Hydrostatic equilibrium of a spherical self-gravitating body (computational lab)

**In Brief:** an Excel spreadsheet exercise to solve the equation of hydrostatic equilibrium numerically for the variation of the pressure in the interior of the Earth.

For a body in hydrostatic equilibrium, the pressure at any point within the body supports the weight of what’s above. For a fluid (or solid) in a uniform external gravitational field (due to some large external mass), this gives \( \frac{dP}{dy} = -\rho \ g \) where the gravitational field is in the y direction (Wolfson and Pasachoff Eq. 18-2). This equation is also valid for a body “in its own gravitational field” as will be briefly discussed in class. The pressure balances the gravitational attraction of any mass part to the rest of the mass of the body, supporting the body against gravitational collapse. For a body with a spherically symmetric mass density \( \rho(r) \), the generalization is

\[
\frac{dP(r)}{dr} = -\rho(r)g(r) = -\rho(r) \frac{GM(r)}{r^2},
\]

where \( M(r) \) is the amount of mass contained within a sphere of radius \( r \), so that

\[
\frac{dM(r)}{dr} = 4\pi r^2 \rho(r).
\]

To help you understand equation (1), I should tell you that the gravitational field \( g(r) \) satisfies Gauss’ law in analogy to the electric field. See eqn. 24-4 of WP. Simply take the Gaussian surface to be a sphere of radius \( r \) concentric with the Earth to get the gravitational field strength \( g(r) \) given in equation (1).

In this lab exercise you are to create a spreadsheet program to solve these two differential equations numerically to obtain the pressure \( P(r) \), given \( \rho(r) \), and will apply it to a few interesting cases. This solution is greatly simplified by noting that if you are given \( \rho(r) \), you can evaluate \( M(r) \) from equation (2) by simple integration. Armed with the resulting \( M(r) \), the right-hand side of (1) is a known function, so you can then also solve (1) by simple integration. You don't need any truly differential-equation procedures. Simple integration perhaps numerical is enough if \( \rho(r) \) is a complicated function.

**General Procedure**

1. First, understand equations (1) and (2) above.

2. The case of an incompressible body, \( \rho = \text{constant} \), can be solved exactly. Integrate the differential equations to find \( P(r) \) analytically for this case. This will be your check: whatever spreadsheet program you write must, when applied to the incompressible case, agree with your analytical solution. This is such an important consideration that you MUST get the analytical solution right. Do this first, and don't continue very far until the instructor or TA has approved and initialed a preliminary written presentation of the analytical solution. Include this initialed preliminary presentation as an attachment to your lab report.

3. Assume you’re given \( \rho(r) \) at equally spaced points along a radius, center to outer edge, of a spherical body like a star, planet, etc. Figure out and set up an Excel spreadsheet program to
produce the pressure $P(r)$, via a numerical integration of the differential equations. Use a rectangular rule, trapezoidal rule, or something fancier if you wish, but *don’t reinvent the wheel! A lot is known about numerical integration; see the end of this write-up*).

(4) Apply your spreadsheet to the incompressible body case: find $P(r)$ *numerically* for this case and test your spreadsheet thoroughly to see how it does vs. the exact solution.

(5) An attached graph gives the variation of density within the Earth. Use it in your spreadsheet to generate the pressure variation within the Earth. (The graph is from Cook, A. H., *Physics of the Earth and Planets*, Halsted Press/J. Wiley, 1973). It is the first of these two graphs that is relevant for the numerical calculation, of course.

Aside: One of the standard density models of the Earth is known as the “Preliminary Reference Model” by Dziewonski and Anderson (1981). You might look it up if you are so motivated. Much work on this topic has been done since then which also takes into account the angular dependence of the density.

*Very important:* you should write your spreadsheet so that exactly the same procedure does both the incompressible test case and the Earth, changing just the values in the density column. If you have to reprogram your spreadsheet to do the Earth then you are defeating the purpose of using the incompressible case as a known test case to debug your procedure.

**A few remarks about numerical integration**

The simplest numerical integration algorithm is the “trapezoidal rule.” Assume that the function $f(x)$ is known at the points $x_0, x_1, \ldots, x_{N-1}, x_N$ where the spacing between the points is $h$, with $x_0 = a$ and $x_N = b$. Let $f(x_i) = f_i$. Then if $h$ is small enough and if $f(x)$ is continuous the trapezoidal rule reads:

$$
\int_a^b f(x) \, dx \approx h \left( \frac{1}{2} f_0 + f_1 + f_2 + \ldots + f_{N-1} + \frac{1}{2} f_N \right)
$$

A more accurate formula is given by Simpson’s rule:

$$
\int_a^b f(x) \, dx \approx \frac{h}{3} \left( f_0 + 4 f_1 + 2 f_2 + 4 f_3 + \ldots + 2 f_{N-2} + 4 f_{N-1} + f_N \right)
$$

Note the alternating coefficients 2 and 4. This approximation requires an odd number of points (so $N$ is even). If the function $f(x)$ is piecewise continuous (for example, continuous except for a few discontinuities), you can use either of these methods for each of the subintervals in which the function is continuous.

**The Write-up**

Things you should include:

- a derivation and discussion of the hydrostatic equilibrium condition, eqns. (1) and (2).
- presentation of the analytical solution for the constant density incompressible case.
- a thorough description of your spreadsheet program, including discussion of any issues you feel are relevant.
- a formula dump of enough of your Excel spreadsheet so that we can see *exactly* how it works (select Tools, Options, select the View tab, check the Formulas box; adjust column widths so all the
• presentation of a numerical solution for the incompressible case, and comparison with the exact analytical solution.
• a graph of the pressure variation for the incompressible case.
• application to the Earth: graph, center pressure, discussion, comments, etc.
• your initialed preliminary written presentation of the analytical solution, as an attachment.

Figure 5.1 The variation of density within the Earth: (a) actual density; (b) reduced to zero pressure.