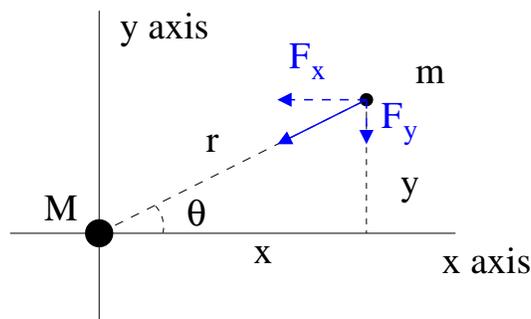


As a second example of solving for the motion of a particle by numerical means, we will solve for the orbital motion of a small mass m around a much larger mass M (taken to be stationary, and at the origin of our coordinate system). This is now a two-dimensional problem, so we will work with both x and y versions of the Second Law. Only gravity is acting, so the x -component of the net force acting on the smaller mass is $\Sigma F_x = -(GMm/r^2)(x/r)$, where (x/r) is the proper expression for the value of $\cos(\theta)$, as defined by the figure to the right. Similarly, the y -component of the net force is $\Sigma F_y = -(GMm/r^2)(y/r)$, and thus the x - and y -components of the Second Law yield $a_x = -(GM/r^3)x$ and $a_y = -(GM/r^3)y$. Begin by converting the value of GM into units more convenient for motions in our solar system, namely years and astronomical units (take one year to be 365.25 days and 1.0 au to be 1.496×10^{11} m).



1. Show how to find the value of GM in units of years and au. The mass of our Sun (our M) is 1.989×10^{30} kg, and the value of G is 6.674×10^{-11} N·m²/kg².

$$GM = \qquad \qquad \qquad = 39.485 \frac{au^3}{y^2}$$

2. The parameters for our calculation will be the value of Δt , the value of GM , and the initial x -location and y -velocity of our orbiting mass (we will always use $y_0 = 0$ and $v_{x0} = 0$). Enter the labels for these parameters in cells N1-N4, and then their values in cell O1-O4; begin with $\Delta t = 0.01$ y, $x_0 = 1.0$ au, and $v_{y0} = 2\pi$ au/y; enter the values of GM and v_{y0} to five significant figures.
2. Make the column for time. Follow the instructions from our previous spreadsheet (except here the fixed location of the value of Δt is cell O1). In this case, apply the appropriate formula to cells A3-A101.
3. Make the columns for x , v_x , a_x , y , v_y , a_y , and finally a column for r and one for $1/r^3$. Enter the appropriate labels in cells B1-I1, then enter the initial conditions for v_x and for y in cells C2 and E2, namely that at time zero the mass has zero x velocity and has y -position zero. Our value for x at time zero is the parameter in cell O3; enter this with a formula in cell B2. Our value for v_y at time zero is the parameter in cell O4; enter this with a formula in cell F2. For a_x , use a formula, namely $a_x(t) = -(GM/r^3)x(t)$ (in the formula bar for cell D2, enter $=(-1)*O\$2*I2*B2$, and then apply to cells D2-D101). For a_y , use a formula for $a_y(t) = -(GM/r^3)y(t)$, and apply to the entire column. For r use a formula for $r(t) = \sqrt{x^2(t) + y^2(t)}$ and apply to the entire column, and similarly for $1/r^3$.

4. Now we need formulas for x and v_x (and then y and v_y) for each subsequent time. As we did in the 1D spreadsheet, for our velocity values we will use an estimate of the average velocity during each time interval. For cell C3, we use the formula $v_{x,avg} = v_x(t) + a_x(t) * 0.5 * \Delta t$; (*i.e.* =C2+D2*0.5*\$O\$1). For cell C4, use $v_x(t + 0.5\Delta t) = v_x(t - 0.5\Delta t) + a_x(t) * \Delta t$ (*i.e.* =C3+D3*\$O\$1) and apply it to cells C4-C101. Then repeat the process for v_y using the appropriate cells. Finally, we can enter the formulas for x and for y ; for x , in cell B3 we use =B2+C3*\$O\$1 and then apply the formula to all of column B. For y , apply the same idea.
5. Check your work. If you have entered everything correctly, you should have the following results in row 101: $t=0.99$ y, $x=0.997617$ au, $v_x=0.630321$ au/y, $a_x=-39.3907$ au/y², $y=-0.06902$ au, $v_y=6.2546$ au/y, $a_y=2.725416$ au/y², etc.
6. Now make a motion diagram of your results. Choose INSERT Chart > Scatter, then right click on the chart and Select Data. Remove any data automatically placed on the chart by Excel, then Add data. For the Series name, type =“motion diagram”, then select cells B2-B101 (x position) for Series X values, and cells E2-E101 (y position) for Series Y values (for the Y values, you must first delete anything automatically placed there by Excel). Your result should be a motion diagram of a circular orbit; you will probably need to manually adjust the dimensions of the Excel chart to insure that the x and y axes have the same scaling.
7. The actual orbit of Earth is not quite circular; instead the aphelion distance is 1.0167 au, and the speed at aphelion is 6.1796 au/y. Change the parameters for x_0 and v_{y0} so that the calculation produces Earth’s slightly elliptical orbit.
8. The table below contains aphelion distances and speeds for three Earth-crossing asteroids. Use your spreadsheet to display the motion diagrams for the orbits of these asteroids and enter the calculated perihelion distances. Adjust the dimensions of your chart to maintain equal scaling on the x and y axes, and adjust your value of Δt so that approximately one full orbit is shown. You might like to compare your calculated perihelion distances with the accepted distances as posted on Wikipedia.

	Aphelion Distance	Aphelion Speed	Perihelion Distance
Asteroid	(au)	(au/y)	(au)
3362 Khufu	1.4531	3.8003	
25143 Itokawa	1.695	4.0948	
3753 Cruithne	1.5114	3.5601	

To make your spreadsheet more flexible, add parameters for the initial values of y position and x velocity. Then, given the location and velocity of launch, and with suitable units for the relevant value of GM , you can calculate the orbit of any $m \ll M$.