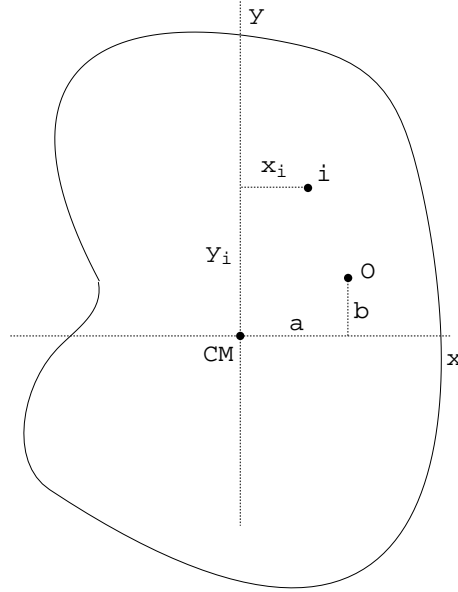


Proof of the Parallel Axis Theorem

Consider a rigid system of particles of mass $M = \sum_i m_i$, rotating about a fixed axis O . We place the origin of our coordinate system at the center-of-mass (CM) of our system of particles. The general point i has coordinates (x_i, y_i) , the x coordinate of O is a , and the y coordinate of O is b . Since the CM is at $(0,0)$, we have, by the definition of the center-of-mass, that

$$x_{CM} = \frac{\sum_i m_i x_i}{M} = 0 \quad \text{and} \quad y_{CM} = \frac{\sum_i m_i y_i}{M} = 0$$



By definition, $I_O = \sum_i m_i r_{iO}^2$, where r_{iO} is measured from O , not from CM. Thus, $r_{iO}^2 = (x_i - a)^2 + (y_i - b)^2$. Therefore,

$$\begin{aligned} I_O &= \sum_i m_i r_{iO}^2 \\ &= \sum_i m_i [(x_i - a)^2 + (y_i - b)^2] \\ &= \sum_i m_i [x_i^2 - 2ax_i + a^2 + y_i^2 - 2by_i + b^2] \end{aligned} \quad (1)$$

But since $x_{CM} = \frac{\sum_i m_i x_i}{M} = 0$, and similarly for y , the cross terms in equation (1) go to zero once the sum over i is done. Therefore,

$$I_O = \sum_i m_i (x_i^2 + y_i^2) + \sum_i m_i (a^2 + b^2)$$

Since $(x_i^2 + y_i^2)$ is just r_i^2 , where r_i is now measured from the CM, and $(a^2 + b^2)$ is just R_O^2 , where R_O is the distance from O to the CM, we have

$$I_O = I_{CM} + MR_O^2$$