WORK AND THE WORK-KINETIC ENERGY THEOREM

We now know the Laws of Nature within the system of Newtonian Mechanics, namely the force laws, the Universal Law of Gravity and Newton's Third Law, and the Second Law of Motion, which connects the idea of forces with the mathematics of motion. All of these Laws are for particles. Eventually, we will want to apply these Laws to systems of particles. Before we do that, we would like to ask "What is the effect of the Second Law acting on a particle as it moves along a path from some starting location to some ending location?" To answer this question, we need to be able to "integrate the Second Law over the path."

To see what we mean by "integrate the Second Law," let's integrate the Second Law over a path in time, *i.e.* from some starting time $t_i$ to some ending time $t_f$, the times which will mark the beginning and end of some experiment. We start with the Second Law for a particle of mass $m$:

$$\Sigma\vec{F}_{\mathrm{on\,m}} = m\vec{a}_{\mathrm{of\,m}}, \tag{1}$$

and we integrate both sides of the equation over the elapsed time of our experiment:

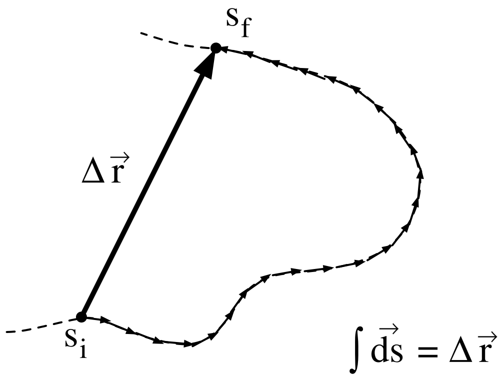
$$\int_{t_i}^{t_f} \Sigma\vec{F}_{\mathrm{on\,m}} dt = \int_{t_i}^{t_f} m\vec{a}_{\mathrm{of\,m}} dt. \tag{2}$$

On the left-hand side of the resulting equation, we have the cumulative effect of the net force on $m$ acting over time; note that the units of the left-hand side are newtons times seconds (N·s), and remember that we expect that the net force (as well as the acceleration on the right-hand side) will be varying with time $t$. We will simply give the quantity on the left-hand side a name, the "net impulse on $m$". The right-hand side is easy to interpret, since velocity is the anti-time-derivative of the acceleration; therefore:

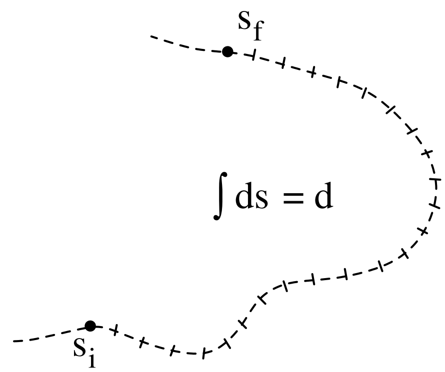
$$\int_{t_i}^{t_f} \Sigma\vec{F}_{\mathrm{on\,m}} dt = m\Delta\vec{v}_{\mathrm{of\,m}}. \tag{3}$$

and we call this result (which comes from "integrating the Second Law over time") the "Impulse-Momentum Theorem" (the right-hand side is called the "change in momentum").

Having seen what it means to "integrate the Second Law over time," we wish now to integrate the Second Law over a path in space; this calculation will allow us to see the cumulative effect of a net force on our particle of mass $m$ acting over that path. First we need to draw a random path in space (we will limit our path to two dimensions of space) and select a "path variable" or "path coordinate".




The sum of the vector steps is the displacement.




The sum of the step lengths is the pathlength, i.e. the distance traveled.

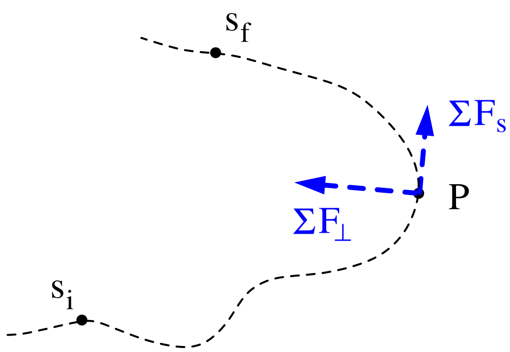
The most popular symbol for the path coordinate is $s$. Imagine a measuring tape (or string) laid out along the path; $s_i$ marks the position of our particle (on the path) at the beginning of our experiment, and $s_f$ marks our particle's position at the end of our experiment. The distance $d$ moved by our particle is simply $d = s_f - s_i$. We divide our distance along the path into an infinite number of infinitely small steps, each of length $ds$, in other words $d = \int_{s_i}^{s_f} ds$. If we include the direction information for each tiny step then our notation for each of these infinitesimal vector steps becomes $\vec{ds}$, and the sum of the infinite number of infinitely small vector steps is the displacement $\Delta\vec{r}$ of our particle over the course of our experiment, *i.e* $\Delta\vec{r} = \int_{s_i}^{s_f} \vec{ds}$.

These integrals over the path coordinate $s$ (whether vector or scalar) are called "path integrals" or "line integrals". For a non-trivial path, finding a method by which to actually analytically compute such an integral can be a notoriously difficult task; by the time you get to the end of your third university-level calculus class, you should be well aware of the difficulties inherent in path integrals, and of the solutions to those difficulties offered by calculus. However, the physical meaning of integrals such as $\int_{s_i}^{s_f} ds$ or $\int_{s_i}^{s_f} \vec{ds}$ should be relatively easy to grasp. It is those physical meanings that you will need to understand in this physics class.

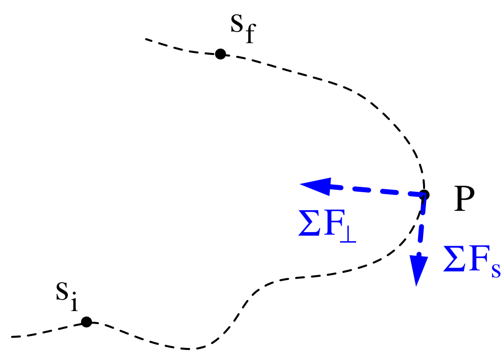
We wish to "integrate the Second Law over our path". Let us begin by asking "what will it mean to integrate the net force on our particle over the path?" Since the 2D path that we have drawn is curving in places, the net force on our particle in those places must have a component which is perpendicular to the path (in order to provide the "change-direction" acceleration of the particle). We know beforehand that the cumulative effect of these perpendicular net-force components, acting over the path, will be to keep the particle moving along the path, and the cumulative result will be the particle's change in direction from the beginning of our experiment to the end; therefore, we do not actually wish to include these perpendicular components of the net force in our integration. So for the integration over the path, we choose to begin with only the $s$-component of the Second law (we could call this the Second Law for the tangential direction, *i.e.* for the direction which is always tangent to the path):

$$\Sigma F_s \;=\; ma_s \tag{4}$$

If you will compare this starting equation to eq. (1), you will see that this time we have omitted some subscripting. These omissions were just to remove clutter; this net force component is still **on** our particle of mass $m$, and the acceleration component is still **of** that particle, but hopefully these reminders-by-subscript are no longer necessary.



Possible transverse (⊥) and tangential (s) components of the net force, assuming particle speed–up at point P.



Possible transverse (⊥) and tangential (s) components of the net force, assuming particle slow–down at point P.

Now we integrate this Second Law, $s$-component, over the path, from the start of our experiment at path location $s_i$ to the end of our experiment at path location $s_f$:

$$\int_{s_i}^{s_f} \Sigma F_s ds \;=\; \int_{s_i}^{s_f} \; ma_s ds. \tag{5}$$

This equation is the "integration of the Second Law over the path", and what remains is for us to make sense of the result. First of all, the equation is more popularly written in terms of dot products, or scalar products, like so:

$$\int_{s_i}^{s_f} \Sigma \vec{F} \cdot \vec{ds} \;=\; \int_{s_i}^{s_f} \; m\vec{a} \cdot \vec{ds}, \tag{6}$$

but the physical meaning of the equation is exactly the same whether written in the dot-product form or the $s$-component form.

On the left-hand side of the resulting equation (either eq. (5) or eq. (6)), we have the cumulative effect of the $s$ component of the net force on $m$ acting over our path; note that the units of the left-hand side are newtons times meters (N·m), which are called joules (J), and remember that we expect that the $s$ component of net force (as well as the $s$-acceleration on the right-hand side) will be varying with path location $s$. For the cumulative effect of net force acting over **time** we chose the name “net impulse”; in this case, we elect to name the left-hand side of eq. (5) (or of eq. (6)) the “net work done on the particle as it moved along the path from $s_i$ to $s_f$”. Thus, the “net work” (or the “work done by the net force”) is, by definition, the result of a path integral (or line integral). We still need to interpret the right-hand side of eq. (5) (or of eq. (6)); but, before we do that, let us spend some time understanding this new definition, the definition of work.

Formally, our definition of the “work done by a force $\vec{F}$ acting on some particle as it moves along a path from $s_i$ to $s_f$” is, in symbols:

$$W_{F,s_i \to s_f} \;\equiv\; \int_{s_i}^{s_f} \vec{F} \cdot \vec{ds}. \tag{7}$$

Note that we choose to make our definition for any particular force; the **net** work on our particle will then be the sum of the works done on the particle by each and every force acting on our particle (or we could first add the forces together and then find the work done by the net force). So our definition of work is a path integral, and for a beginning class, we choose to make these path integrals for work as simple as possible by considering only physical situations for which the integral does not require complex mathematical manipulations. Here are three such physical situations.

1. When the $s$-component of the force $\vec{F}$ is constant, regardless of how much the path twists or turns. If we use the symbol $F$ for the magnitude of the constant value of $F_s$, then $F_s = \pm F$; the plus sign is appropriate when the constant $F_s$ is in the positive $s$ direction (the direction of movement), and the negative sign is appropriate when the constant $F_s$ is opposed to the direction of movement. In other words, for this simplification, the work done by the force $\vec{F}$ turns out to be simply plus or minus force times distance. Here is the argument in symbols:

$$\text{Constant } F_s: \qquad W_F \equiv \int_{s_i}^{s_f} F_s ds = F_s \int_{s_i}^{s_f} ds = \pm F d \tag{8}$$

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Example: A cart full of lab equipment is being moved to a new location in a physics building. The path of the cart takes it 20 m East to the spacious lobby of the building and then through a gentle left-hand turn in the lobby followed by 12 m North to the new location of the lab. The gentle turn is a quarter circle of length 4 m, so the total length of the roughly L-shaped path is 36 m. A lab assistant maintains a horizontal 9-N force on the cart, in the direction of motion, for the entire length of the “experiment” on the path. Find the work done by that 9-N force during the movement of the cart.

Set Up: We have knowledge of the path, and we have enough knowledge of the force for which the work is requested, so our objective is simply to apply the definition of work,

treating the moving cart like a particle. Since the $s$ component of the force is constant, and since it is in the direction of motion of our cart, we simply need to compute positive force times distance.
Execute: $W_{9\,\mathrm{N}} = +(9\,\mathrm{N})(36\,\mathrm{m}) = 324\,\mathrm{N}\cdot\mathrm{m} = 324$ J
Evaluate: We now know the work done by the 9 N force applied by the lab assistant. We know nothing about the net work done on the cart. We do know that there must have been leftwards (or sideways, or perpendicular, or transverse) forces acting on the cart during its left turn, but the question asked us nothing about those forces, and anyway, those transverse force components cannot possibly have contributed to any net work, since they are transverse components and not parallel (or $s$) components.

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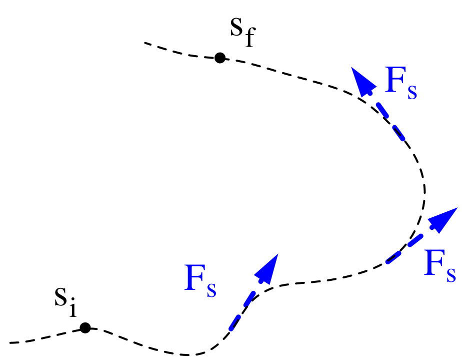
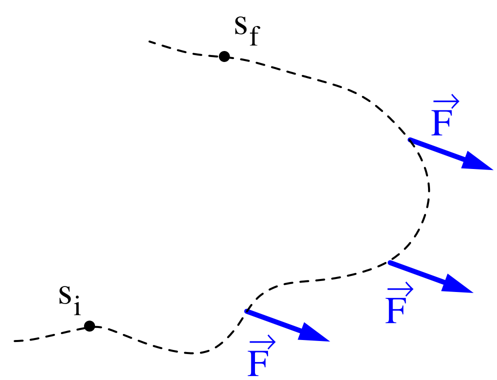


A constant $F_s$, shown at three randomly chosen locations along the path.



A constant $\vec{F}$, shown at three randomly chosen locations along the path.

2. When the force $\vec{F}$ is constant over the path (instead of the $s$ component of $\vec{F}$ being constant). For this case, we will only need to take the dot product of the constant force and the displacement of our particle as it moved along the path from $s_i$ to $s_f$. Here is the argument in symbols:

$$\text{Constant } \vec{F}: \qquad W_F \equiv \int_{s_i}^{s_f} \vec{F}\cdot\vec{ds} = \vec{F}\cdot\int_{s_i}^{s_f}\vec{ds} = \vec{F}\cdot\Delta\vec{r} \tag{9}$$

You will find nice examples of the "work done by a constant force" in the reading from our text (Examples 6.1 and 6.2).

3. When the path is as simple as possible, *i.e.* one dimensional and in one direction, and the $s$ component of the force, though possibly varying along the path, has been measured at each location. In such a case, the customary choice is to change the path coordinate simply to $x$. We can then plot the $x$-component of the force of interest versus $x$ and the integral that we need will be the "area under the curve" of $F_x$ versus $x$ from $x_i$ to $x_f$. Here's the argument in symbols:

$$\text{Simple path, measured force}: \qquad W_F \equiv \int_{s_i}^{s_f} F_s ds = \int_{x_i}^{x_f} F_x(x)dx \tag{10}$$

Example: A string of a homemade crossbow has been measured to exert a force of 200 N on the bolt (the bolt is the "arrow" for a crossbow) when the bow has been drawn back by 10 cm. The force on the bolt (B) by the string (S) has been measured to decrease linearly to 0.0 N as the bow is relaxed. The measurements have been reported as $F_{\mathrm{S\,on\,B,x}} = -(2000\,\mathrm{N/m})x$ for $x = 0$ to $-10\,\mathrm{cm}$ (the $x$-component of the force is in the positive $x$ direction when the location of the notch in the bolt is on the negative side of $x = 0$). Find the work done on the bolt by the string if the drawn bow pushes the bolt from $x = -10\,\mathrm{cm}$ to $x = 0$.
Set Up: We have a case of a one-dimensional and one-directional path with the force along the path measured at each location. So we will need to compute the area under the reported force-versus-position curve.
Execute:

$$W_{F_{\mathrm{S\,on\,B}}} = \int_{x_i}^{x_f} F_x(x)dx = \int_{-0.1\,\mathrm{m}}^{0} (-2000\,\mathrm{N/m})xdx = 0 - ((-1000\,\mathrm{N/m})(0.01\,\mathrm{m}^2) = 10\ \mathrm{J}$$

Evaluate: We have calculated the work done on the bolt by the string (or by the force on the bolt by the string) over the path from −10 cm to zero. We haven't been given whether the bolt was shot horizontally, vertically, or perhaps at some angle, so once again we know nothing about the net work done on the bolt. If the situation were arranged so that the force on the bolt by the string were the only significant force on the bolt during the travel along the path, only then would the calculated 10 J be the net work done on the bolt.

Now that we have some understanding of the physical meaning of the definition of work, let us return to the problem of interpreting the right-hand side of eq. (5) or (6). Begin by simply restating eq. (5)

$$\int_{s_i}^{s_f} \Sigma F_s ds \; = \int_{s_i}^{s_f} \; ma_s ds. \tag{5}$$

Since distance equals speed multiplied by elapsed time, the infinitesimal step length $ds$ is just the speed $v$ (at that particular location along the path) times the infinitesimal elapsed time $dt$ during which the step will occur. So let's make that replacement, which will convert the integral on the right-hand side from a path integral to a time integral:

$$\int_{s_i}^{s_f} \Sigma F_s ds \; = \int_{t_i}^{t_f} \; ma_s v\, dt. \tag{11}$$

Next, we recognize that $a_s$ is the speed-up or slow-down component of acceleration (*i.e.* the parallel or tangential component of acceleration), and so the product $a_s dt$ must produce the amount of speed change $dv$ which will occur during the infinitesimal time $dt$. So, let's make that replacement, which will convert our integral from a time integral to a speed integral, like so:

$$\int_{s_i}^{s_f} \Sigma F_s ds \ = \int_{v_i}^{v_f} \ mv\,dv. \tag{12}$$

Success! Our two changes of variable have converted our integral on the right-hand side to something that is easy to interpret, namely:

$$\int_{s_i}^{s_f} \Sigma F_s ds \ = \frac{1}{2} m(v_f^2 - v_i^2). \tag{13}$$

This is the final result for our "integration of the Second Law over a path in space". We name the right-hand side the "change in kinetic energy ($K$)" for our particle of mass $m$, and we call our resulting relationship the "Work-Kinetic Energy Theorem" or simply the "Work-Energy Theorem", *i.e.*

$$\Sigma W_{\mathrm{on\,m,s_i \to s_f}} = \Delta K_{\mathrm{of\,m}} \tag{14}$$

It is worthwhile to state this result in words: the **net** work on a particle of mass $m$ moving on a path from $s_i$ to $s_f$ is always equal to the change in kinetic energy of that particle. This important result is the foundation of the energy perspective on dynamics (dynamics is the explanation of motions), and the starting point for what is perhaps the crowning achievement of Newtonian dynamics, the principle of the Conservation of Energy.

Your reading on the topic of Work and Kinetic Energy now continues with Section 6.1 and Section 6.3 (page 183) from Young and Freedman.