} - r_{ym} r_{mx}}{\sqrt{\left(1 - r_{ym}^2\right)\left(1 - r_{mx}^2\right)}}$$ - difference between the raw and partial correlation

ADDENDUM - TYPOGRAPHICAL ERROR IN OLKIN AND FINN A3

If Olkin and Finn $a3* = \frac{r_{yx} r_{mx} - r_{ym}}{1 - r_{mx}^2}$, note that two r_{ym} elements have been replaced with r_{mx}

Assuming the typographical error : $\frac{(a3*)^2}{\left(1 - r_{mx}^2\right)\left(1 - r_{ym}^2\right)} = \left(\frac{\partial f}{\partial r_{mx}}\right)^2$

$$\frac{(a3*)^2}{\left(1 - r_{mx}^2\right)\left(1 - r_{ym}^2\right)} = \frac{\left(\frac{r_{yx} r_{mx} - r_{ym}}{1 - r_{mx}^2}\right)^2}{\left(1 - r_{mx}^2\right)^3\left(1 - r_{ym}^2\right)}$$

$$= \frac{\left(r_{yx} r_{mx} - r_{ym}\right)^2}{\left(1 - r_{mx}^2\right)^3\left(1 - r_{ym}^2\right)}$$

$$\left(\frac{\partial f}{\partial r_{mx}}\right)^2 = \left(\frac{r_{ym} - r_{mx} r_{yx}}{\left(1 - r_{mx}^2\right)^{3/2}\left(1 - r_{ym}^2\right)^{1/2}}\right)^2$$

$$= \frac{\left(r_{ym} - r_{mx} r_{yx}\right)^2}{\left(1 - r_{mx}^2\right)^3\left(1 - r_{ym}^2\right)} = \frac{\left(r_{mx} r_{yx} - r_{ym}\right)^2}{\left(1 - r_{mx}^2\right)^3\left(1 - r_{ym}^2\right)}$$

APPENDIX E

SAS PROGRAM FOR THE METHOD OF FINITE DIFFERENCES

```
title 'method of finite differences verification of derivatives';
data a;
input rxy rmy rxm;
/*Function for the difference between raw and partial correlation;*/
diffr=rxy-((rxy-rxm*rmy)/((sqrt(1-rmy*rmy))*(sqrt(1-rxm*rxm))));

/*Our derivative; */
us=(rmy-rxm*rxy)
/((sqrt(1-rmy*rmy))*((1-rxm*rxm)**(3/2)));

/*derivative from Olkin and Finn 1995 page 160';*/
olkin=(rxy*rmy-rmy)/((sqrt(1-rxm*rxm))*((1-rmy*rmy)**(3/2)));

do i=.00001 to .0001 by .00001;
rxm=.8+i;
fn=rxy-((rxy-rxm*rmy)/((sqrt(1-rmy*rmy))*(sqrt(1-rxm*rxm))));
fdiff=(fn-diffr)/i;
output;
end;

cards;
.1 .2 .8
;
proc print;
run;
```

```
                   method of finite differences verification of derivatives                  6

OBS    RXY    RMY      RXM       DIFFR        US        OLKIN        I         FN         FDIFF

  1    0.1    0.2    0.30001    0.057204    0.19987    -0.20061    .00001    0.057206    0.19987
  2    0.1    0.2    0.30002    0.057204    0.19987    -0.20061    .00002    0.057208    0.19987
  3    0.1    0.2    0.30003    0.057204    0.19987    -0.20061    .00003    0.057210    0.19987
  4    0.1    0.2    0.30004    0.057204    0.19987    -0.20061    .00004    0.057212    0.19987
  5    0.1    0.2    0.30005    0.057204    0.19987    -0.20061    .00005    0.057214    0.19987
  6    0.1    0.2    0.30006    0.057204    0.19987    -0.20061    .00006    0.057216    0.19987
  7    0.1    0.2    0.30007    0.057204    0.19987    -0.20061    .00007    0.057218    0.19987
  8    0.1    0.2    0.30008    0.057204    0.19987    -0.20061    .00008    0.057220    0.19987
  9    0.1    0.2    0.30009    0.057204    0.19987    -0.20061    .00009    0.057222    0.19988
 10    0.1    0.2    0.30010    0.057204    0.19987    -0.20061    .00010    0.057224    0.19988
                    method of finite differences verification of derivatives                  7

 OBS    RXY    RMY      RXM       DIFFR        US        OLKIN        I         FN        FDIFF

   1    0.1    0.2    0.80001    0.20206    0.56701    -0.31894    .00001    0.20207    0.56703
   2    0.1    0.2    0.80002    0.20206    0.56701    -0.31894    .00002    0.20207    0.56704
   3    0.1    0.2    0.80003    0.20206    0.56701    -0.31894    .00003    0.20208    0.56706
   4    0.1    0.2    0.80004    0.20206    0.56701    -0.31894    .00004    0.20208    0.56708
   5    0.1    0.2    0.80005    0.20206    0.56701    -0.31894    .00005    0.20209    0.56709
   6    0.1    0.2    0.80006    0.20206    0.56701    -0.31894    .00006    0.20210    0.56711
   7    0.1    0.2    0.80007    0.20206    0.56701    -0.31894    .00007    0.20210    0.56713
   8    0.1    0.2    0.80008    0.20206    0.56701    -0.31894    .00008    0.20211    0.56714
   9    0.1    0.2    0.80009    0.20206    0.56701    -0.31894    .00009    0.20211    0.56716
  10    0.1    0.2    0.80010    0.20206    0.56701    -0.31894    .00010    0.20212    0.56718
```