PHY 132 – Summer 2000
LAB 10: Resonance in LRC Circuit[1]

Introduction

In this lab we will measure the steady-state behavior of a resonant system. Specifically we will look at the forced response of a series LRC circuit to a sine wave input. This builds on the previous lab where we looked at the phase behavior of each component in the same circuit. There also is a close link with mechanical resonant behavior, which many of you studied in PHY122, in context of “damped oscillations”. We will see that the power transfer to the load resistor is maximal at resonant frequency. We will also look at the transient behavior, and analyze the digitized waveform to estimate the time constant τ and the resonance quality factor Q.

Theory:

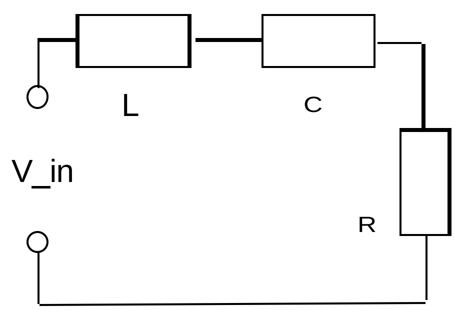

Fig. 1 Generic series LRC circuit.

Consider the series LRC circuit as shown in Fig. 1. We start with Kirchhoff’s law (sum of instantaneous voltages around a closed loop is zero):

$$\sum_j v_j = 0 \qquad \text{Eq. 1}$$

This takes the form

$$L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt + i(t)R = v_{in}(t) \qquad \text{Eq. 2}$$

[1] Adapted by R. J. Jacob from P. Bennett, PHY-132 Lab Manual© (ASU)

The terms correspond to voltage on: the coil (Faraday's Law $V = -Ldi/dt$), capacitor ($V_c=Q/C$), and resistor ($V_R=iR$). This equation has a relatively simple solution provided $v_{in}(t)$ is a sine wave. Thus we assume

$$v_{in}(t) = V_0 \sin(\omega t) \qquad \text{Eq. 3}$$

where V_0 is a constant amplitude fixed by the power supply. Current flows with the same frequency (of course!) and is given by

$$i(t) = I(w)^* \sin(\omega t - \Phi) \qquad \text{Eq. 4}$$

Notice that the amplitude depends on frequency, and is given by Ohm's law as

$$I(w) = V_0/X_{LRC} \qquad \text{Eq. 5}$$

where the reactance of the series circuit X_{LRC} is given by

$$(X_{LRC})^2 = R^2 + (X_L - X_C)^2 \qquad \text{Eq. 6}$$

Putting in for X_L and X_C we then have

$$I(w) = \frac{V_0}{\left(R^2 + \left(wL - \frac{1}{wC}\right)^2\right)^{1/2}} \qquad \text{Eq. 7}$$

It is useful to look now at the time-averaged power consumed in the circuit. This is given by $\langle P(w)\rangle = \langle i^2(t)R\rangle = \frac{1}{2} I_0^2 R$ or

$$P(w) = \frac{V_0^2 R/2}{\left(R^2 + \left(wL - \frac{1}{wC}\right)^2\right)} \qquad \text{Eq. 8}$$

This function is sharply peaked at the resonant frequency ω_0 defined by $X_L=X_C$ or $\omega_0 L = 1/(\omega_0 C)$ or

$$\omega_0^2 LC = 1 \qquad \text{Eq. 9}$$

At this frequency, the total impedance is minimum, and given by $X_{LRC}(\omega_0) = R$. The current flow is maximized at $I(w_0) = V_0/R$ and the phase shift is zero.

It is useful to characterize the sharpness of the P(w) function in terms of $\Delta\omega$ its full width at half maximum (FWHM). It is readily shown that

$$\Delta\omega = R/L \qquad \text{Eq. 10}$$

Furthermore we define a dimensionless "quality factor" Q as

$$Q = \omega_0/\Delta\omega = \omega_0 L/R \qquad \text{Eq. 11}$$

Evidently, a large Q corresponds to a sharply peaked P(w) function. We can now rewrite Eq. 8 in terms of ω_0 and Q rather than circuit parameters L, C thusly:

$$P(w) = 1/2 I_0^2(w)R = \frac{V_0^2}{2R}\left[1 + Q^2\left(\frac{w}{w_0} - \frac{w_0}{w}\right)^2\right]^{-1} \qquad \text{Eq. 12}$$

Suppose next we suddenly disconnect the oscillator (or equivalently hold $v_{in}(t)$ constant). Kirchhoff's law is now

$$L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt + i(t)R = 0 \qquad \text{Eq. 13}$$

This equation has the solution

$$i(t) = I_0 \exp(-t/\tau)\sin(\omega' t + \Phi) \qquad \text{Eq. 14}$$

where the new frequency ω' is slightly lower than the undamped frequency ω_0 as

$$(\omega')^2 = \omega_0^2 - (1/\tau)^2 \qquad \text{Eq. 15}$$

and the decay time τ is given as

$$\tau = 2L/R \qquad \text{Eq. 16}$$

This function is a damped sine wave as shown in Fig. 2. It is usefully considered as a sine wave confined by a simple decaying exponential "envelope function",
$f(t) = \exp(-t/\tau)$. As with RC circuits, we can determine τ from the time for the envelope function to drop to exp(-1) = 0.368 of its maximum value. This can readily be done by

eyeball of the live scope trace. Likewise, we can eyeball the "quality" factor Q directly from the damped sinewave since we have

$$Q=\omega_0/\Delta\omega = \omega_0\tau/2 = \pi f_0\tau \qquad \text{Eq. 17}$$

This can be considered as pi times the number of cycles required for the waveform (envelope) to decay by a factor of exp(+1), since we have 2π*(cycles/sec)*(sec to decay). Evidently, a large Q corresponds to slow damping (many oscillations) and vice versa.

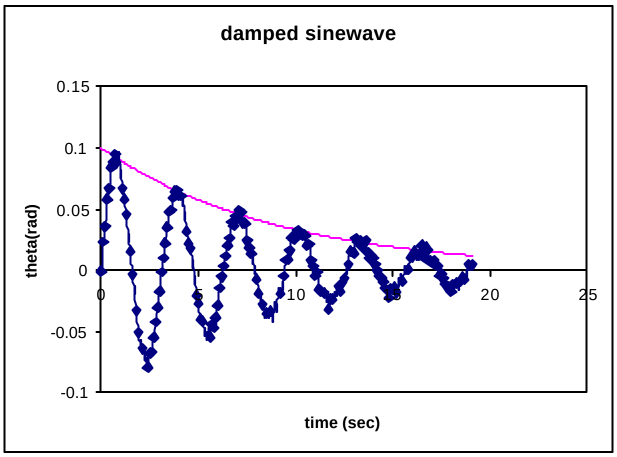

Fig. 2 Damped sine wave with constants: w=2 rad/sec, tau = 9 sec, amplitude = 0.1 rad. Note the solid line representing the envelope function exp(-t/τ).

The decay time τ is closely related to the frequency spread $\Delta\omega$. Indeed we have

$$(\Delta\omega*\tau)/2 = 1 \qquad \text{Eq. 18}$$

This is an identity property of the Fourier transform, which in this context, is a mathematical operation that transforms a time function into an equivalent a frequency function. On a higher level this behavior is closely analogous to the uncertainly principle, which states that the product of uncertainties for energy (frequency) and (life-)time of any particle is fixed at a minimum finite value, Planck's constant.

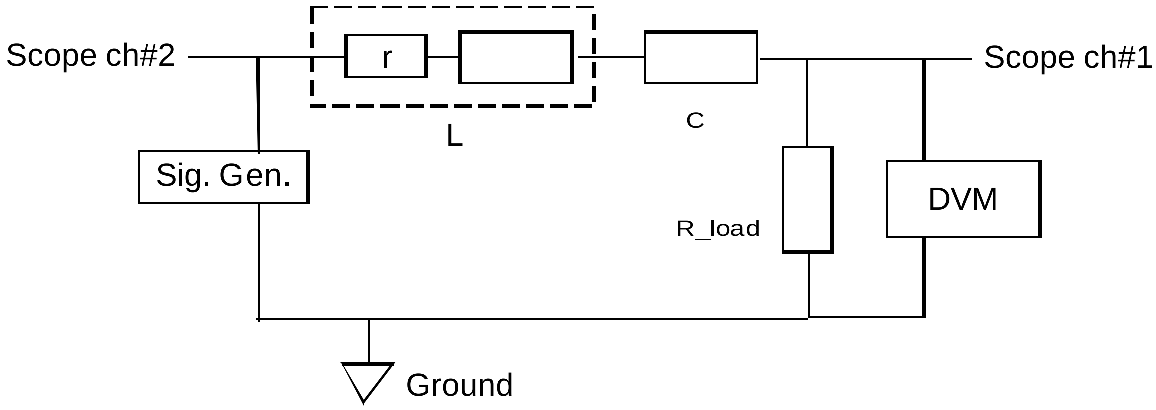

Fig. 2 Circuit used for measuring LRC behavior.

The circuit we will use is shown in Fig. 2. Note that the coil is non-superconducting and has an unavoidable non-zero internal resistance “r”. This circuit element is only experimentally accessible as the combination (L+r). The internal “r” contributes to the total circuit resistance R in the equations above. Thus we have

$$R = R_{load} + r \qquad \text{Eq. 19}$$

Scope ch #2 senses the input voltage, while scope ch #1 senses the series current, given by voltage across the series resistor R_{load}. This is also measured by the DVM (rms values, of course) set on the AC voltage scale (NOT current scale!). You can calculate the current from Ohm’s law.

<u>Procedure</u>

1. Before connecting the circuit, measure the coil internal resistance “r” using the DVM.
2. Connect the circuit of Fig. 2 using R_{load} = 50 ohms, C = 0.1 μfd and L = 85mH. Use the “Low ohms” output of the signal generator. Get both waves showing on the scope simultaneously. We will use the DVM for most of the data, but its useful to “see” the waveforms on the scope, nonetheless. Input a sine wave about 1 V_{rms} at 2.0 kHz. (Temporarily move the DVM over to the input).

3. Move the DVM back to the resistor. Find and record the resonant frequency f_0 where the current through R_{load} is maximum. Take 3 independent readings to get ±.
4. Measure the current vs. frequency I(w) as you tune through resonance. Take about 15 points total, going on both sides of resonance, far enough away so I_0 drops to ~ 20% of its value at resonance.
5. (transient) Change the input to a square wave about 30 Hz. The output wave (scope ch#1) should show a damped sine wave starting with each up/down of the square-wave. Open the setup file "phy132/lab12" and capture the waveform. You need not export this data file, we will do all analysis by eyeball of the v(t) plot.

Analysis

1. Compare your resonant frequency f_0 (procedure part 3) with theory. Component values are accurate to R(1%), C(10%) and L(5%).
2. Working in GA, calculate and plot $P(\omega)$ derived from current measurements. Mark the FWHM points on the $P(\omega)$ curve, and determine Q from this.
3. Fit $P(\omega)$ to determine Q, and w_0 and R. You should put in V_{rms} as a known constant. See notes in "fits.pdf".
4. Do eyeball fits for Q, and w' based on print-out of the transient waveform v(t).
5. Compare your values (and errors) of Q and ω_0 from the various methods.