Permutations in Concatenated Zigzag Codes

California State Polytechnic University, Pomona

and

Loyola Marymount University

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Shawn Abernethy Jr.,[*] Cindy Lee,[†] Jasmin Uribe,[‡] Sai Michael Wentum Jr.,[§]
Laura Smith,[¶] Edward Mosteig[‖]

Applied Mathematical Sciences Summer Institute
Department of Mathematics & Statistics
California State Polytechnic University Pomona
3801 W. Temple Ave.
Pomona, CA 91768

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[*]Elizabethtown College
[†]Loyola Marymount University
[‡]University of Arizona
[§]College of Charleston
[¶]University of California, Los Angeles
[‖]Loyola Marymount University, Los Angeles

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Abstract

Coding theory is a branch of mathematics, computer science and electrical engineering that explores the transmission of information across noisy channels. Coding theory is used in data transmission, data storage, and telecommunications. The focus of this project is on concatenated zigzag codes, which are constructed using permutations. We study the effects of permutations on the error-correcting capabilities of the coding scheme. In conjunction, we explore the behavior of average dispersion in order to further our understanding of randomness of a permutation and find correspondence with error-correction.

1 Introduction

Coding theory is the study of the transmission of data across a noisy channel with the ultimate goal of successfully recovering the data to its original form from an error-ridden message. Since a message may be corrupted upon transmission across the noisy channel, error-correcting measures must be taken in order to ensure that the receiver can reconstruct the original codeword. This study works to improve the error-correcting capabilities of certain codes while maintaining reasonable transmission time.

It is important to note that coding theory is neither cryptography nor steganography. Although the term coding theory is often confused with both of these forms of communication, coding theory is an entirely different field of study. Steganography is literally translated as *covered writing* and is not largely used in practice. It simply refers to sending messages hidden from the naked eye. One example is tattooing a message on someone's head and then letting their hair grow, thus covering up the message. The receiver, however, knows where the message is located and simply removes the hair and reads the message. Steganography is not secure because an interceptor only needs to find the location of the message. Cryptography, on the other hand, is much more secure. It literally means *hidden writing*, which means that even if the message is intercepted only the sender and receiver have the predetermined algorithm that encrypts and decrypts the message. In coding theory, however, one does not attempt to hide the message; rather, one adds redundancy to a message to overcome errors that are introduced during transmission.

The performance of an encoding scheme is measured by its error-correcting capabilities and its rate, which is defined as the ratio of the number of information bits to the number of total bits sent.

$$\text{rate} = \frac{\#\text{ information bits}}{\#\text{ total bits}}. \tag{1}$$

Higher rates correspond to faster transmission since more information bits are transmitted per total bits. However, transmissions with high rates are more prone to uncorrectable errors because there is less redundancy present in the transmission. Thus, it is desirable to find an ideal rate that minimizes transmission time but maximizes error correction.

Coding theory is used in many places. An example of this is NASA, which uses coding theory for sending and receiving transmissions across space. It can also be found in everyday life ranging from data storage in CDs and DVDs to wireless communications. Currently, engineers from Nokia are applying for patents regarding specific components of a class of codes called concatenated zigzag codes. The focus of our research is on optimizing the error-correcting capabilities of concatenated zigzag codes.

2 Objective

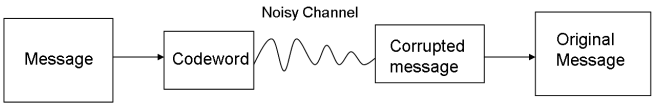
This project explores the relationship between permutations and their role in the error-correcting capabilities of concatenated zigzag codes. Through examining permutations and analyzing their effects on error-correction, we hope to find a specific type of permutation that reduces error rates. One of the properties we focus on is the behavior of the dispersion of permutations. Dispersion is a measure of the randomness of a permutation. In conjunction,

we explore the behavior of average dispersion in order to further our understanding of the randomness in a permutation and find a correspondence with error-correction, if any. Finally, we create a new way of analyzing permutations using variance and group dispersion tables.

3 Zigzag Codes and Concatenated Zigzag Codes

The coding process is made up of three basic parts: encoding, transmission through noisy channels, and decoding.

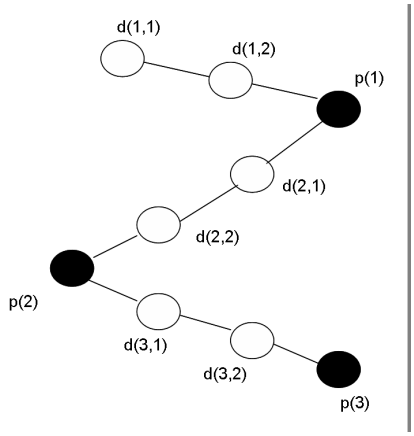
A message is encoded using a predetermined coding scheme, which can be as simple or as complex as the sender desires. A **codeword** is defined as any output of an encoding scheme. The codeword is what is transmitted. During transmission, noise may introduce errors to the codeword, resulting in corruption of the message. This error-ridden message is then received by the intended recipient and decoded using specific decoding algorithms.



3.1 Zigzag Codes

Zigzag codes can be described using the following graph, where the white nodes are information bits and the black nodes are parity bits. We use I to denote the number of segments in the graph and J to denote the number of information bits on each segment. In other words, I denotes the total number of parity bits and J denotes the number of information bits per parity bit. In this particular example, $I = 3$ and $J = 2$.

The information bits are denoted using the notation $d(i, j)$, where i is the number of the segment on which the node lies and j refers to the specific position within that segment. The parity bits are denoted by $p(i)$, where the corresponding node lies at the end of segment i.



In the case of zigzag codes the encoding process consists of computing the redundant parity bits. These parity bits are calculated by adding all previous nodes on the same segment using the following equations:

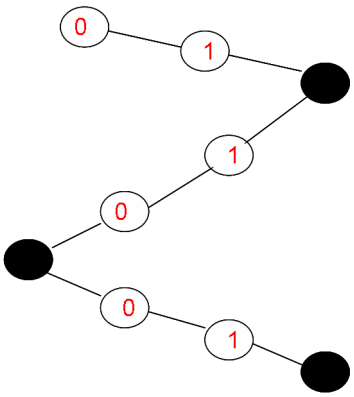
$$p(1) = \sum_{j=1}^{J} d(i,j) \bmod 2;$$

$$p(i) = \sum_{j=1}^{J} d(i,j) + p(i-1) \bmod 2, \quad \text{for } i = 2, 3..., I.$$

For example, consider the message [011001]. With the aid of the diagram, we compute the parity bits.

To calculate the first parity bit, the bits of the first segment are added together modulo two:

$$p(1) = d(1,1) + d(1,2) = 0 + 1 = 1.$$



The second parity bit is the sum of the first parity bit and the information bits lying on the segment:

$$p(2) = p(1)+d(2,1)+d(2,2) = 1+1+0 = 2 \bmod 2 = 0.$$

The subsequent parity bits are calculated in a similar manner.

Redundancy introduced by the parity bits ensures that more errors will be corrected during the decoding process, thus increasing the error-correcting capabilities of the code.

To facilitate the encoding and decoding process, the information bits and parity bits are stored in the matrix D and the vector P, respectively.

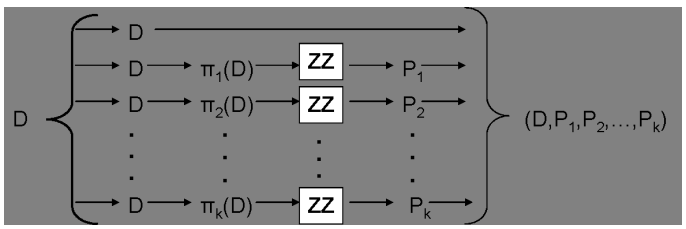
$$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Referring to the zigzag code diagram, we create matrices for calculations. The $I \times J$ matrix, D, is derived by taking each of the information bits from a given segment of the diagram and placing them in the corresponding row of D. The column vector P is obtained by reading the parity bits from the zigzag diagram top to bottom.

3.2 Concatenated Zigzag Codes

Zigzag codes by themselves have poor error-correcting capabilities. However, they have excellent transmission rates, namely $\frac{J}{J+1}$. Concatenated zigzag codes are used to increase error-correction. Although These codes have excellent error-correcting capabilities, they have a lower rate.

Concatenated zigzag codewords are created by first creating K copies of the message D. Each replica is fed through one of the permutations, $\pi_1, \pi_2, ...\pi_K$ (for background info on permutations, see Section 4.1). For each $\pi_k(D)$, the parity vector P_k is computed by using a constituent zigzag code. Finally, the codeword is formed by concatenating the matrix D with the parity vectors $P_1, P_2, ..., P_K$.



The figure above is a graphical representation of the encoding process using a concatenated zigzag encoder. The message, D, is replicated, permuted, and encoded. Then the parity vectors, $P_1, P_2, ..., P_K$, are concatenated with D and sent through the noisy channel as the codeword.

To decode, we implemented the Max-Log-APP (MLA) process of decoding zigzag codes originally presented by Li Ping in [Pi(2)] and later presented by Tejas Bhatt and Victor Stolpman in [Bh]. Using MLA, the decoding process computes forward and backward extrinsic information for the parity bits of the k^{th} permutation for each of the sub-iterations. This constructs a single iteration of the decoding process.

To form the codeword, the $I \times J$ message matrix D and the parity vectors $P_1, P_2, ..., P_K$ are concatenated. The 0s of the codeword are then changed to 1s and the 1s are then changed to -1s. It is this modulated codeword that is transmitted to the receiver.

The receiver then reconstructs the message matrix, and the parity vectors received with possible corruption. The received message matrix is denoted $\widetilde{D}$ and the parity vectors as $\widetilde{P}_1, \widetilde{P}_2, ..., \widetilde{P}_K$. Let $\widetilde{P}_k$ be the received parity vector corresponding to the k^{th} permutation. The receiver then uses the parity vectors to correct any errors introduced to the received message with the following equations:

- Max-log approximation:

$$W(z_1, z_2, ..., z_n) = \left[\Pi_{j=1}^{n} \text{sign}(z_j)\right] \cdot \min_{1 \le j \le n} |z_j|$$

- Forward recursion:

$$F^{[q]}(p_k(i)) = \widetilde{p}_k(i) + W\left(F^{[q]}(p_k(i-1)), L_o^{[q]}(d_k(i,1)), ..., L_o^{[q]}(d_k(i,J))\right)$$

  where $i = 1, 2, \ldots, I$ and $F^{[q]}(p_k(0)) = +\infty$

- Backward recursion:

$$B^{[q]}(p_k(i-1)) = \widetilde{p}_k(i-1) + W\left(L_o^{[q]}(d_k(i,1)), ..., L_o^{[q]}(d_k(i,J)), B^{[q]}(p_k(i))\right)$$

  where $i = I-1, ..., 2, 1$ and $B^{[q]}(\widetilde{p}_k(I)) = \widetilde{p}(I)$

- Extrinsic Information

$$L_e^{[q]}(d_k(i,j)) = W\left(\begin{array}{c} F^{[q]}(p_k(i-1)), L_o^{[q]}(d_k(i,1)), ..., L_o^{[q]}(d_k(i,j-1)), \\ L_o^{[q]}(d_k(i,j-1)), ..., L_o^{[q]}(d_k(i,J)), B^{[q]}(p_k(i)) \end{array}\right)$$

$$L_o^{[q]}(d_k(i,j)) = \pi_k\left[\widetilde{d}(i,j) + \sum_{k'<k}[\pi_{k'}^{-1}[L_e^{[q]}(d_{k'}(i,j))]] + \sum_{k'>k}[\pi_{k'}^{-1}[L_e^{[q-1]}(d_{k'}(i,j))]]\right],$$

  where L_o is initialized as an $I \times J$ matrix of zeros.

- Final Log Likelihood Ratio Computation

$$L^{[q]}(d(i,j)) = \widetilde{d}(i,j) + \sum_{k=1}^{K} \pi_k^{-1}\left[L_e^{[q]}(d_k(i,j))\right]$$

The receiver then takes the sign of all the entries in the $I \times J$ matrix $L^{[q]}$. The larger the magnitudes of the entries in $L^{[q]}$, the more probable that entry is a 1 or -1. The receiver then demodulates the matrix to construct the message – hopefully the original message sent.

4 Permutations, Dispersion and Variance

As we previously stated, the focus of our project is to determine which sets of permutations optimize error-correction in concatenated zigzag codes. One of the focuses of this section is on dispersion, a measure of the randomness of a permutation. Since we are working with a set of permutations, using the dispersion of a single permutation will not suffice as a predictor of performance of concatenated zigzag codes. This requires the creation of what we call group dispersion tables. In addition, we study the variance of a permutation, which is a possible way of measuring how a certain row is permuted with respect to other rows of the message matrix.

4.1 Permutations and Dispersion

Definition 1. A permutation of set size n is defined to be any bijection $\pi : [n] \to [n]$ where $[n]$ denotes $\{1, 2, 3, ..., n\}$.

Consider the following permutation of [5]:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix}.$$

This notation represents $\pi(1) = 4$, $\pi(2) = 3$, $\pi(3) = 1$, $\pi(4) = 5$ and $\pi(5) = 2$.

To measure the randomness of a permutation we use a metric known as dispersion. To properly define the dispersion of a permutation, we need to make a preliminary construction.

Definition 2. Given a permutation $\pi : [n] \to [n]$, the **list of differences** of π is given by

$$D_L(\pi) = \{(j - i, \pi(j) - \pi(i)) \mid 1 \leq i < j \leq n\}. \tag{2}$$

Let $|D_L(\pi)|$ denote the number of unique elements of D_L.

Definition 3. The **dispersion** of π is a normalized measure of the number of unique ordered pairs that appears in the list of differences, computed as:

$$\text{disp}(\pi) = \frac{|D_L(\pi)|}{\binom{n}{2}}. \tag{3}$$

Note that $0 < \text{disp}(\pi) \leq 1$ for any permutation, where a dispersion of 0 indicates a highly structured permutation and 1 is a random permutation. To illustrate this definition, we compute the dispersion of the following permutation:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

The elements of $D_L(\pi)$ are given in the following table:

Finally, we find the dispersion by counting the number of distinct ordered pairs in $D_L(\pi)$ and dividing by $\binom{n}{2}$. In this particular case, there are two repeated ordered pairs, (1,1) and (2,-2), and so there are only four unique ordered pairs. Therefore,

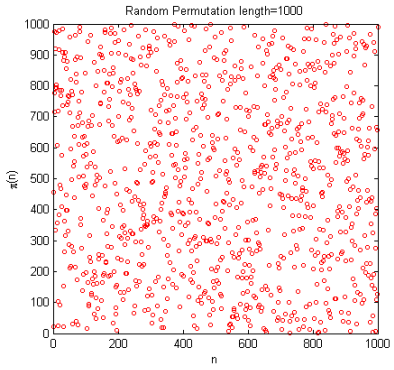
$$\text{disp}(\pi) = \frac{|D_L(\pi)|}{\binom{n}{2}} = \frac{4}{\binom{4}{2}} = \frac{4}{6} = \frac{2}{3}.$$

		i			
		1	2	3	4
j	1	*	*	*	*
	2	(1,1)	*	*	*
	3	(2,-2)	(1,-3)	*	*
	4	(3,-1)	(2,-2)	(1,1)	*

Figure 1: Table 4.1

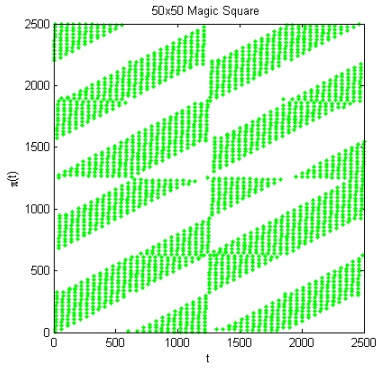
4.1.1 Permutations Generated from Matlab's `randperm` Command

An example of a permutation with a high dispersion, is a random permutation. One way to generate a random permutation is to use the built-in command `randperm` of Matlab. The graph below shows a random permutation with set size 1000 and dispersion 0.8133. The average dispersion yielded by a random permutation of Matlab is approximately 0.81.



4.1.2 Permutation Generated from Matlab's magic command

A permutation can also be created by using the magic square command, `magicsquare`, provided in Matlab. It produces a $n \times n$ matrix such that each integer from 1 to n^2 appears exactly once as an entry and all the row sums and column sums are identical. To create a permutation from a magic square, we reshape the matrix into a row vector of length n^2. This is done by taking consecutive rows and placing them one after another to form a row vector. The following is a graphical representation of a permutation constructed from a 50×50 magic square. This permutation has a dispersion of 0.016. The graph demonstrates the structure of the magic square and its correspondence to a low dispersion.



4.2 Theoretical Results Concerning Permutations

While working with permutations, we found the following results.

Theorem 4. *Define $\pi_1 : [n] \to [n]$ to be the identity permutation and define $\pi_2 : [n] \to [n]$ by $\pi_2(i) = n - i$. Then for any permutation $\pi : [n]$ to $\mathbb{Z}$,*

$$disp(\pi_1) = disp(\pi_2) \leq disp(\pi).$$

Proof. Due to the construction of a triangular difference table, the minimum number of distinct pairs in $D_L(\pi)$ is $n - 1$ for any permutation π. We see that

$$\begin{aligned} \pi_1(j - i) - \pi_1(j) &= -i, \\ \pi_2(j - i) - \pi_2(j) &= i, \end{aligned}$$

and so the entries of each diagonal are identical; thus π_1 and π_2 have the smallest possible dispersion. □

Corollary 5. *For any block length n, the minimum dispersion of a permutation is always $\frac{2}{n}$.*

Proof. By Theorem 4 the identity permutation has the smallest dispersion, and it is enough to show it has dispersion $\frac{2}{n}$. The triangular difference table of the identity has $n - 1$ unique entries, and so the dispersion of the identity is

$$\frac{n-1}{\frac{n(n-1)}{2}} = \frac{2}{n}.$$

□

4.3 Behavior of Average Dispersion

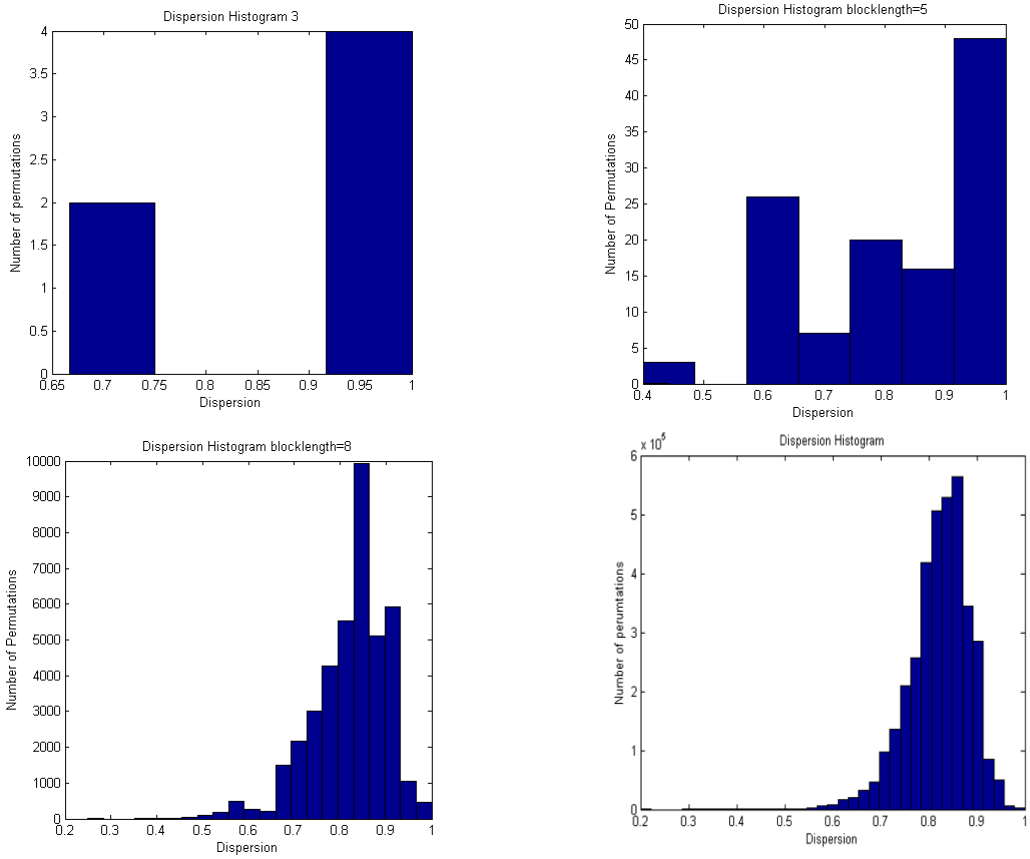
To further understand dispersion and any possible correlation it has with error-correction, we examine the histograms of permutations with respect to dispersion. Average dispersion is computed by summing the values of dispersion over all permutations of a given set size and

then dividing by the total number of permutations of that set size. The following equation is used to calculate the average dispersion of all permutations of set size n:

$$\text{average disp}(n) = \frac{1}{n!} \sum_{\pi \in S_n} \text{disp}(\pi).$$

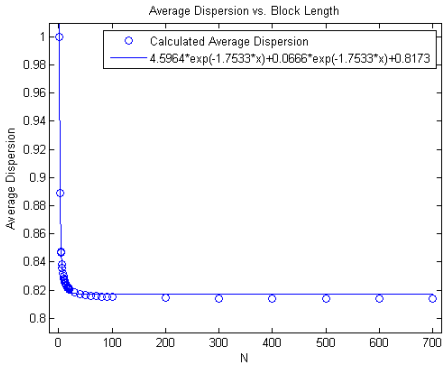
For set size 2 the average dispersion of a given permutation is 1; since there are 2 possible permutations both with dispersion 1, thus the average dispersion is $\frac{1+1}{2} = 1$. For set sizes up to 10, it is possible to compute the dispersion of all permutations. However, for larger set sizes, only a sample of the total number of permutations is used to calculate the average due to constraints on computing power and time.

The following histograms show the distribution of permutations with respect to dispersion.



From these distributions, it can be seen that as the set size increases, the proportion of permutations with dispersion between 0.8 and 0.9 increases. Since producing a histogram for the dispersion of all permutations of set size $n > 10$ is computationally infeasible, sampling is necessary due to the memory constraints of our computers.

The graph below demonstrates the asymptotic behavior of the average dispersion with respect to set size. In fact, the average dispersion approaches a constant around 0.81.



The behavior demonstrated by the distribution of average dispersion can be approximated by a sum of two exponential functions: $\text{disp}(n) = 4.5964^{-1.7533n} + 0.0666^{1.7533n} + 0.8173$. It is unknown whether average dispersion has an asymptote at a value around 0.81, or whether it approaches 0. The following table shows the set size, the average dispersion, the standard deviation and the sample size of each set size examined (if n >10). The table suggests the standard deviation is approaching 0 and the dispersion is approaching a value around 0.81.

n	Average Dispersion	Standard Deviation	Sample Size
2	1	0	2
3	0.8888888	0.1721326	6
4	0.8472222	0.1766343	24
5	0.8466666	0.1494715	120
6	0.8383333	0.1273568	720
7	0.8355253	0.0656522	5040
8	0.8318895	0.0866602	40320
9	0.8300064	0.0737352	362880
10	0.8280427	0.0656522	3628800
11	0.8268012	0.0581352	30000000
12	0.8254789	0.0534279	30000000
13	0.8246220	0.0488241	30000000
14	0.8237070	0.0451643	30000000
15	0.8230517	0.0419172	30000000
16	0.8223984	0.0391362	30000000
17	0.8218721	0.0366868	30000000
18	0.8213694	0.0345478	30000000
19	0.8209739	0.0326291	30000000
20	0.8205578	0.0309247	30000000
30	0.8182114	0.0203068	30000000
40	0.8170504	0.0151259	10000000
50	0.8163426	0.0120574	10000000
60	0.8158753	0.0100037	10000000
70	0.8155449	0.0085911	10000000
80	0.8152998	0.0075097	10000000
90	0.8151062	0.0066752	10000000
100	0.8149514	0.0060031	10000000
200	0.8142764	0.0030067	10000000
300	0.8140503	0.0020198	10000000
400	0.8139387	0.0015189	10000000
500	0.8138702	0.0012231	10000000
600	0.8138251	0.0010271	10000000
700	0.8137924	0.0008843	10000000

4.4 Variance of a Permutation

In general, there are a lot of ways to construct permutations. For various classes of permutations, their performance in concatenated zigzag codes is well-known. For example, as we shall see in this section, any of the classical block permutations have very poor performance. In contrast, randomly-generated permutations have excellent performance. Since there are so many different permutations it would be useful if there were a way to quantify how well a permutation would perform before running time-intensive simulations.

Definition 6. Define the symmetric group, S_n, to be the collection of all the permutations of $[n] = \{1, 2, \ldots, n\}$.

Definition 7. Given integers I, J and a set S, denote the set of all $I \times J$ matrices whose entries take on values from the set S by

$$\mathrm{Mat}_{I,J}(S).$$

Definition 8. Given positive integers I, J, define

$$\rho_{I,J} : S_{IJ} \to \mathrm{Mat}_{I,J}(\{1, \ldots, IJ\})$$

to be the function that takes a permutation $\pi \in S_{IJ}$ and produces the $I \times J$ matrix whose (i, j) entry is $\pi((i-1)J + j)$. That is, $\rho_{I,J}$ places the values $\pi(1), \pi(2), \ldots, \pi(IJ)$ in an $I \times J$ matrix row-by-row.

Definition 9. Given positive integers I, J, define

$$\beta_{I,J} : S_{IJ} \to \mathrm{Mat}_{I,J}(\{1, \ldots, IJ\})$$

to be the function that takes in a permutation π and produces the $I \times J$ matrix, the **row correlation matrix** $RCM_{I,J}(\pi)$ whose $(i, j)^{th}$ entry is

$$\left\lceil \frac{\rho_{I,J}(\pi)}{I} \right\rceil.$$

Definition 10. Let $\pi \in S_n$ be a permutation, and let I, J be positive integers such that $n = IJ$. Then the (I, J)-**variance of a permutation** is the sum of the population variance of the rows of $RCM_{I,J}(\pi)$; that is,

$$\mathrm{var}_{I,J}(\pi) = \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{1}{J} (m_{ij} - \frac{1}{j} \sum_{j=1}^{J} m_{i,j})^2,$$

where $M = (m_{i,j}) = \beta_{I,J}(\pi)$.

Example 11. Consider the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 2 & 9 & 4 & 7 & 6 & 3 & 8 & 1 & 5 \end{pmatrix}.$$

Evaluating the mapping $\beta_{2,5}$ at the permutation π, we get

$$\beta_{2,5}(\pi) = \begin{pmatrix} 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 & 1 \end{pmatrix}.$$

The mean of the first row is

$$\frac{2+1+2+1+2}{5} = 1.6,$$

and the mean of the second row is

$$\frac{2+1+2+1+1}{5} = 1.4.$$

The population variance of the first and second rows of $\beta_{2,5}(\pi)$ are

$$\frac{(2-1.6)^2+(1-1.6)^2+(2-1.6)^2+(1-1.6)^2+(2-1.6)^2}{5} = .24$$

and

$$\frac{(2-1.4)^2+(1-1.4)^2+(2-1.4)^2+(1-1.4)^2+(1-1.4)^2}{5} = .24,$$

respectively. Thus, the $(2,5)$-variance of π is 0.48.

A problem with the variance of a permutation is that unlike dispersion, variance does not have a natural normalized value between 0 and 1. Since variance tends to increase with set size, it is hard to compare permutations of significantly different set sizes. Thus, for a given set size, we wish to create an upper-bound for the variance. We would then be able to divide the variance by the upper bound, thus producing a normalized value that ranges between 0 and 1. We make a conjecture concerning this upper bound in the case $I = J$.

Proposition 12. *Let $\pi : [n^2] \to [n^2]$ be a permutation such that every row of $\beta_{n,n}(\pi)$ consists of the entries 1 through n in some order. Then the (n,n)-variance of $\beta_{n,n}(\pi)$ is*

$$\frac{n^3-n}{12}.$$

Proof. Let x be a row that appears in $\beta_{n,n}(\pi)$. Then x is a vector with n entries and it has one occurrence of each of the numbers $1, 2, \ldots, n$. By Definition 10, the variance of one row is

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

where $\bar{x} = \frac{1}{n}\sum x_i$. Thus, the total variance for the n rows is

$$n \cdot \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}(i - \bar{x})^2 \tag{4}$$

It is well known that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$

and so

$$\begin{aligned}
\sum_{i=1}^{n}(i-\bar{x})^2 &= \sum_{i=1}^{n}\left(i-\frac{(n+1)}{2}\right)^2 \\
&= \sum_{i=1}^{n}\left(i^2-i(n+1)+\frac{(n+1)^2}{4}\right) \\
&= \sum_{i=1}^{n}i^2-\sum_{i=1}^{n}i(n+1)+\sum_{i=1}^{n}\frac{(n+1)^2}{4} \\
&= \sum_{i=1}^{n}i^2-(n+1)\sum_{i=1}^{n}i+\frac{(n+1)^2}{4}\sum_{i=1}^{n}1 \\
&= \frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)(n+1)}{2}+\frac{n(n+1)^2}{4} \\
&= n(n+1)\left(\frac{2(2n+1)-6(n+1)+3\,(n+1)}{12}\right) \\
&= n(n+1)\left(\frac{(4n+2)-(6n+6)+(3n+3)}{12}\right) \\
&= \frac{n(n+1)(n-1)}{12} \\
&= \frac{n^3-n}{12}.
\end{aligned}$$

□

Conjecture 13. For any $\pi \in S_{n^2}$ the maximum (n,n)-variance possible for π is $\frac{n^3-n}{12}$.

4.5 Group Dispersion Table

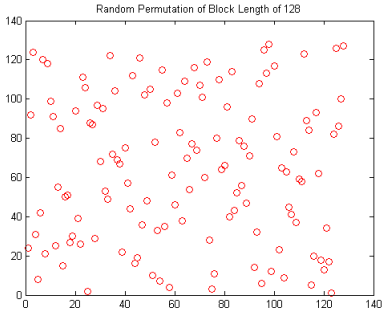
Since concatenated zigzag codes utilize a set of permutations, the dispersion of a single permutation is not the defining measure of the performance of the set. A possible predictive measure of the performance of a set of permutations, however, may be found in the group dispersion table.

Definition 14. Given a set of K permutations, the **group dispersion table** is a lower-triangular matrix M such that

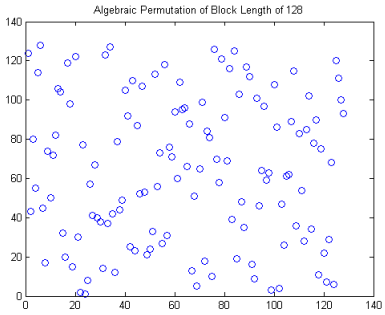
$$M(i,j)=\begin{cases} D(\pi_i\pi_j^{-1}) & \text{for } i<j; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 15. Given permutations $\pi_1,\ldots,\pi_k \in S_n$, the **average dispersion distance** of $\{\pi_1,\ldots,\pi_k\}$ is the average of the nonzero entries of the group dispersion table.

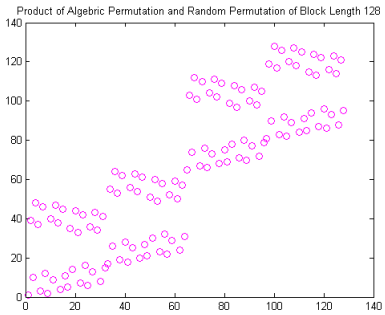
The following graphs demonstrate that two permutations π_1, π_2 with high dispersion can yield a permutation $\pi_1\pi_2^{-1}$ with low dispersion.



Example of a randomly generated permutation, π_1 with dispersion of 0.8155.



Example of an algebraic permutation, π_2 with dispersion 0.8178.



Example of the composition of the two permutations, $\pi_1\pi_2^{-1}$ with a dispersion of 0.3410.

Thus the average dispersion distance is defined so that we have a measure of how different the permutations in a collection are from one another.

Example 16. Consider the set of 7 permutations in S_{10} defined as

$$
\begin{aligned}
\pi_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}, \\
\pi_2 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 3 & 5 & 7 & 9 & 2 & 4 & 6 & 8 & 10 \end{pmatrix}, \\
\pi_3 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 4 & 6 & 8 & 10 & 1 & 3 & 5 & 7 & 9 \end{pmatrix}, \\
\pi_4 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}, \\
\pi_5 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 4 & 6 & 7 & 9 & 10 & 2 & 3 & 5 & 8 \end{pmatrix}, \\
\pi_6 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \\
\pi_7 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 10 \end{pmatrix}.
\end{aligned}
$$

As an intermediate step toward constructing the group dispersion table, we place the following permutations in a table before computing the values of dispersion:

	i						
	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0
2	$\pi_1\pi_2^{-1}$	0	0	0	0	0	0
3	$\pi_1\pi_3^{-1}$	$\pi_2\pi_3^{-1}$	0	0	0	0	0
4	$\pi_1\pi_4^{-1}$	$\pi_2\pi_4^{-1}$	$\pi_3\pi_4^{-1}$	0	0	0	0
5	$\pi_1\pi_5^{-1}$	$\pi_2\pi_5^{-1}$	$\pi_3\pi_5^{-1}$	$\pi_4\pi_5^{-1}$	0	0	0
6	$\pi_1\pi_6^{-1}$	$\pi_2\pi_6^{-1}$	$\pi_3\pi_6^{-1}$	$\pi_4\pi_6^{-1}$	$\pi_5\pi_6^{-1}$	0	0
7	$\pi_1\pi_7^{-1}$	$\pi_2\pi_7^{-1}$	$\pi_3\pi_7^{-1}$	$\pi_4\pi_7^{-1}$	$\pi_5\pi_7^{-1}$	$\pi_6\pi_7^{-1}$	0

j

Taking the dispersion of every nonzero entry, we obtain the following table:

	i						
	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0
2	0.2889	0	0	0	0	0	0
3	0.2889	0.4000	0	0	0	0	0
4	0.2000	0.2889	0.2889	0	0	0	0
5	0.5778	0.5778	0.5111	0.5778	0	0	0
6	0.2889	0.4000	0.4000	0.2889	0.6000	0	0
7	0.3778	0.4667	0.4667	0.3778	0.4889	0.3111	0

j

Summing the entries in the group dispersion table, we obtain a value of 8.4667, and so the average of the nonzero entries is $\frac{8.4667}{21} = 0.4032$. Thus, for this particular set of permutations, the average dispersion distance is 0.4032.

5 Triangular Difference Tables

To simplify calculations and to examine patterns among the ordered pairs of table 4.1, the table can be condensed to include only the second component of each ordered pair. To find the number of unique elements of $D_L(\pi)$, consider any repetitions in each diagonal. This condensed $D_L(\pi)$ table is known as the triangular difference table (TDT). To compute dispersion using the TDT, sum the counts of the number of unique entries in each diagonal and then divide by the total number of integer entries in the table. This is described more formally in Definition 17 below and in Lemma 41.

Definition 17. Let $\pi : [n] \to [n]$ be a permutation. We define the *triangular difference table* of π to be the lower-triangular (i.e., below the main diagonal) portion of the $n \times n$ matrix whose (i, j) entry (where $i < j$) is given by

$$\pi(j+1) - \pi(i). \tag{5}$$

Thus, the (i, j) entry of the TDT is $\pi(j+1) - \pi(i)$. Note that the triangular difference table has $n-1$ rows and $n-1$ columns with different numbers of numerical entries.

Because the triangular distance table can be found using the list of differences, where the first components in each diagonal are the same, the values are only compared within its own diagonal. In other words, when looking for distinct values, or the number of distinct values in a TDT, only the comparisons within diagonals are taken into consideration.

Example 18. Given the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix},$$

the corresponding triangular TDT is

$$\begin{matrix} -1 & & \\ -3 & -2 & \\ -2 & -1 & 1 \end{matrix}.$$

In this case, although -1 appears in both the 1^{st} and 3^{rd} rows, because they are in different diagonals, they are considered unique. (Same for the -2s).

Lemma 19. *Given a permutation π of $[n]$ and an integer $1 \leq a \leq n-1$, the number of entries in the triangular difference table of π whose absolute value is a must be $n-a$.*

Proof. Let π be a permutation in S_n and let $a \in [n-1]$. There are $n-a$ pairs i, j such that $|j-1| = a$, namely, $(1, a+1), (2, a+2), ..., (n-a, n)$. Thus, there are $n-a$ pairs of the form $(\pi(i), \pi(j))$ such that $|\pi(j) - \pi(i)| = a$. □

Lemma 20. *Let $\pi : [n] \to [n]$ be a permutation and let $A = (a_{i,j})$ be the corresponding triangular difference table. Then $a_{i,j}$ is the sum of the i^{th} through j^{th} entries of the diagonal; that is,*

$$a_{i,j} = \sum_{\ell=i}^{j} a_{\ell,\ell}. \tag{6}$$

Proof. By Definition 17, we know that

$$a_{i,j} = \pi(j+1) - \pi(i).$$

Thus,

$$\begin{aligned} a_{i,j} &= [\pi(j+1) - \pi(j)] + [\pi(j) - \pi(j-1)] + ... + [\pi(i+1) - \pi(i)] \\ &= a_{jj} + a(j-1)(j-1) + ... + a_{ii}. \end{aligned}$$

□

Using this result, the relationship between the main diagonal and the rest of the numerical entries in the table can be seen.

Lemma 21. *Given a permutation $\pi_1 : [n] \to [n]$, define $\pi_2 : [n] \to [n]$ by*

$$\pi_2(i) = n - \pi_1(i).$$

Then the triangular difference tables of π_1 and π_2 are the negative transposes of one another.

Proof. Let $A = (a_{ij})$ and $B = (b_{ij})$ be the TDTs of $\pi_1 \text{and} \pi_2$, respectively. Then

$$\begin{aligned} b_{ij} &= \pi_2(j+1) - \pi_2(i) && (7) \\ &= (n - \pi_1(j+1)) - (n - \pi_1(i)) && (8) \\ &= -(\pi_1(j+1) - \pi_1(i)) && (9) \\ &= -a_{ij}. && (10) \end{aligned}$$

□

By the definition of a triangular difference table, it is known that for each TDT there exists a permutation that generates it. An interesting question is whether or not a TDT can be generated by more than one permutation.

Theorem 22. *Each triangular difference table is generated by a unique permutation.*

Proof. Let $\Delta(\pi)$ be the main diagonal of a TDT, M. By Lemma 19, for some k, l, the (k, l) entry of the TDT, M, is $n - 1$. So for any permutation π with the given TDT, M,

$$\pi(l+1) - \pi(k) = n - 1,$$

and so,

$$\pi(l+1) = n.$$

In other words, $\pi(l+1)$ is determined by M. Since $(\pi(2)-\pi(1), \pi(3)-\pi(2), ..., \pi(n)-\pi(n-1))$ is determined by M, one can use this information in conjunction with the fact $\pi(l+1) = n$ to solve for $\pi(1), \pi(2), ..., \pi(n)$. □

Definition 23. Given a permutation π of set size n with TDT $A = (a_{ij})$, the *main diagonal* of π is the diagonal from $a_{1,1}$ to $a_{n-1,n-1}$ of A. In particular, it is defined as

$$\Delta\pi = [\pi(2) - \pi(1), \pi(3) - \pi(2), ..., \pi(n) - \pi(n-1)]. \quad (11)$$

Definition 24. Given a vector $v = (v_1, ..., v_n) \in \mathbb{R}^n$, define the *absolute value of* v by

$$|v| = (|v_1|, ..., |v_n|) \in \mathbb{R}^n.$$

If d is the diagonal of a TDT, then $(-d)$ is the diagonal of another TDT. The relationship of permutations that generate such TDTs is given in Lemma 21. Studying the absolute values of the entries in the diagonals and the signs of the entries in the diagonals, we pose the following two conjectures.

Conjecture 25. Let $\pi \in S_n$, where n is odd. Let $(d_1, ..., d_n) = \Delta\pi$. If $|d_i| = |d_{n-i}|$ for all $1 \leq i \leq n-1$, then $d_i = d_{n-i}$ for all $1 \leq i \leq n-1$.

Conjecture 26. The number of the permutations whose TDTs have palindromic diagonals is divisible by 4.

Now, we count the number of potential candidates for the diagonal of a TDT. We call a vector $v \in [n-1]$ a **candidate** if for any $1 \leq i \leq n-1$, at most $n-i$ of the components of v have absolute value i (i.e, the vector v satisfies the condition of Lemma 19).

Note that not all candidates are actually diagonals of TDTs.

To find these candidates, we use exponential generating functions, which are defined as follows.

Definition 27. The **exponential generating function** of the sequence $(a_o, a_1, a_2, ...)$ is defined by

$$g(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}. \tag{12}$$

Definition 28. The number of candidates of set size n is denoted

$$d_n = \{v \in \mathbb{R}^{n-1} | \text{ v is a candidate}\}.$$

We calculate d_n using exponential generating functions. The exponential generating function can calculate the number of the absolute values of the entries of $\Delta\pi$. Since there are up to $n-j$ possible occurrences of j in the TDT, we use the following as a factor of our exponential generating function given by (12):

$$g_j(x) = \sum_{i=0}^{n-j} \frac{x^i}{i!}.$$

By the theory of exponential generating functions (see [Tu]), d_n is the coefficient of x^{n-1} in

$$\prod_{j=1}^{n-1} g_j(x).$$

Example 29. Let $n = 4$. In any candidate vector, there may be up to 3, 2, and 1 occurrences of 1,2, and 3, respectively. In the exponential generating function, the number 1 is represented by $(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!})$, 2 is represented by $(1 + x + \frac{x^2}{2!})$, and 3 is represented by $(1 + x)$.
Thus, we have the following exponential generating function:

$$g(x) = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!})(1 + x + \frac{x^2}{2!})(1 + x) = \sum_{q=0}^{\infty} \frac{a_q x^q}{q!}.$$

The coefficient of $\frac{x^3}{3!}$ is $a_3 = 19$, which means that there are 19 candidates of vector length 3 for $n = 4$.

6 Comparisons of Permutations in Concatenated Zigzag Codes

In this section we outline the results obtained by running simulations that compare the performance of various permutations when implemented in concatenated zigzag codes.

6.1 Background

Permutations are a main component in the construction of concatenated zigzag codes. One of the primary goals of this project is to determine what kinds of permutations and which sets of permutations improve error-correcting capabilities. In order to do this, `Zigzagsimulator`, a Matlab program, was created. The simulator takes in the dimensions of the message, the permutation(s), the signal to noise ratios (SNR) to be tested and the number of iterations the decoding process is to perform. The program returns a graph of the signal to noise ratio versus the bit-error-rate (BER) which is defined as the number of uncorrectable bits to the total bits transmitted.

`Zigzagsimulator` includes a random message generator, an encoder that is fed permutations, a noise generator, a decoder and a way of checking the uncorrectable errors remaining after the decoding process.

To create a message, a simple algorithm was written to generate a random message of specified dimensions by the user. The encoding process was simulated by a program that replicated the message matrix a specified number of times, K, then permuted Kreplicas with permutations given by the user. Each is run through the zigzag encoding process and the parity matrices are computed. Finally, these K parity matrices are concatenated with the original message and transmitted as a codeword through the noise generating algorithm. This specific noise generation algorithm simulates additive white Gaussian noise.

Finally, the codeword is received and decoded. The decoding algorithm uses the decoding equations presented by Tejas Bhatt and Victor Stolpman in [Bh] from work previously done by Li Ping in [Pi(2)]. These algorithms use the redundancy provided by parity bits to find errors in the received word. Belief propagation is used to determine the most probable message. Belief propagation is an algorithm that calculates the probability that a certain bit is either a 0 or a 1. The number of iterations tells the decoding algorithm how many times

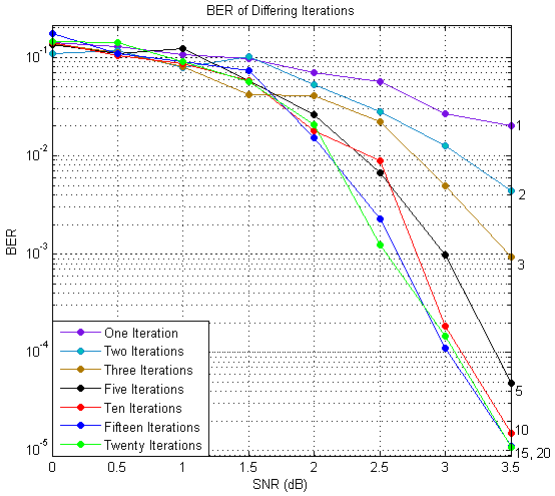
to cycle through the decoding process, calculate probabilities and allow each permutation to check itself against the others. This process ensures that only the most probable message is returned after decoding.

The last step of the simulation compares the decoded message to the original message, counting the number of differences, or uncorrectable errors. The lower the BER, the better error-correcting capabilities of the code, which means there are less uncorrectable bits per total bits, and the original message is more accurately reconstructed.

6.2 Iterations of the Decode Process

Since the decoding algorithm for concatenated zigzag codes involves an iterative process, we must choose how many iterations to use in our own simulations. In [Pi(2)], performance was measured for the case where eighteen iterations were used. To determine how many iterations to use in our simulations, we considered runtime. Although using fewer iterations yields faster runtimes, we wanted to make sure that accuracy was not sacrificed for faster runtimes.

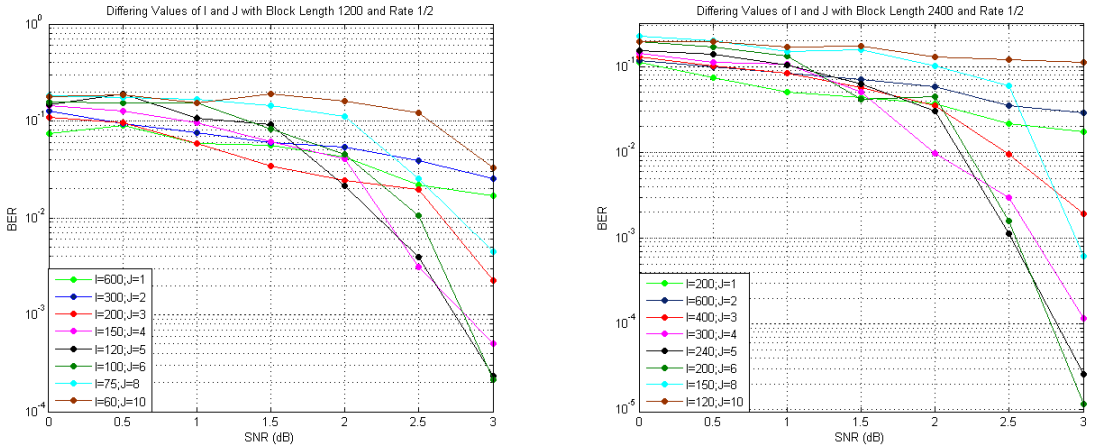
We ran simulations for various numbers of iterations between 1 and 30, testing values of SNR between 0 and 3 dB. We used parameters $I = 128$ and $J = 4$ and implemented $K = 4$ randomly-generated permutations. Thus all codes considered for this comparison have rate $\frac{1}{2}$ with block length 1024. The resulting graph is as follows:



The graph shows us that as the number of iterations increases, the BER decreases. Although the performance gain due to increasing the number of iterations from one to ten is significant, the payoff is comparatively negligible when transitioning from ten to twenty iterations. Based on these findings, further simulations are run with thirteen iterations, producing comparable results to a larger number of iterations but minimizing computing time.

6.3 Reshaping the Information Matrix D

The following graphs illustrate the effects of reshaping the information matrix D (by varying the paramters I, J, and K) while holding fixed the block length and the coding rate. For example, for block length 200, let $I = 50$ and $J = K = 2$. The number of parity bits in this case is $IK = 50 \cdot 2 = 100$. Keeping the same block length, let $I = 25$, then $J = K = 4$. The number of parity bits would still be 100, or $25 \cdot 4$. The first graph used a block length of 1200, and the second graph used one of 2400.



The graphs show that the optimal values of J and K for rate $\frac{1}{2}$ are 4, 5, and 6 for block lengths 1200 and 2400. Suggesting that when holding the number of parity bits fixed for rate $\frac{1}{2}$ the size of the set of permutations used should be 4, 5, or 6.

Conjecture 30. Let N be the number of information bits. When applying a concatenated zigzag coding scheme of rate $\frac{1}{2}$, the optimal BER is obtained when D is an $\frac{N}{J} \times J$ matrix where $J \in \{4, 5, 6\}$ and the number of permutations is $K = J$.

6.4 Classical Block Permutations

To test how well highly structured permutations perform, we used the four classical block permutations as a set of permutations from section 3.1.2, see [Jo]. The four permutations all have a dispersion equal to 0.0095, showing how structured these permutations are. In all our simulations we compared the performance of these permutations with randomly generated permutations, each of which has a dispersion of approximately 0.81.

Classical block permutations are constructed by reading the entries of a matrix in various ways. Consider the following 3×3 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

We can obtain the classical block permutations of A by reading the matrix A as the permutation is titled:

- Left to Right/Top to Bottom (LR/TB) Permutation on A:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 9 \end{pmatrix}$$

- Left to Right/Bottom to Top (LR/BT) Permutation on A:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \end{pmatrix}$$

- Right to Left/Top to Bottom (RL/TB) Permutation on A:

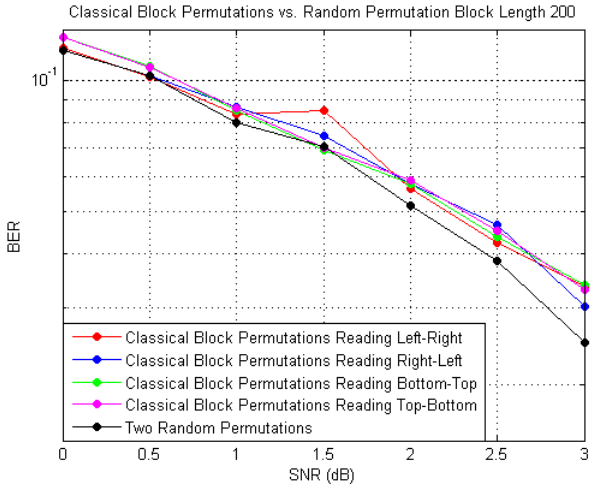
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & 9 & 2 & 5 & 8 & 1 & 4 & 7 \end{pmatrix}$$

- Right to Left/Bottom to Top (RL/BT) Permutation on A:

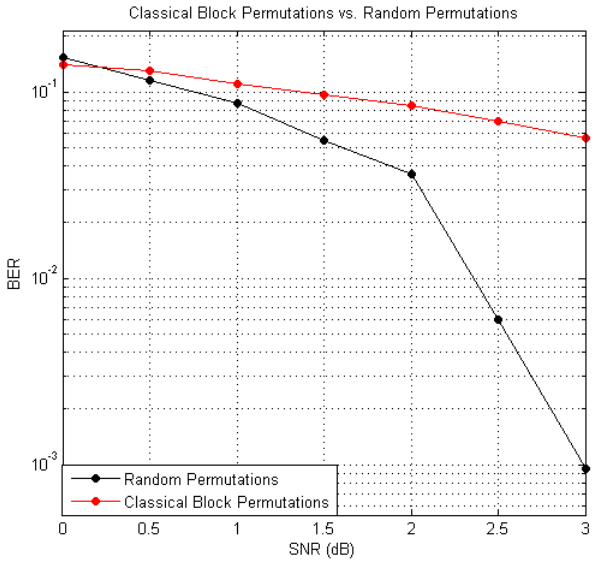
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 6 & 3 & 8 & 5 & 2 & 7 & 4 & 1 \end{pmatrix}$$

All of these permutations have a dispersion of 0.0095.

Our first comparison involved running simulations with $I = 100$ and $J = K = 4$ so that the block length was 800 with rate $\frac{1}{2}$. We compared the performance of using all four classical block permutations against four randomly generated permutations. The graph below shows that the set of four classical block permutations do not perform as well as a set of four randomly-generated permutations.



Next we compared various pairs of classical block permutations to randomly generated permutations. The block length used was 200 with $I = 50, J = K = 2$ and a rate of $\frac{1}{2}$. The graph below shows that in each case the performance of the various pairs of different classical permutations were outperformed by a pair of randomly generated permutations, which suggests that permutations with a great deal of structure do not have the same error-correcting capabilities as randomly generated permutations.



6.5 Algebraic Permutations

Is it possible to create a set of structured permutations with a wide range of values of dispersion that perform as well as a set of randomly-generated permutations? To address this question, we constructed algebraic permutations. A brief background of their construction follows.

6.5.1 Fields and Monomial Orders

Definition 31. A **field** $\mathbb{F}$ is a set with two operations defined on the set,"addition" and "multiplication," such that the following properties hold:
(i) Closure: $\forall a, b \in \mathbb{F},$ if $a \cdot b \in \mathbb{F}$, then $a + b \in \mathbb{F}$
(ii) Commutativity: $\forall a, b \in \mathbb{F}, a + b = b + a$, and $a \cdot b = b \cdot a$
(iii) Associativity: $\forall a, b, c \in \mathbb{F}, a + (b + c) = (a + b) + c$
(iv) Distributivity: $\forall a, b, c \in \mathbb{F}, a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
(v) Identity: $\exists\ 0, 1 \in \mathbb{F}\ \forall a \in \mathbb{F}$ such that $a + 0 = a,\ a \cdot 1 = a$
(vi) Additive Inverse: $\forall a \in \mathbb{F}, \exists(-a) \in \mathbb{F}$ such that $a + (-a) = 0$
(v) Multiplicative inverse: $\forall a \in \mathbb{F}, a \neq 0, \exists b \in \mathbb{F}$ such that $a \cdot b = 1$

Theorem 32. *The set of integers $\mathbb{Z}_p$ is a **field** if and only if p is prime.*

Definition 33. A field is a **finite field** if it contains only a finite number of elements.

Theorem 34. *There exists a field with q elements if and only if $q = p^r$ for some p and positive integer r.*

The collection of all polynomials in indeterminate x with coefficients in the field $\mathbb{F}$ is denoted by $\mathbb{F}[x]$.

Definition 35. A polynomial $f(x)$ is said to be **reducible** over $\mathbb{F}$ if there exist nonconstant polynomials $g(x), h(x) \in \mathbb{F}[x]$ such that $f(x) = g(x)h(x)$. A polynomial is **irreducible** if it is not reducible.

Definition 36. Let $h(x)$ be an irreducible polynomial over $\mathbb{F}_p[x]$ and let $r = \deg\ h(x)$. The set $\mathbb{F}_p[x]/\langle h(x)\rangle$ is defined to be the collection of all polynomials over $\mathbb{F}_p$ of degree at most $r-1$; that is,

$$\mathbb{F}_p[x]/\langle h(x)\rangle = \{\beta_{r-1}\alpha^{r-1} + \ldots + \beta_0 \mid \beta_i \ \in \ \mathbb{F}_p\}, \text{ where } \alpha \text{ is an indeterminate.}$$

Define addition as regular addition between two polynomials modulo p and multiplication as follows:

(i) Fix an irreducible polynomial $h(x)$ of degree r with coefficients in $\mathbb{F}_p$. This is to be used for the multiplication of any pair of polynomials.

(ii) For two polynomials f and g, compute $f \cdot g$ as normal polynomial multiplication.

(iii) Divide $f \cdot g$ by h. The remainder is defined to be the product of $f(x)$ and $g(x)$ in $\mathbb{F}_{p^r}$.

Theorem 37. *Given two irreducible polynomials of the same degree, $h_1(x),\ h_2(x) \in \mathbb{F}_p[x]$, $\mathbb{F}_p[x]/\langle h_1(x)\rangle$ and $\mathbb{F}_p[x]/\langle h_2(x)\rangle$ are isomorphic.*

Consequently, since $\mathbb{F}_p[x]/\langle h_1(x)\rangle$ and $\mathbb{F}_p[x]/\langle h_2(x)\rangle$ are isomorphic whenever $r = \deg\ h_1(x) = \deg\ h_2(x)$, the notation $\mathbb{F}_{p^r}$ is used for all isomorphic copies of this field.

Constructing permutations from $\mathbb{Z}_q$ to $\mathbb{Z}_q$ using finite fields requires the use of a construct called monomial orders. Let S be a set. A **total order** on S is a binary relation "$<$" where the following hold for all $\alpha,\ \beta,\ \gamma \in \mathbb{Z}^n$.

(i) For all $\alpha \neq \beta \in \mathbb{Z}^n, \alpha < \beta$ or $\beta < \alpha$.

(ii) If $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.

(iii) $\alpha \not< \alpha$.

Definition 38. A **monomial ordering** on $\mathbb{Z}^n$ is a total order " $<$ " such that

a. For all $\alpha \in \mathbb{N}^n, \alpha \geq \overrightarrow{0}$.

b. $\alpha > \beta \implies \alpha + \gamma > \beta + \gamma$.

Two types of monomial orders are the Lexicographic and Graded Lexicographic Orders. Comparisons under the lexicographic order are given by the following rule: $\alpha < \beta$ if and only if the first nonzero entry of $\beta - \alpha$ is positive. Comparisons under the graded lexicographic order are given by the following rule: $\alpha < \beta$ if and only if $\sum_{i=1}^n\ \alpha_i < \sum_{i=1}^n\ \beta_i$ or $\sum_{i=1}^n\ \alpha_i = \sum_{i=1}^n\ \beta_i$ and the first nonzero entry of $\beta - \alpha$ is positive.

6.5.2 The Construction of Algebraic Permutations

To construct the algebraic permutations used in later simulations, a computer program was used, see [Do]. The program uses a polynomial permutation $\pi : \mathbb{Z}_{p^r} \to \mathbb{Z}_{p^r}$.

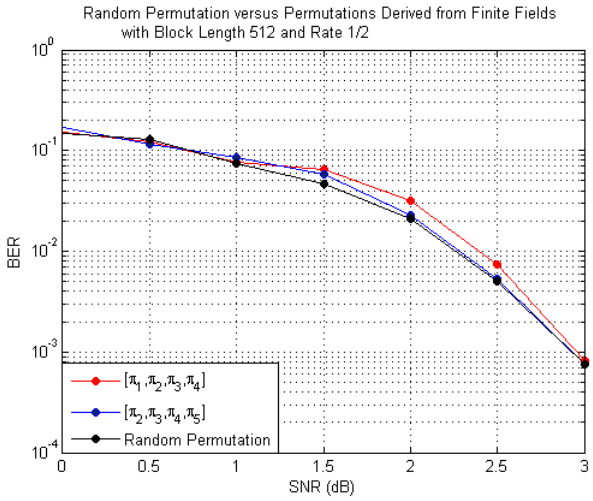
$$\mathbb{Z}_{p^r} \xrightarrow{\pi_1} (\mathbb{Z}_p)^r \xrightarrow{\pi_2} \mathbb{F}_{p^r} \xrightarrow{\pi_3} \mathbb{F}_{p^r} \xrightarrow{\pi_2^{-1}} (\mathbb{Z}_p)^r \xrightarrow{\pi_1^{-1}} \mathbb{Z}_{p^r} \tag{13}$$

The function π_1 uses a monomial ordering to order the vectors of $(\mathbb{Z}_p)^r$. It orders the vectors of $(\mathbb{Z}_P)^r$ from smallest to largest according to this monomial order. That is, $(\mathbb{Z}_P)^r = \{v_0, \ldots, v_{p^r-1}\}$ where $v_i \leq v_{i+1}$. Then π_1 is defined by $\pi_1(i) = v_i$. The function $\pi_2 : (\mathbb{Z}_p)^r \to \mathbb{F}_{p^r}$ is given by $(\beta_{r-1}, \ldots, \beta_0) \mapsto \sum_{i=0}^{r-1} \beta_i \alpha^i$. The function $\pi_3 : \mathbb{F}_{p^r} \to \mathbb{F}_{p^r}$ is given by $x \mapsto p(x)$, where $p(x) \in \mathbb{F}_{p^r}[x]$.

Algebraic permutations were chosen since dispersion varies much more widely for algebraic permutations than for randomly generated permutations. One algebraic permutation may have a relatively low dispersion, whereas another may have a dispersion around 0.81. The algebraic permutations used were generated by the following polynomials in $\mathbb{F}_{512}[t]$:

$$\begin{aligned}
\pi_1(t) &= t, \text{ dispersion} = 0.0039 \\
\pi_2(t) &= t^2, \text{ dispersion} = 0.3357 \\
\pi_3(t) &= t^3, \text{ dispersion} = 0.8160 \\
\pi_4(t) &= t^4, \text{ dispersion} = 0.4993 \\
\pi_5(t) &= t + t^2 + t^3, \text{ dispersion} = 0.8152
\end{aligned}$$

We consider two sets of permutations, $\Pi_1 = \{\pi_1, \pi_2, \pi_3, \pi_4\}$ and $\Pi_2 = \{\pi_2, \pi_3, \pi_4, \pi_5\}$. The set Π_1 consists of permutations whose dispersion ranges from 0 to 0.8160, and the set Π_2 consists of permutations whose dispersion ranges from 0.3357 to 0.8160. We compared the performance of Π_1 and Π_2 with a set of four random permutations, all with dispersion approximately 0.81. Comparing the performance of zigzag codes with parameters $I = 128$, $J = K = 4$, and these three sets of permutations, the following graph was obtained:



The results show that both Π_1 and Π_2 did as well as the four random permutations, which demonstrates that a set of permutations with widely varying dispersion can do as well a set of randomly-generated permutations with dispersion near 0.81.

6.6 Structured Permutations by Tejas Bhatt and Victor Stolpman

After showing it was possible to use a set of permutations with varying values of dispersion and still attain high levels of performance, the next trial was dedicated to using a set of structured permutations described by Tejas Bhatt and Victor Stolpman in [Bh]. These permutations have low dispersion but Bhatt and Stolpman claim that their performance is comparable to that of randomly generated permutations.

Bhatt and Stolpman constructed the following method for producing a collection of permutations $\{\pi_1, \pi_2, ...\pi_K\}$, each of which permutes the entries of an $I \times J$ matrix D. For the sake of completeness, we provide a brief description of the permutations constructed by Bhatt and Stolpman. For more details, refer to [Bh].

- Reshape the original message $I \times J$ matrix D as an $I' \times J'$ matrix, where $I' \leq I$ and $I'J' = IJ$.
- A circular shift of a column $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$ of s units, generates the column $\begin{bmatrix} a_{n-s} \\ \vdots \\ a_1 \\ a_n \\ \vdots \\ a_{n-s-1} \end{bmatrix}$. For each π_k and each column of D we perform a circular shift.
- The following two conditions must be met:
  - No two columns can have the same column shift for any π_k, where $k \in \{1, 2, ...K\}$.
  - For any π_k, where $k \in \{1, 2, ...K\}$, there does not exist a row $u \in \pi_k(D)$ and a row $v \in \pi_{k'}(D)$ such that u and v have entries in common.
- The next step bit-reverses the rows of the permuted matrix. Consider the tuple $R = (1, 2, \ldots, I')$, which indexes the rows. First convert the entries of this tuple into binary and reverse-order the bits of each entry. Then after converting these entries back to decimal to produce a vector R', reorder the rows of the matrix according to how R' is obtained from R by permuting the entries. The following example illustrates this process. Begin with the 4×3 matrix below:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12}. \end{bmatrix}$$

$$\begin{aligned} R = \{1,2,3,4\} &\rightarrow \{001,010,011,100\} \text{ (R converted to binary)} \\ &\rightarrow \{100,010,110,001\} \text{ (R bit-reversed)} \\ &\rightarrow \{4,2,6,1\} \text{ (R converted to decimal)} \\ &\rightarrow \{1,2,4,6\} \text{ (bit-reversed R re-ordered)} \end{aligned}$$

This suggests that the new order of the rows should be $\{4,2,1,3\}$. The bit reversed matrix would then be $\begin{bmatrix} a_{10} & a_{11} & a_{12} \\ a_4 & a_5 & a_6 \\ a_1 & a_2 & a_3 \\ a_7 & a_8 & a_9 \end{bmatrix}$.

- Optionally, the columns of the matrix may be swapped. This step was not performed in any of our trials.

For comparison, we developed a special class of hybrid permutations.

Definition 39. A **random column permutation** of the entries of an $I \times J$ matrix (read row-by-row) is a permutation $\pi = [IJ] \rightarrow [IJ]$ such that the entries of a column remain in the same column from which they originated after the permutation is applied. Mathematically this is equivalent to the condition $\pi(l) \equiv l \mod J$ for all $l \in [IJ]$.

Definition 40. Given random column permutations $\pi_1, \pi_2, ..., \pi_K$, we call $\{\pi_1, \pi_2, ..., \pi_K\}$ a set of **restricted random column permutations** if for any i,j, where $i \neq j$, any row u of $\pi_k(D)$, and any row $\mathbf{v}$ of $\pi_{k'}(D)$, no more than two entries of u and v are identical.

Since the previous two permutations are permutations that only permute the columns, Theorem 42 pertains to the upper bound of the dispersion of these permutations. By counting the number of unique entries along each diagonal of a TDT, we can reformulate the definition of dispersion as follows.

Lemma 41. *Given a permutation $\pi : [n] \rightarrow [n]$, the* **dispersion** *of π is given by*

$$\frac{1}{\binom{n}{2}} \sum_{k=1}^{n-1} \beta_k(\pi) \tag{14}$$

where

$$\beta_k(\pi) = |\{\pi(i) - \pi(j) \mid i,j \in [n], i - j = k\}|. \tag{15}$$

Theorem 42. *Let $\pi : [IJ] \rightarrow [IJ]$ be a permutation such that for every $\ell \in [IJ]$, $\pi(\ell) \equiv \ell \mod J$. Then the dispersion of π is at most*

$$\frac{4/J - 3/J^2 + 1/IJ^2 + 2/I^2J^2}{1 - 1/IJ}.$$

Therefore, as $J \rightarrow \infty$, dispersion heads to zero, regardless of the magnitude of I. Moreover, as $I \rightarrow \infty$, dispersion approaches $4/J - 3/J^2$. If $J \leq I$, then

$$disp(\pi) \leq \frac{4I^2J - 3I^2 + I - J + 3}{I^2J^2 - IJ}.$$

Proof. Let $k \in [IJ]$. Since $\pi(\ell + k) \equiv \ell + k \mod J$ and $\pi(\ell) \equiv \ell \mod J$, it follows that $\pi(\ell + k) - \pi(\ell) \equiv k \mod J$ and so $\pi(\ell + k) - \pi(\ell) = k + mJ$ for some $m \in \mathbb{Z}$. Moreover, $|\pi(\ell+k) - \pi(\ell)| < IJ$, and so $-IJ < \pi(\ell+k) - \pi(\ell) < IJ$. Therefore, $-IJ < k + mJ < IJ$, and so $-I - \frac{k}{J} < m < I - \frac{k}{J}$. If $\frac{k}{J}$ is an integer, then m can take on $2I - 1$ possible values; that is, $\beta_k \leq 2I - 1$. If $\frac{k}{J}$ is not an integer, then m can take on $2I$ possible values; that is, $\beta_k \leq 2I$. For $1 \leq k \leq I(J-1)+1$, we use the upper bound $\beta_k \leq 2I$. Thus,

$$\sum_{k=1}^{IJ-I+1} \beta_k \leq \sum_{k=1}^{IJ-I+1} 2I = (IJ - I + 1)(2I) = 2I^2J - 2I^2 + 2I. \tag{16}$$

The number of pairs (i, j) such that $i, j \in [IJ]$ and $i - j = k$ is $IJ - k$, and so $\beta_k \leq IJ - k$. Therefore,

$$\sum_{k=IJ-I+2}^{IJ-1} \beta_k \leq \sum_{k=IJ-I+2}^{IJ-1} IJ - k = \sum_{k=1}^{I-2} k = (1/2)(I-2)(I-1) = (1/2)(I^2 - 3I + 2). \tag{17}$$

By combining (16) and (17), and then simplifying, we obtain

$$\sum_{k=1}^{IJ-1} \beta_k \leq \frac{1}{2}(4I^2J - 3I^2 + I + 2). \tag{18}$$

Dividing this sum by $\binom{IJ}{2}$ and simplifying, we obtain

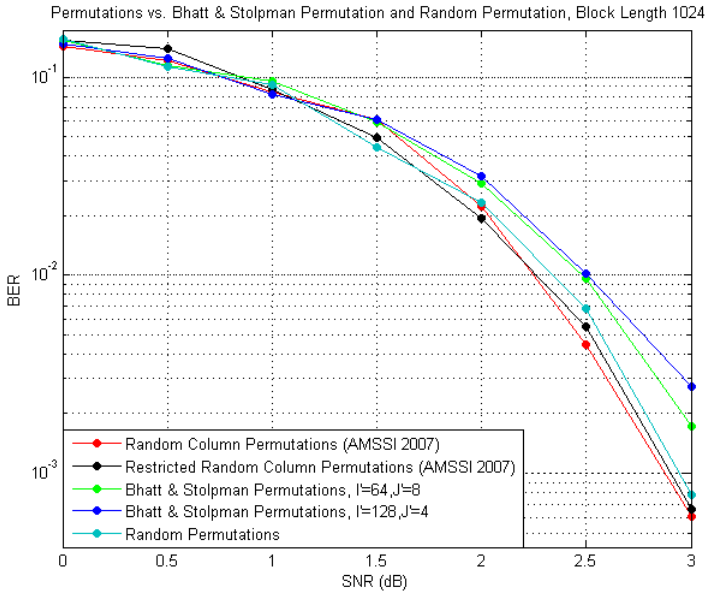
$$\mathrm{disp}(\pi) \leq \frac{4I^2J - 3I^2 + I + 2}{I^2J^2 - IJ} = \frac{4/J - 3/J^2 + 1/IJ^2 + 2/I^2J^2}{1 - 1/IJ}.$$

Note that for one out of every J terms in (16), we can use the upper bound $2I - 1$ in place of $2I$. If $J \leq I$, then $IJ - I + 1 \geq J(J-1) + 1$, in which case we can subtract $J - 1$ from the bound given in (16). In this case, we obtain

$$\mathrm{disp}(\pi) \leq \frac{4I^2J - 3I^2 + I - J + 3}{I^2J^2 - IJ}.$$

□

When testing these types of permutations we used a block length of 1024, $I = 128$ and $J = K = 4$ and rate $\frac{1}{2}$. For our simulations, we tested concatenated zigzag codes with $I = 128$ and $J = K = 4$, which yields a block length of 1024 and a code rate of $\frac{1}{2}$. The graph below compares four classes of permutations.



Bhatt and Stolpman's permutations with $I' = 64$, $J' = 8$ possess an average dispersion distance of 0.1853 and when $I' = 128$, $J' = 4$ the average dispersion distance is 0.1486. When looking at the individual permutations generated for each set, the dispersion values are quite low (less than 0.1), thus showing that the dispersion of individual permutations is not the only measure of the performance of a set of permutations. The random column permutations have an average dispersion distance of 0.5088 and the restricted random column permutations have an average dispersion distance of 0.5084. The set of random permutations have an average dispersion distance of 0.8141. The graph shows that a set of permutations that permute only the values within their respective columns have a BER comparable to a set of random permutations. Furthermore, the proximity of all five permutations suggests that there exist structured permutations that outperform random permutations. The results also show that both hybrid permutations outperformed Bhatt and Stolpman's structured permutations.

This trial also showed that there is no correlation between the performance of a set of permutations in concatenated zigzag codes and average dispersion distance. The reason is that the random column permutations and restricted random column permutations did the best, and their average dispersion distance was around 0.5. A set of random permutations had a average dispersion distance of 0.8141, and had excellent BER as well. In addition, the a two sets of the Bhatt and Stolpman permutations had great BER, but had a low average dispersion distance of 0.1486 and 0.1853. Since all of these sets of permutations had great error correction, but varying average dispersion distance, we can conclude that there is no strong correlation between the two present. However, there may be a weak positive correlation present. In fact, we have no examples where the average dispersion distance is high and yet the corresponding permutations yield poor performance

7 Future Work

In further studies, we would like to determine if variance is an accurate predictor of the error-correcting capabilities of set permutations. If variance is not an accurate predictor, we would like to find another predictor for sets of permutations. In addition, we hope to determine if the average dispersion of the permutations of S_n approaches a constant around 0.81 as n increases and if that value can be written in terms of common constants or a sum of a known sequence. To aid in future simulations, we would like to improve the efficiency of the zigzag simulator by either streamlining the Matlab code or implementing a version in C++. We would also like to include frame error rates, the amount of uncorrectable frames to the total number of frames transmitted. Finally, we would like to run more simulations using random column permutations versus random permutations.

8 Acknowledgements

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9 Programs

This section documents the Matlab code and the C++ code used throughout this project.

9.1 Zigzagsimualator

This program is the main function that calls upon all the other subfunctions to calculate BERs for given SNRs and Permutations. The user sends it the dimensions of the message they would like to produce.

```
function BER = Zigzagsimulator2(I,J,SNR,PI,NUMITERS) %%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%             Zigzag Code Used to Simulate BER             %
% The second draft of the main function that relies on     %
% user input for the dimensions of the message, I, J,      %
% a vector of values for the SNR, Pi for the Permutations,%
% and the number of iterations the decoder will go though %
%                      created: July 2, 2007               %
%                     modified: July 9, 2007               %
%            Applied Mathematical Science Institute        %
%                          Team Special Ed                 %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

diary 'BERexecute.txt'; %%
%1.) Declaration of Variables
NUMERRORS = 500;            % Tolerance of Errors
BER = zeros(1,length(SNR)); % initialization of Bit-Error-Rate
FRAME = zeros(1,length(SNR));
[K x] = size(PI);           % determines how many subiterations
BLOCKLENGTH = I*(J);      % The number of bits for each codeword
fprintf(' === Variables Declared === \n'); %%
%2.)    Main loop: runs the whole simulation for
%       each value of snr placed in the SNR array
fprintf(' === Loop Started === \n') for s=1:length(SNR)
    loops = 0;              % initializes loop count at 0
    errors = 0;             % initializes error count at 0 for each pass through
    frameerror = 0;         % initializes frame error count to 0
    IP = zeros(K,I*J);      % loop used to determine the inverse permutation
    for k=1:K               % for each of the K permutations
        IP(k,PI(k,:)) = 1:I*J;
    end

    while(errors < NUMERRORS)% when the number of errors is greater than our
        % allowed limit the loop ends
        D = RandMessage(I,J);% randomly generates a message
```

```
        %%% ENCODING BEGINS
        P = Parities(D,PI);  % calculates the parities for each permutation
        C = codeword(D,P);   % merges D and P into a 1xI*(J+K) vector
        C = tilde(C);        % maps 0->1 and 1->(-1)
        %%% ENCODING ENDS

        %%% SIMULATION OF ERRORS BEING INTRODUCED
        % USING GAUSSIAN NOISE
        [R, sigma] = whitenoise(C,SNR(s),I*J, I*(J+K));  % R is the received word
        %R = C;  %<-Check's encoding scheme-test for the DECODING
        %%% END SIMULATION OF NOISEY CHANNEL

        %%% DECODING BEGINS
        Dtilde = R(1:I*J);          % separates the received message bits from
        % the received word
        Ptilde = R(I*J+1:I*(J+K));  % separates the parity bits for each permutation
        % from the received word

        % the loops talk to eachother....sup?...nothing much you?...same
        % old...how about those 1's and -1's?...O, they are ok, but they are
        % a bit different from yours...O, really?  How about I use some of
        % yours...sure...
        received = sign(Decode(PI,IP,I,J,K,Dtilde,Ptilde,NUMITERS,sigma));
        % only concerned with whether the or not if the sign is positive or
        % negative

        % sums the errors from the decoded message to the original
        errors = errors + sum(abs(C(1:I*J) - received))/2;
        if errors~=0
            frameerror = frameerror+1;
        end
        %errors = sum(abs(C - sign(R)))
        loops = loops + 1;          % increments the loop
    end
    BER(s) = errors/(loops*BLOCKLENGTH);    %calculates the Bit Error Rate
    fprintf('\n');
    fprintf('%s%1.2f%s%1.10f','For SNR = ',SNR(s),' the BER is ', BER(s));
    fprintf('\n');
end %%
% Graphs the different Bit Error Rates for the different SNR
ZigzagBERgraph(SNR,BER);
diary off;
```

9.1.1 RandMessage.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                        RandMessage.m                                  %
%                  Creates a IxJ Random Message                         %
%                     created: July 2, 2007                             %
%                                                                       %
%       Applied Mathematical Science Institute 2007                     %
%                         Team Special Ed                               %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function D = RandMessage(I,J)
    D = zeros(I,J);             % create a I x J matirx of zeros
    for i=1:I
        for j=1:J
            if rand <= .5       % if rand is less than .5
                D(i,j) = 1;     % replace the d(i,j) with
            end                 % with a 1
        end
    end
```

9.1.2 Parities.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                         Parities.m                                    %
%   Function used to compute the parities for each PI                   %
%                     created: July 2, 2007                             %
%                                                                       %
%       Applied Mathematical Science Institute 2007                     %
%                         Team Special Ed                               %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Function used to compute the parities for each Pi for D
function P = Parities(D,Pi)

[K IJ] = size(Pi);

for k=1:K
   Temp = Permute(D,Pi(k,:));
   P(k,:) = CalcParities(Temp);
end
```

9.2 Permute.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          Permute.m                                    %
%                       created: July 2, 2007                           %
%       Applied Mathematical Science Institute 2007                     %
%                          Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function dk = Permute(D,Pi)
%%%%%%%%%%%%%%%%%%%%%
% this is how we permute
% pi = [3 4 5 2 6 1]
%(a1 a2 a3 a4 a5 a6)
% pi(a) = (a6 a4 a1 a2 a3 a5)
[I J] = size(D);

k = J;

for i=1:I
    d(k-J+1:k) = D(i,:);
    k = k +J;
end temp = d(Pi);

k=1;

for i=1:I
    for j=1:J
        dk(i,j) = temp(k);
        k = k+1;
    end
end
```

9.2.1 codeword.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                            Codeword.m                                 %
%   Function used to create the codeword sent through the               %
%                           noisey channel                              %
%                       created: July 2, 2007                           %
%                                                                       %
%         Applied Mathematical Science Institute 2007                   %
%                          Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function c = codeword(D,P)
% D is the message
% P are the parities
% Returns 1*IJ+IK codeword
[I J] = size(D); [K IJ] = size(P);

d = reshape(D', 1, I*J); p = reshape(P', 1, K*IJ);

c = [d p];
```

9.2.2 tilde.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                            tilde.m                                    %
%  Function used to modulate the matrix to 1's, and -1's                %
%        from the original codeword of 0's and 1's                      %
%                       created: July 2, 2007                           %
%                                                                       %
%         Applied Mathematical Science Institute 2007                   %
%                          Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function modc = tilde(c)
modc = (-1).^c; %converts 1's to -1's and 0's to 1's
```

9.2.3 whitenoise.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                        whitenoise.m                                    %
%      Function used to simulate Gaussian white noise                    %
%         and corrupts bits sent from the codeword                       %
%                     created: July 2, 2007                              %
%                                                                        %
%         Applied Mathematical Science Institute 2007                    %
%                          Team Special Ed                               %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [r sigma] = whitenoise(c, snr, k, n)
% c: codeword
% snr: signal to noise ratio
% k/n is the rate
% returns sigma and the distorted vector
en = 10 ^(snr/10); sigma = 1/(sqrt(2*en*(k/n))); r = c; for
i=1:length(c)
    x = rand();
    if x < 10^(-50)
        x = 10^(-50);
    end
    r(i) = c(i)+ sigma*sqrt(-2*log(x))*cos(2*pi*rand());
end
```

9.2.4 Decode.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                             Decode.m                                   %
%        Function that decodes the received codeword                     %
%                      created: July 2, 2007                             %
%                                                                        %
%           Applied Mathematical Science Institute                       %
%                          Team Special Ed                               %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function Final = Decode(Pi,iP,I,J,K,Dtilde,Ptilde,numiters,sigma)

Le = zeros(K,I*J); Lo = zeros(1,I*J);

for iter=1:numiters
        for k=1:K
            Le(k,:) = calcLe(Ptilde((k*I-I+1):(k*I)),Pi(k,:),iP(k,:),Lo,I,J);
            clip(Le);
            Lo = calcLo(Dtilde,Le,k,K,Pi(k,:));
```

```
            clip(Lo);
        end
end Final = FinalLLR(Dtilde,Le,K);
```

9.2.5 calcLe.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          calcLe.m                                     %
%                      created: July 2, 2007                            %
% a function that computes part of extrinsic Information               %
%                        on data bits                                   %
%            Applied Mathematical Science Institute                     %
%                          Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function Le = calcLe(Ptilde,Pi,iP,Lo,I,J)

Lo = Permute(Lo,Pi); F = calcF(Ptilde,Lo,I,J); B =
calcB(Ptilde,Lo,I,J);


for j=1:J
    temp = Lo(1:J);
    temp(j) = Inf;
    Le(j) = calcW([temp B(1)]);
end

for i=2:I
    for j=1:J
        temp = Lo(i*J-J+1:i*J);
        temp(j) = Inf;
        Le((i-1)*J+j) = calcW([temp B(i) F(i-1)]);
    end
end Le = Permute(Le, iP);
```

9.2.6 calcF.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          calcF.m                                      %
%                      created: July 2, 2007                            %
%       Applied Mathematical Science Institute 2007                     %
%                         Team Special Ed                               %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function F = calcF(Ptilde,Lo,I,J)
%calcF
%a function that computes that forward recursion
%06/28/07


    F(1) = Ptilde(1) + calcW([Inf Lo(1:J)]);
    for i=2:I
        F(i) = Ptilde(i) + calcW([F(i-1) Lo((i-1)*J+1:i*J)]);
    end
```

9.2.7 calcB.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          calcB.m                                      %
%                      created: July 2, 2007                            %
%       Applied Mathematical Science Institute 2007                     %
%                         Team Special Ed                               %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function B = calcB(Ptilde,Lo,I,J)
%a function that computes that backward recursion

B(I) = Ptilde(I); for i=I:-1:2
   B(i-1) = Ptilde(i-1) + calcW([B(i) Lo((i-1)*J+1:i*J)]);
end
```

9.2.8 calcW.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          calcW.m                                      %
%                       created: July 2, 2007                           %
%       Applied Mathematical Science Institute 2007                     %
%                         Team Special Ed                               %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function W = calcW(a)
% function that computes W
n = length(a);

tally = 1; min = inf;

for i=1:n
    tally = tally * a(i);
    if abs(a(i))<min
        min = abs(a(i));
    end
end

if tally < 0
    sign = -1;
elseif tally > 0
    sign = 1;
else
    sign = 0;
end

W = sign*min;
```

9.2.9 calcLo.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          calcLo.m                                     %
%        Function that decodes the received codeword                    %
%                       created: July 2, 2007                           %
% a function that computes part of extrinsic Information                %
%                         on data bits                                  %
%       Applied Mathematical Science Institute 2007                     %
%                          Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function Lo = calcLo(Dtilde,Le,k,K,Pi)

[m n] = size(Le); sum = zeros(1,n);
```

```
for i=1:K
    if i ~= k
        sum = Le(i,:)+ sum;
    end
end Lo = Dtilde + sum;
```

9.2.10 clip.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                         clip.m                           %
%                    created: July 2, 2007                 %
%        Applied Mathematical Science Institute 2007       %
%                       Team Special Ed                    %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Function that ensures that NaN is not ever a problem
% clips code and keeps it under control
function C = clip(m) [x y] = size(m);
B = ones(x,y)*10^7;
C = sign(m).*min(abs(m),B);
```

9.2.11 FinalLLR.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                        FinalLLR.m                        %
%                    created: July 2, 2007                 %
%        Applied Mathematical Science Institute 2007       %
%                       Team Special Ed                    %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function Final = FinalLLR(Dtilde,Le,K)
% Takes Dtilde and the Le and makes the final decision on what the message was
[m n] = size(Le); sum = zeros(1,n); Final = zeros(n); for k=1:K
    sum = Le(k,:) + sum;
end Final = sum + Dtilde;
```

9.3 Permutations

This section documents the codes of the various permutations used in our research.

9.3.1 Right to Left/Top to Bottom

Reads the matrix from right to left, top to bottom as the permutation.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                RLTB.m                                 %
% Right to Left/Top to Bottom Classical Block Permutation %
%                                                                       %
%       Applied Mathematical Science Institute 2007       %
%                          Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function temp = RLTB(M) [m n] = size(M); x = 1; y = n;
% Reads the matrix Right to Left/Top to Bottom as the Permutation
for i=1:m
    for j=1:n
        temp(i,j) = M(x,y);
        x = x+1;
        if x > m
            x = 1;
            y = y-1;
        end
    end
end
```

9.3.2 Right to Left/Bottom to Top

Reads the matrix from right to left, bottom to top as the permutation.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                RLBT.m                                 %
% Right to Left/Bottom to Top Classical Block Permutation %
%                                                                       %
%       Applied Mathematical Science Institute 2007       %
%                          Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function temp = RLBT(M) [m n] = size(M); x = m; y = n;

% Reads the matrix Right to Left/Bottom to Top as the Permutation
for i=1:m
    for j=1:n
        temp(i,j) = M(x,y);
        x = x-1;
        if x <= 0
```

```
            x = m;
            y = y-1;
        end
    end
end
```

9.3.3 Left to Right/Top to Bottom

Reads the matrix from left to right, top to bottom as the permutation.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                              LRTB.m                                   %
% Left to Right/Top to Bottom Classical Block Permutation %
%                                                                       %
%       Applied Mathematical Science Institute 2007       %
%                         Team Special Ed                 %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function temp = LRTB(M) [m n] = size(M) x = 1; y = 1;
% Reads the matrix Left to Right/Top to Bottom as the Permutation
for i=1:m
    for j=1:n
        temp(i,j) = M(x,y);
        x = x+1;
        if x > m
            x = 1;
            y = y+1;
        end
    end
end
```

9.3.4 Left to Right/Bottom to Top

Reads the matrix from left to right, bottom to top as the permutation.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                              LRBT.m                                   %
% Left to Right/Bottom to Top Classical Block Permutation %
%                                                                       %
%       Applied Mathematical Science Institute 2007       %
%                         Team Special Ed                 %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function temp = LRBT(M) [m n] = size(M) x = m; y = 1;
% Reads the matrix Left to Right/Bottom to Top as the Permutation
for i=1:m
    for j=1:n
```

```
        temp(i,j) = M(x,y);
        x = x-1;
        if x <= 0
            x = m;
            y = y+1;
        end
    end
end
```

9.3.5 Permutation Presented by Tejas Bhatt and Victor Stolpman

This code models the structured permutation presented by Tejas Bhatt and Victor Stolpman.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                      BhattPermutation.m                                %
%                          July 16, 2007                                 %
%       Applied Mathematical Science Institute 2007                      %
%                         Team Special Ed                                %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function Mat = BhattPermutation(I,J,K, PrimeI, PrimeJ)
%   model of the permutation presented
%   by Tejas Bhatt and Victor Stolpman in 'Structured
%   Interleavers and Decoder Architectures for Zigzag Codes'
%   and in the US patent US 2007/0067697Al
% I,J: Block Length of the message
% K: number of permutations
% PrimeI, PrimeJ: the new shape of the matrix to use for permuting the IxJ
% matrix.

counter = 1; for i=1:PrimeI
    for j=1:PrimeJ
        M(i,j) = counter;
        counter = counter+1;
    end
end
%% Finding p's
counter = 1; for i=1:I
    if gcd(i,I)==1
        p(counter) = i;
        counter = counter + 1;
    end
end
%% Find SjK
bool = 0; ind = 0; maxind =length(p);
```

```
% find the first prime number
while (bool == 0 && ind < maxind)
    ind = ind + 1;
    v = 1:PrimeJ;
    S(1,:) = mod(v.*p(ind),PrimeI);
    bool = conditionA(S(1,:));
end if ind == maxind
    tempM = 'ERROR-out of prime numbers';
    return
end
% find K-1 more prime numbers
if K>1
    for k=2:K
        bool1 = 0;
        bool2 = 0;
        while(bool1==0 && bool2==0 && ind < maxind)
            ind = ind + 1;
            v = 1:PrimeJ;
            S(k,:) = mod(v.*(ind),PrimeI);
            % make sure that there is no repeated Sk values
            bool1 = conditionA(S(k,:));
            j = 1;
            while(j<=length(S)&&bool==0)
            % makes sure that there will be no bits in the same parity more than once
                bool2 = conditionB(S(k,:),S(j,:));
                j = j+1;
            end
            if ind == maxind
                tempM = 'ERROR-out of prime numbers';
                return
            end
        end
    end
end
%% Perform Column Shift--yo
%temp = reshape(M',PrimeI,PrimeJ);
for k=1:K
    for j=1:PrimeJ
        temp(:,j,k) = circshift(M(:,j),S(k,j));
    end
end
%% Bit-reversing
[Y,Mult] = bitrevorder([1:PrimeI]); for i=1:PrimeI
   temp2(i,:,:) = temp(Mult(i),:,:);
end
```

```
%% Flatten the matrix----raaaaaaaaawwwwwwwrrrr---big dino feet!

for k=1:K
    tempM(:,:,k) = reshape(temp2(:,:,k)',1,I*J);
end
%% Make it a better format
Mat = un3d(tempM);
```

9.3.6 conditionA.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                           conditionA.m                                 %
%                          July 16, 2007                                 %
%       Applied Mathematical Science Institute 2007                      %
%                         Team Special Ed                                %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function bool = conditionA(v)
% takes a vector and makes sure there are no repeated values in the vector
bool = 1; for i=1:length(v)-1
   for j=i+1:length(v)
       if v(i) == v(j)
          bool = 0;
       end
   end
end
```

9.3.7 conditionB.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                           conditionB.m                                 %
%                          July 16, 2007                                 %
%       Applied Mathematical Science Institute 2007                      %
%                         Team Special Ed                                %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function bool = conditionB(v1,v2)
% checks and makes sure two infor bits will only act in the same column row
% once
bool = 0; for i=1:length(v1)-1
   for j=i+1:length(v1)
       if v1(i)-v1(j) == v2(i)-v2(j)
          bool = 1;
       end
   end
end
```

9.3.8 un3d.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                un3d.m                                  %
%                            July 16, 2007                               %
%       Applied Mathematical Science Institute 2007                      %
%                           Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function Mat = un3d(v)
% resizes a 3D array into a 2D array
[I J K] = size(v); for k=1:K
   Mat(k,:) = v(:,:,k);
end
```

9.3.9 Randomly Permuting Each Column

Function that randomly permutes each individual column of the matrix, not allowing the the entries each column to leave their respective column.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                            randcolperm.m                               %
%                            July 16, 2007                               %
%       Applied Mathematical Science Institute 2007                      %
%                           Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [temp4 disp]= randcolperm(I,J,K)
% Randomly permutes values within a column for each column, no values are
% allowed to leave their respective column. It then returns a permutation.
%% Make General Matrix
M = zeros(I,J); counter = 1; for i=1:I
    for j=1:J
        M(i,j) = counter;
        counter = counter + 1;
    end
end
%% Start making K permutations
for k=1:K
%%  Declare Random Permutations for Each Column j
    for j=1:J
       Pi(j,:) = randperm(I);
       disp(k,j) = fastdisp(Pi(j,:));
    end
%%  Take the transpose of the matrix and Permute the matrix
    temp1 = M';
    temp2 = M';
    for j=1:J
```

```
        temp2(j,:) = Permute(temp1(j,:),Pi(j,:));
    end
    temp3 = temp2';
%%  Flatten the new Matrix
    temp4(k,:) = reshape(temp3',1,I*J);
end
```

9.3.10 Randomly Permute Each Column with a Restriction

Function that randomly permutes each individual column of the matrix, not allowing the the entries each column to leave their respective column. The restriction is that rows can not have the same two information bits in different permutations' rows more than two times.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                     restrictedrandcolperm.m                           %
%                          July 17, 2007                                %
%       Applied Mathematical Science Institute 2007                     %
%                        Team Special Ed                                %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function perm= restrictedrandcolperm(I,J,K)
% Randomly permutes values within a column for each column, no values are
% allowed to leave their respective column. It then returns a permutation.
% makes sure no more than rows of each permutation has no more than two
% shared information bits.
%% Make General Matrix
M = zeros(I,J); counter = 1; for i=1:I
    for j=1:J
        M(i,j) = counter;
        counter = counter + 1;
    end
end
%% Start making K permutations
k = 1; while k<=K
%%  Declare Random Permutations for Each Column j
    for j=1:J
        Pi(j,:) = randperm(I);
    end
%%  Take the transpose of the matrix and Permute the matrix
    temp1 = M';
    temp2 = M';
    for j=1:J
        temp2(j,:) = Permute(temp1(j,:),Pi(j,:));
    end
    temp3 = temp2';
%% Determine if the no rows have repeat columns
```

```
    if k~=1
        bool = 1;   %assume the permutation is good
        kd = 0;
        while(kd<k-1 && bool == 1)
            kd = kd+1;
            for t1 = 1:I
               t2 = 0;
               while(t2<I&&bool==1)
                  t2 = t2+1;
                  difference = temp3(t1,:)-storage(t2,:,kd);
                  counter = 0;
                  for t3=1:length(difference)
                     if difference(t3)==0
                         counter = counter+1;
                     end
                  end
                  if counter > 2
                     bool = 0;
                  end
               end
            end
        end
        if bool == 1
           storage(:,:,k) = temp3(:,:);
            k = k+1;
        end
    else
        storage(:,:,k) = temp3(:,:);
        k = k+1;
    end
end perm = storage;
%% Flatten the matrix----raaaaaaaaawwwwwwwrrrr---big dino feet!

for k=1:K
    tempM(:,:,k) = reshape(storage(:,:,k)',1,I*J);
end
%% Make it a better format
perm = un3d(tempM);
```

9.3.11 swapperperm.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          swapperperm.m                                 %
%                                                                        %
%       Applied Mathematical Science Institute 2007                      %
%                        Team Special Ed                                 %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function result = swapperperm(v,K)
% takes a vector, and for every ten values the first five values of that
% block are switched with the last five values of the block. Same idea with
% the remainder if the block length is not an a factor of ten, but switches
% the last five values with the remaining values from the last block.

n = size(v);

r = mod(n(2),10);   % find the remainder of the size
                    % divided by 10

j = 6;              % initial value of 10
k = 0;              % k is initially 1

for i=1:(n(2)-r)    % look at the values up to the remainder
   if j-k <= 10     % ensures the values are only swapped once
      a(j) = v(i);  % assign v(i)'s value to a temp variable
      a(i) = v(j);  % assign a(i)
      j = j+1;
   end
   if mod(i,10) == 0% resets the process for each multiple of 10
      k = k + 10;
      j = 6 + k;
   end
end

j = n(2) - r + 1;   % parameters for dealing with the remainder
b = 0;              % j is where the remainder starts
while j <= n(2)
    a(j) = v(n(2)-b);   %assigns value to array
    j = j+1;
    b = b+1;
end for k =1:K result(k,:)= a(1,:); end
```

9.3.12 Magic Square Permutation

Creates a Magic Square and returns the square as a row vector used as a permutation.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                        magicsquareperm.m                               %
%                                                                        %
%       Applied Mathematical Science Institute 2007                      %
%                         Team Special Ed                                %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function ft = magicsquareperm(n)
% a generates a n by n magic square and turn it into a 1 dimensional vector
% used as a permutaiton.
ft = zeros(1,n); v = magic(n);
%first start with a zero vector of length 1000000.
for j = 1:n
    ft(n*j - (n-1):n*j)=v(j,:);
end;
```

9.3.13 throwupperm.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                           throwupperm.m                                %
%                        Dated: June 24, 2007                            %
%       Applied Mathematical Science Institute 2007                      %
%                           Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function T = throwupperm(N)

%given a random message throwupperm swaps the rows of a matrix bottom up so
%that we get diffrent parodies



[A B]= size(N) M = zeros(A,B) k=1 for i= 1:A
    for j = 1:B
    M(i,j) = k;
    k = k+1;
    end
end

T(1,:) = flatten(M);

k = 2; for i =  A:-1:2
    M = swaprows(M,i,1)
    c = T(k-1,:) - flatten(M)
    for i = 1: (ceil(mod(rand*10000,B)) + 2)
       %pick a random row and shift it
```

```
        w = (ceil(mod(rand*10000,A)))
        x=M(w,:);
        M(w,:)=horazontalswrill(x');
        %pick a random row and swril it
    end
    if find (c) ~= 0
    T(k,:) = flatten(M); %turn it into a permutation
    k=k+1;
    end
end
```

9.3.14 flatten.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          flatten.m                                    %
%                       created: June 30, 2007                          %
%       Applied Mathematical Science Institute 2007                     %
%                          Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function T = flatten(M)

%takes a message and flattens it
[A B] = size(M)

T =reshape(M', 1,A*B)
```

9.3.15 swapprows.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          swapprows.m                                  %
%                       created: June 30, 2007                          %
%       Applied Mathematical Science Institute 2007                     %
%                          Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function r = swaprows(M, a, b)
% given a matrix M swaps rows a and b

temp(1,:) = M(a,:); M(a,:) = M(b,:); M(b,:) = temp(1,:); r =
M;
```

9.3.16 horazontalswrill.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          horazontalswrill.m                           %
%                       created: June 30, 2007                          %
%       Applied Mathematical Science Institute 2007                     %
%                          Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


function T = horazontalswrill(n)

%takes a perumtation n and swirls it around by a random amount

[A B] = size(n)

c = ceil(mod(rand*rand*1000,B-1))

T = circshift(n,c);
```

9.4 Dispersion and their Measurements

9.4.1 sumvarows.m

This function calculates variance of a permutation.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          sumvarows.m                                  %
%                       created: June 30, 2007                          %
%       Applied Mathematical Science Institute 2007                     %
%                          Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

function [varsum, T] = sumvarows(m,rows,cols)

```
[A B] = size(m); % create said matrix
k=1; w  = zeros(rows,cols); m(1,:); for a  = 1:rows
    for b  = 1:cols
        w(a,b) = (m(1,k)); %puts the elements of the permutation in the matrix
        %as if they were apart of the identity function
        k = k+1;
    end
end

if (A> 1)
    for t = 2:A
        k = 1;
```

```
        temp =zeros(rows,cols);
        for a  = 1:rows
            for b  = 1:cols
                temp(a,b) = ceil(m(t,k)/cols);
                k = k+1;
            end
        end
        w = [w;temp]; %append the new matrix on top of the old one
    end
end [Q U]= size(w);

for i = 1: Q
    T1(i)  =  var(w(i,:)); %take the variance
    T(i) = T1(i)*(U-1)/(U);%multiply by constat to turn it into population variance
end
```

9.4.2 dispoflistofpermswithinvers.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                   dispoflistofpermswithinvers.m                        %
%                     created: June 30, 2007                             %
%       Applied Mathematical Science Institute 2007                      %
%                         Team Special Ed                                %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


function R = dispoflistofpermswithinverse(n)
%this function given a list of permutations
%computes the dispersion every permutations with the inverse
%of another permutation.

[A,B] = size(n); T=zeros(A,B);

for i = 1:A
for j = 1:B
if(j<i)
temp =  compute2perms(n(i,:),inverseperm(n(j,:)));
%compute the values of the matrix
T(i,j)=fastdisp(temp(1,:));
end
end
 end

R = sum(sum(T))/(A*(A-1)/2); %take the average
```

9.4.3 inverseperm.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          inverseperm.m                                %
%                       created: June 30, 2007                          %
%       Applied Mathematical Science Institute 2007                     %
%                          Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function IP = inverseperm(PI)
%computes the inverse perm
[A B] = size(PI);

IP = zeros(1,B);      % loop used to determine the inverse permutation
IP(PI(1,:)) = 1:B;
```

9.4.4 compute2perms.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                          compute2perms.m                              %
%                       created: June 30, 2007                          %
%       Applied Mathematical Science Institute 2007                     %
%                          Team Special Ed                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


function T = compute2perms(a,b)
%given two permutations of the same block length this function puts the
%identity through 1 permutation the the other

[A B] = size(a);

for  i = 1:B
    T(b(i)) = a(i);
end
```

9.5 Triangular Distance and Random Walks

This program uses Triangular Distance Equations generated by Random Walks along the number line from -n to n, defined by user input. Random Walks on the set $\begin{pmatrix} -n & \dots & -2 & -1 & 0 & 1 & 2 & \dots & n \end{pmatrix}$ produced more than the Triangular Distance tables for the permutations on $\mathbb{S}_n$. The program determines whether or not the steps taken by the Random Walk are in fact a Triangular Distance, and prints out each solution and total number of solutions. C++ was used due to the abstract data type Doubly Linked Lists and the flexibility of this kind of data type.

```
/*****************************************************/
/*          Random Walks Using Linked Lists          */
/*                           by                      */
/*      Applied Mathematical Science Institute       */
/*                   Team Special Ed                 */
/*                 created: 7/12/2007                */
/*                modified: 7/12/2007                */
/*      This program uses random walks under set     */
/*      conditions to generate all of the random     */
/*      permutations of the Symmetric Group n        */
/*      using Triangular Distance Tables             */
/*****************************************************/

#include <iomanip> #include <iostream> #include <fstream>

//  Create a structure to store each point on
//  the numberline.
struct Node {
    int n,visit;
    bool visited;
    Node *Next,*Prev;
};
//FUNCTIONS
void initialize();                  // initializes Head and Tail

Nodes void create(int sample);             // creates the number line

from n void marknode(int n, bool value);   // marks the specified

node true or false Node next(Node *temp, int step);
// moves to next node specified by step

Node prev(Node *temp, int step);
//moves to prev node specified by step
bool isaperm();
// verifies that a permutation was created
```

```
void search(Node *t, int max, int counter);
// searches for permutations

//GLOBAL VARIABLES
Node Head;                          // The top node represents 0 on
the number line Node Tail;          // The tail node set to 1000000
                                    // as "wall" between -n and n.

int *iter,*use;                     // pointers that correspond to
                                    // the amount each step
                                    // can be used.

int *diag;                          // pointer that corresponds to
                                    // the steps or distances
                                    // on the diagonal

int *visitednodes;                  // pointer that corresponds to
                                    // the nodes visited
int *perm;
// pointer that corresponds with the permutation int solutions, sample;
// number of solutions and sample size respectively

using namespace std;

void initialize() {
    Head.n = 0;                     // set Head equal to 0
    Head.Next = Head.Prev = &Tail;  // set the Head pointers to point to the Tail
    Tail.Next = Tail.Prev = &Head;  // set the Tail pointers to point to the Head
    Tail.n = 1000000;               // create a really tall "wall"
    Tail.visited = true;   // set Tail's visited value to true--no one likes Tail
    solutions = 0;                  // no solutions...yet
}

void create(int sample) {
    Node *temp1,*temp2;             // temp Node pointers
    temp1 = &Head;                  // set temp1 to point at Head
    int i;                          // loop variable
    for(i=1;i<=sample;i++)          // create the positive side of the number line
    {
        temp2 = new Node;
        temp2->Next = temp1->Next;
        temp1->Next->Prev = temp2;
        temp1->Next = temp2;
        temp2->Prev = temp1;
```

```
        temp2->n = i;
        temp2->visited = false;
        temp2->visit = 0;
        temp1 = temp2;
    }
    temp1 = &Head;                  // reset temp1 to point back at Head
    for(i=1;i<=sample;i++)          // create the negative side of the number line
    {
        temp2 = new Node;
        temp2->Prev = temp1->Prev;
        temp1->Prev->Next = temp2;
        temp2->Next = temp1;
        temp1->Prev = temp2;
        temp2->n = (-1)*i;
        temp2->visited = false;
        temp2->visit = 0;
        temp1 = temp2;
    }
}

void marknode(int n, bool value) {
    Node *temp;
    temp = Tail.Prev;
    while(temp->n!=n)
    {
        temp = temp->Next;
    }
    temp->visited = value;
    if(value)
        temp->visit += 1;
    else
        temp->visit -= 1;
}

Node next(Node *temp, int step) // where to start and how many
                                 // "steps" forward
{
    if(temp->n == 1000000)       // make sure the "wall" isn't climbed
    {
        return Tail;
    }

    int i;
    Node *temp2;
    temp2 = temp;
```

```
    for(i=1;i<=step;i++)       // "step" though each node
    {
        temp2 = temp2->Next;
        if(temp2->n==1000000)  // loop hit the "wall"
        {
            return Tail;
        }
    }
    return *temp2;                  // return end point
}

Node prev(Node *temp, int step)     // where to start and how many
                                    // "steps" backwards
{
    if(temp->n == 1000)             // make sure the "wall" isn't climbed
    {
        return Tail;
    }
    int i;
    Node *temp2;
    temp2 = temp;
    for(i=1;i<=step;i++)            // "step" though each loop
    {
        temp2 = temp2->Prev;
        if(temp2->n==1000)
        {
            return Tail;
        }
    }
    return *temp2;                  // return end point
}

bool isaperm()
{
    int i;                         // i is the first number to use in the permutations
    int ind;                         // navigator of the indexes of the pointer arrays
    bool flag;                       // determines if permutation is found
    for(i=1;i<=sample;i++)           // start at one and go to n
    {
        ind = 1;                     // start at index 1 for diag and perm
        flag = true;                 // initialize flag as true
        perm[0] = i;                 // initialize first perm entry as i
        while(flag)
        {
            perm[ind] = perm[ind-1] + diag[ind-1];
```

```
            // determine the next entry using the corresponding diag entry and prev perm

            // make sure perm value is within its limits
            if(perm[ind] <= 0 || perm[ind]>sample)
                flag = false;
            if(ind==sample-1 && flag)               // PERMUTATION FOUND!
            {
                return true;
            }
            ind++;                                  // increment the index
        }
    }
    return false;                                   // no permutation found
}

void search(Node *t, int max, int counter)      // recursion!
{
// takes a starting position, max steps and largest step and counter
    if(counter == max)              // took max amount of steps
    {
        if(abs(visitednodes[max-1]-visitednodes[0]) <= max && isaperm())
        {
    // two conditions needed to be meet: abs val of largest step can be no larger than t
     // max step size and it must be a permutation
            solutions++;            // increase solutions counter
        //print solution
            cout << "\nSolution " << solutions << " is: " << "\nThe Diagonal is: ";
            for(int i=0;i<=max-1;i++)
            {
                cout << diag[i] << " ";
            }
            cout << "\nThe Permutation is: ";
                for(int j=0;j<=sample-1;j++)
                {
                    cout << perm[j] << " ";
                }
            cout << "\nThe Nodes visited were: ";
            for(int i=0;i<=max-1;i++)
            {
                cout << visitednodes[i] << " ";
            }
            cout << "\n";
        }
        return;
    }
```

```
    // start recursion
    for(int i=1;i<=max;i++)             // step sizes
    {
        if(use[i-1]+1<=iter[i-1])
        {
            use[i-1] += 1;              // mark the node used one more time
            marknode(t->n,true);        // mark the node visited
            Node *temp;
            temp = &next(t,i);
            if(temp->visited!=true)     // if next node isn't visited go to it
            {
                diag[counter] = i;      // set diag to the i step value
                visitednodes[counter] = temp->n;    // mark visit
                search(temp,max,counter+1); // recurse
            }
            temp = &prev(t,i);
            if(temp->visited!=true)
            {
                diag[counter] = (-1)*i; // set diag to the i step value
                visitednodes[counter] = temp->n;    // mark visit
                search(temp,max,counter+1); // recurse
            }
            marknode(t->n,false);       // remark the node false--I was never there
            use[i-1] -= 1;              // take back of the use
        }
    }
}

void main() {
    //ask for the set size and place it into sample
    cout << "Please input the set size:";
    cin >> sample;
    initialize();
    create(sample-1);
    // create temp. arrays
    iter = new int [sample-1];
    use = new int [sample-1];
    diag = new int [sample-1];
    visitednodes = new int [sample-1];
    perm = new int [sample];
    // initialize the arrays
    for(int i=0;i<=sample-1;i++)
    {
        iter[i] = abs((sample-1)-i);
        use[i] = 0;
```

```
        perm[i] = 0;
    }
    search(&Head,sample-1,0);       // begin long search
    // final output
    cout << "\nThe number of solutions: " << solutions << "\n";
    cout << "\n";
    int x = 1;
    Node temp2 = Head;
    // message to prevent the program from closing at the end when running the
    // executible file
    cout << "\n\n Reached the end of the program.
             Enter any character then 'Enter' to exit: ";
    cin >> x;
    //delete arrays--deny all evidence of their existence
    delete [] iter;
    delete [] use;
    delete [] diag;
    delete [] visitednodes;
    delete [] perm;
}
```

References

[Bh] Bhatt, Tejas; Stolpman, Victor. *Structured Interleavers and Decoder Architectures for Zigzag Codes*, IEEE Conference Proceedings on Signals, Systems and Computers, Oct.-Nov. 2006, pp.99-104.

[Do] Dolloff, Jason; Kenz, Zackary; Rische, Jacquelyn; Rogers, Danielle Ashley; Smith, Laura; Mosteig, Edward. *Algebraic Interleavers in Turbo Codes*. California State Polytechnic University, Pomona and Loyola Marymount University.

[He] Heegard, Chris; Wicker, Stephen B. *Turbo Coding*, Kluwer Academic Publishers, USA, 1999.

[Jo] Jones, Alaina; Moreno, Benjamin; Smith, Laura; Viteri, Andrea; Yao, Kouadio David; Mosteig, Edward. *Exploring Interleavers in Turbo Coding*, 2005 AMSSI Technical Report, available at www.amssi.org.

[Li] Little, John B.; Mosteig, Edward. *Error Control Codes from Algebra and Geometry–Notes for SACNAS Minicourse*, Oct. 2003.

[Tu] Tucker, Alan. *Applied Combinatorics, 5 ed.*, Wiley, 2006.

[Pi(1)] Ping, Li; Chan, Sammy; Yeung, Kwan L. *Iterative Decoding of Multi-Dimensional Concatenated Single Parity Check Codes*, 1998 IEEE International Conference on Communications, vol. 1, pp. 131-135.

[Pi(2)] Ping, Li; Huang, Xiaoling; Phamdo, Nam. *Zigzag Codes and Concatenated Zigzag Codes*, IEEE Transactions on Information Theory, vol. 47, no. 2, Feb. 2001, pp. 800-807.