

10.5.0 Angular Momentum in Two Dimensions

Young and Freedman only discuss the topic of angular momentum in its most general, fully three-dimensional, formulation; however, most of the interesting applications, at our level, involve rotation in only two dimensions. Here is an introduction to the topic of angular momentum suitable for two-dimensional rotation only; *i.e.* when every particle in the system is rotating in a single plane, and the plane of rotation for any particle is parallel to the plane of rotation for any other particle in the system.

So far, during our lessons on rotational dynamics, we have had to learn two definitions, the definition of torque and the definition of rotational inertia (the rotational analogues of force and mass). We now need to define angular momentum, *i.e.* the rotational analog of momentum. To make the right definition, we need to think about the momentum perspective on Newton's Second Law. We learned in Young and Freedman Chapter 8 that, when an impulsive net force is not constant, it is useful to rewrite the Second Law in the following form: $\Sigma \vec{\mathbf{F}} = d\vec{\mathbf{p}}/dt$, or in one dimension, $\Sigma F_x = dp_x/dt$. We want to perform the analogous operation (*i.e.* rewriting the Second Law, this time for rotation) for an impulsive net torque, using the rotational analog of momentum (called angular momentum and symbolized by L). In other words, we want $\Sigma \tau_z = dL_z/dt$, where the z subscript indicates an axis perpendicular to the xy plane, with all particle motion being parallel to the xy plane. We can therefore define angular momentum by analogy; L_z must have the same relationship to $\vec{\mathbf{p}}$ that τ_z has to $\vec{\mathbf{F}}$. All we have to do is to copy down the definition of torque from Young and Freedman Chapter 10.1, change $\vec{\mathbf{F}}$ in that definition to $\vec{\mathbf{p}}$, and we will have defined angular momentum. The fully three-dimensional definition of torque is $\vec{\mathbf{r}} \times \vec{\mathbf{F}}$ and so the fully three-dimensional definition of angular momentum is $\vec{\mathbf{r}} \times \vec{\mathbf{p}}$; but we don't need that fully correct form for rotation in two dimensions only.

For motion confined to two dimensions (parallel to an xy plane), the definition of torque can be taken as $\tau_z \equiv F_t r$, where F_t is the tangential component of the force $\vec{\mathbf{F}}$, which is applied at distance r from axis z . Notice that the label "tangential" in this context is referring to a direction which is "perpendicular to radial"; the choice of "tangential" for this label is reasonable because, in the examples we have done so far, the particles which comprise our rigid systems have always been moving in circular paths around a fixed axis, and so the perpendicular-to-radial force components are also tangential to the paths of the particles. Compare this definition with the three ways of finding the magnitude of torque as given in Young and Freedman equation 10.2,

$$\tau = F\ell = rF \sin|\phi| = |F_t|r,$$

where the absolute value signs are added to emphasize that these expressions give the magnitude of the torque. To define L_z for two dimensional motion, we only need to replace the F in these expressions with p . Here's that definition:

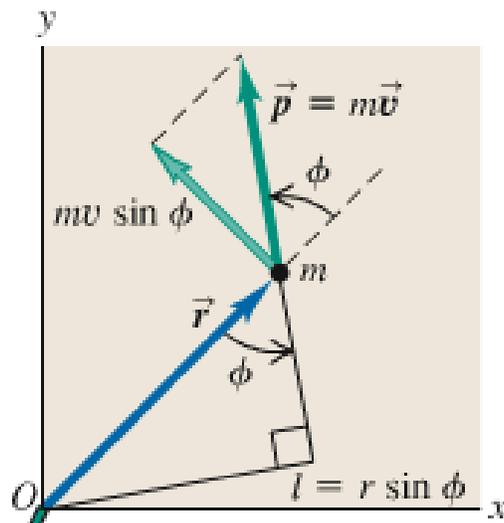
The ANGULAR MOMENTUM (L_z) about axis z for a particle having a momentum \vec{p} , and located at a distance r from z is

$$L_z \equiv (p_{\perp\text{to radial}})r \quad \text{definition of L for two dimensions}$$

where $p_{\perp\text{to radial}}$ is the component of \vec{p} which is perpendicular to the radial direction; we now use “perpendicular-to-radial” instead of tangential because the momentum vector *itself* always points in the direction which is tangential to the path of the particle, and so perpendicular-to-radial is no longer necessarily tangent to the path (see the figure below). With this definition, L_z is a 1D vector with CCW usually taken as positive. The units of angular momentum are $\text{kg}\cdot\text{m}^2/\text{s}$. We only need to agree on a simpler subscript for “perpendicular-to-radial”. Let’s appeal to a common knowledge of polar coordinates (r, θ) ; the “ θ direction” in polar coordinates is always perpendicular to radial. So, we choose to write $p_{\perp\text{to radial}}$ more simply as p_θ , *i.e.* the definition of angular momentum is $L_z \equiv (p_\theta)r$.

Just like every force with a nonzero perpendicular-to-radial component can produce a torque about an axis, every momentum with a nonzero perpendicular-to-radial component can produce an angular momentum about an axis. Calculating angular momentum is just exactly like calculating torque; as shown in the figure, you can use the same three different ways of doing the calculation:

$$|L_z| = |p_\theta|r = (p \sin|\phi|)r = p(r \sin|\phi|) = p\ell.$$

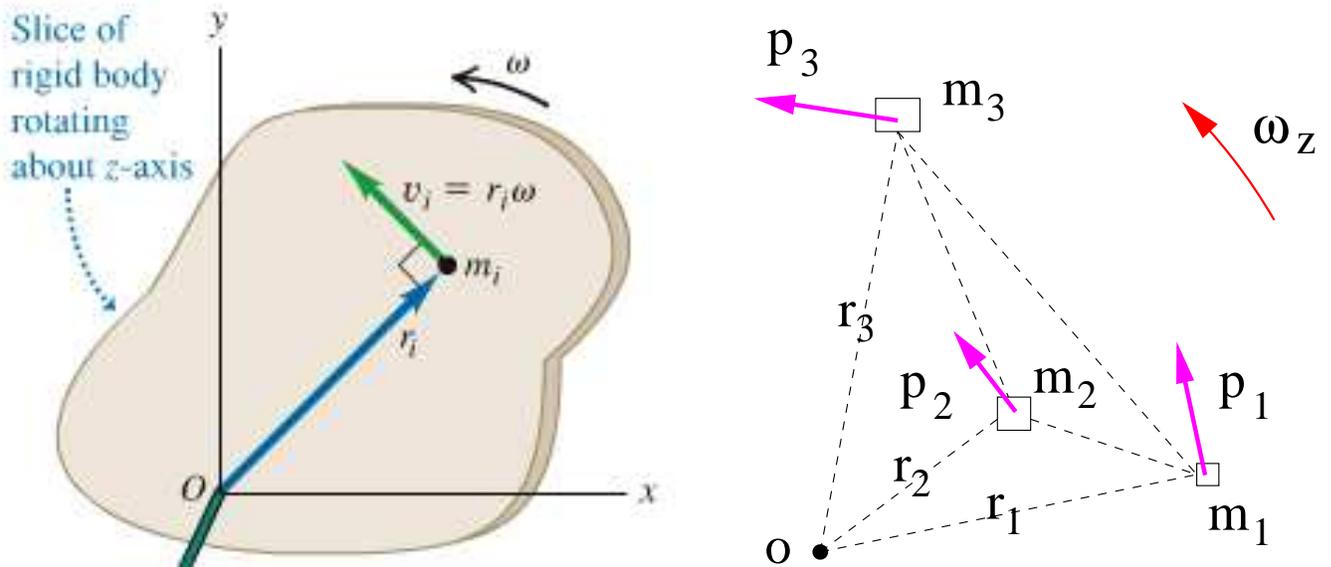


It is important to notice that neither the definition of torque ($\tau_z \equiv F_\theta r$), nor the definition of angular momentum ($L_z \equiv p_\theta r$), requires a rotating rigid object. Any force with a non-zero perpendicular-to-radial component produces a torque, and any momentum with a non-zero perpendicular-to-radial component produces an angular momentum. Of course, it is true that the force must act on a particle (all forces are between particles), and the momentum must be associated with a particle, but there is no requirement in the definitions that the particle’s motion be confined to rotation about the axis; the particle might be moving in a straight line, but if the force in question has a perpendicular-to-radial component then a non-zero torque exists, and if the momentum in question has a perpendicular-to-radial component then a non-zero angular momentum exists.

One more warning – whereas rotational kinetic energy IS a type of kinetic energy (translational kinetic energy in joules can be converted into rotational kinetic energy in

joules), angular momentum IS NOT a type of momentum. Angular momentum is the rotational analog of momentum, just as torque is the rotational analog of force. The units of angular momentum are $\text{kg}\cdot\text{m}^2/\text{s}$ and the units of momentum are $\text{kg}\cdot\text{m}/\text{s}$; there can be no conversion of momentum into angular momentum or vice versa.

While the definition of angular momentum does not require rotation in any way, we will often be concerned with the angular momentum of a rigid system of particles rotating about a fixed axis; so, as an example, we will next calculate the angular momentum of such a system. Assume that the system has rotational inertia I_z about axis z ; also assume that the system is rotating with angular velocity ω_z . The figures show two possible drawings of such a system. The figure on the left shows an object with a continuous, non-symmetric, mass distribution which is rotating about the z axis at the origin O . The figure on the right shows a system consisting of only three masses, connected by rigid massless sticks, but also rotating about the z axis at the origin O . Our calculation of angular momentum can be done for either system; for the system on the left there are an infinite number of infinitesimal masses; for the system on the right there are only three masses. The system on the right explicitly shows the momentum vectors of each particle. For our calculation, the important feature of such a system is that these momentum vectors are now purely perpendicular-to-radial; it is this feature that makes our calculation easy.



We wish to calculate $L_{z,sys} = \sum_i L_{z,i}$. To get each $L_{z,i}$ we use our definition of L_z . We use two facts: (1) each momentum vector is purely perpendicular-to-radial, and (2) $v_t = r\omega_z$. Here's the calculation of $L_{z,i}$ (note that the subscript θ is now replaced with the subscript t , since perpendicular-to-radial is again purely tangential):

$$L_{z,i} \equiv (p_{t,i})r_i = m_i(v_{t,i})r_i = m_i(r_i\omega_z)r_i = (m_i r_i^2)\omega_z$$

You should check each step above to make sure that you understand the reason for that

step. We are now ready to calculate $L_{z,sys}$ about fixed axis z passing through the origin O . We use two facts: (1) ω_z is the same for every particle i , and therefore can be factored out of the sum over all particles, and (2) $\Sigma_i(m_i r_i^2)$ is the definition of the rotational inertia of our rigid system of particles about axis z :

$$L_{z,sys} = \Sigma_i L_{z,i} = \Sigma_i((m_i r_i^2)\omega_z) = (\Sigma_i(m_i r_i^2))\omega_z = I_z \omega_z.$$

Once again, you should make sure that you understand the reason for each step. This result, $L_{z,sys} = I_z \omega_z$ is analogous to the definition of momentum, *i.e.* momentum is mass times velocity; however, it is important to remember that this result **ONLY** holds for a rigid system of particles rotating about a fixed axis z (it does hold in one other case, mentioned below). In general, to find the angular momentum of a system of particles, one must find the angular momentum of every particle from the definition $L_z \equiv p_\theta r$ and then find the system angular momentum from $L_{z,sys} = \Sigma_i L_{z,i}$. The result $L = I\omega$ does hold for one case in which the axis is not fixed. We describe this case here, but we will not do the proof. When the axis passes through the center of mass of a rigid system of particles (without a system that can be considered rigid, at least for some time, the idea of rotational inertia, $I = \Sigma(mr^2)$, makes little sense), and the axis cannot change direction, then $L_{z,cm} = I_{cm}\omega_z$ is true, even if the z axis though the center of mass is translating (as in a rolling wheel). By taking the time derivative of either $L_{z,sys} = I_z \omega_z$ (for a fixed axis) or $L_{z,cm} = I_{cm}\omega_z$ (where the axis is either fixed or translating, but can't change direction), we can easily arrive at either $\Sigma\tau_{z,ext} = I_z \alpha_z$ or $\Sigma\tau_{z,cm} = I_{cm} \alpha_z$ (Young and Freedman equations 10.7 and 10.13).

So we have now selected our definition for angular momentum, namely $L_z \equiv p_\theta r$. It remains to prove that, given this definition, $\Sigma\tau_z = dL_z/dt$. In fact, without the full three-dimensional definitions ($\vec{\tau} = \vec{r} \times \vec{F}$ and $\vec{L} = \vec{r} \times \vec{p}$), this proof is quite tedious. With the full three-dimensional definitions the proof is quite easy, and it appears in Young and Freedman on page 316.

We would like to apply $\Sigma\tau_z = dL_z/dt$ to a system of N particles in a general way, not requiring the system of particles to be rigidly-connected. All that we must do is to add N equations, one equation for each particle i , namely $\Sigma\tau_{z,i} = dL_{z,i}/dt$. On the right-hand side of the sum of equations, we end up taking the time derivative of the angular momentum of the system, *i.e.* $dL_{z,sys}/dt$. The left-hand side of the sum of equations is a little more difficult, but we can simplify by using the figure 10.8 from Young and Freedman page 306. As illustrated in that figure, on the left-hand side of the sum of equations, all the torques due to internal forces drop out, leaving only the torques due to external forces. Therefore, the result of adding the N equations becomes simply

$$\Sigma\tau_{z,ext} = \frac{d}{dt}(L_{z,sys}). \tag{1}$$

This result is applicable for any system of particles whatsoever, rigidly-connected or not, as long as the axis about which the calculation is done is regarded as fixed. The particles need not be rotating about the axis at all. If we wish to find a similar result for a moving axis, then it should not be surprising that the moving axis must pass through the center-of-mass of the system of particles, *i.e.*

$$\Sigma\tau_{cm,ext} = \frac{d}{dt}(L_{cm,sys}).$$

For interested parties, this proof, which uses the full three-dimensional definitions of torque and angular momentum, is provided on our class webpage.

Equation (1) is the two-dimensional version of Young and Freedman's equation 10.29 on page 317. An important and useful special case of equation (1) occurs when the left-hand side is zero. If, for any reason, the sum of the external torques about a selected axis is zero, then the system angular momentum about that axis cannot change. This rule is known as the principle of conservation of angular momentum, and it is the third great conservation principle of classical physics.

THE PRINCIPLE OF CONSERVATION OF ANGULAR MOMENTUM.

For any system of N particles, and with respect to fixed axis z , the $\Sigma\tau_{z,ext}$ acting on the system is zero if and only if the total angular momentum of the system, $\Sigma L_{i,z}$, is constant. In particular, if the system is closed, so that there can be no external torques acting on the system, then the total system angular momentum is constant about any axis. Therefore, if the Universe is a closed system, then the total angular momentum of the Universe is constant.

Your reading continues with the three examples 10.10, 10.11 and 10.12 from Young and Freedman pages 320-321.