5 Error Propagation

We start from eq. 1.4.2, which shows the explicit dependence of g on the measured variables "t" and "h". Thus

$$g(t,h) = 2h/t^2 \qquad \text{eq.1.5.1}$$

The simplest way to get the error in g from the error in t is to use extreme values. That is, we calculate two values of g using first the lowest value of t (the mean minus the error), then the highest (mean plus error). This gives us the range of g values to expect. Lets work this out for Larry's data, assuming that the drop height is h = 10.0 meters, free from errors. The drop time is

$$t = t_0 \pm \delta t \qquad \text{eq.1.5.2}$$

where t_0 = 1.46 sec is the mean and δt = 0.02 sec is the standard deviation of the mean σ_m. We want the error in g that results from the error in t. We can get this by directly plugging in the extreme values, and using the finite difference method. This gives

$$\Delta g = |g(t_0+\delta) - g(t_0)| \qquad \text{eq1.5.3}$$

This number is

$$\Delta g =| 2*10.0/(1.46+0.024)^2 - 2*10.0/1.46^2 | = 0.30 \text{ m/sec}^2. \qquad \text{eq.1.5.4}$$

Note that we always consider absolute values of errors, and a similar value would result for $g(t_0\text{-}\delta t)$. Alternatively, we can use derivative calculus to give

$$\Delta g \approx \left|\partial g/\partial t\right| \delta t \qquad \text{eq.1.5.5}$$

where the partial derivative is evaluated at t_0. Partial derivative means that one evaluates the derivative holding all other variables constant. This evaluates as

$$\Delta g \sim | -(4h/ t_0^3)| \delta t = (40/1.46^3)(0.024) = 0.31 \text{ m/sec}^2 \qquad \text{eq.1.5.6}$$

The finite difference vs derivative methods will be precisely equal in the limit of δt ~ 0. The results for all three Stooges are shown in Table 4.

	value	(%)	Comment
Measured variables			
t1	1.46 ± 0.024 sec	1.6%	Larry, 12 trials
t2	1.50 ± 0.022 sec	1.5%	Moe, 12 trials
t3	1.37 ± 0.029 sec	2.1%	Curly, 12 trials
Derived parameters			
g1	866 ± 27 cm/sec^2	3.2%	Larry
g2	888 ± 31 cm/sec^2	3.5%	Moe
g3	982 ± 44 cm/sec^2	4.5%	Curly

Table 4. Summary of results for g(t).

At this point, since we have a known model (value for g), we can evaluate the results to assess systematic errors and try to recommend improvements in the experiment. We see that Larry and Moe do not agree with the known value of $g=981$ cm/sec^2, while Curly does. These statements, as usual, mean "within the error bars at the 68% confidence level". It would appear that Curly did it right, and the others did not. They brainstorm a while and list out the following possible systematic effects.

1. wind (sideways)
2. Coriolis effect
3. air resistance

They decide that 1 and 2 are negligible, while 3 could be important, since air resistance would slow their large balls, lengthening the drop time and decreasing the result for "g". They repeat the whole experiment using small steel balls instead of tennis balls. Now Larry and Moe's "g" is right, while Curly's is too high. After consult with the instructor, Curly confessed that his girlfriend had helped. She dropped the ball, while he ran the timer. Curly could anticipate the end of the fall, but was always delayed at the start by his reaction time, about 0.2 sec. The others worked alone, and anticipated both the start and the end of the drop. Hence, they were right and Curly was wrong.

This example was chosen to illustrate the process of assessing and improving an experiment through an understanding of random and systematic errors. **By simply repeating enough trials, they made random errors small enough to expose systematic errors.** At this level, they found that the $1000 timer (with ±0.001 sec precision) could be replaced by a $10 Casio watch (with ±0.1 sec precision). The effects of systematic errors were more subtle, and required a good model, as well as informed judgement. In this case, the original procedure (in Curly's case) contained systematic errors that tended to cancel. So the first "improvement" actually made his result worse! In the end the group is confident that systematic errors are smaller than their random errors and they report

> "We find a value for the acceleration of gravity of $g = 9.6 \pm 0.3 \times 10^2$ cm/sec^2 (3.1%) This result agrees with the known value $g = 981$ cm/sec^2."

Note the format in this case, where the result is stated in "scientific notation" so the number of significant figures is unambiguous.

6 Uncertainties in multiple variables

Most experiments involve measuring multiple variables. Continuing the above example, we can allow for an error in drop height $h = h_0 \pm \delta h$. We now want to evaluate the error in "g" resulting from errors in both "t" and "h". The naive approach would be to simply evaluate g(t,h) with extremal errors, thus

$$\Delta g = |g(t_0 \pm \delta t,\ h_0 \pm \delta h) - g(t_0,\ h_0)| \qquad \text{eq.1.6.1}$$

Using example values of $t = 1.56 \pm 0.02$ sec and $h = 10.0 \pm 0.1$ m, we have

$$\Delta g = 2(10.0)/(1.58)^2 - 2(9.9)/(1.56)^2 = 8.08 - 8.14 = 0.06\ m/sec^2 \qquad \text{eq.1.6.2}$$

This is not quite right, since h and t are not always both too large. That is, if many teams set out to do the experiment, they would likely mix large t with small h, etc. In the calculation above, the errors from h and t are inadvertently cancelled, since we take the ratio. The correct result is actually larger. The correct procedure for adding statistically independent errors is given as

$$\Delta g_{tot} = \sqrt{(\Delta g_t)^2 + (\Delta g_h)^2} \qquad \text{eq.1.6.3}$$

Master Equation for Error Propagation

The total error Δg_{tot} is the quadratic sum of errors from each variable separately. This "undoes" the inadvertant correlations. The separate terms can be calculated either as finite differences or derivatives as in the example above. In our example we have

$$\Delta g_t = |-(4h/\ t_0^3)|\ \delta t = (40/1.46^3)(0.024) = 0.31\ \text{m/sec}^2 \qquad \text{eq.1.6.4}$$

$$\Delta g_h = (2/\ t_0^2)\delta h = (2/1.46^2)(0.1) = 0.09\ \text{m/sec}^2 \qquad \text{eq.1.6.5}$$

The quadratic sum gives

$$\Delta g_{tot} = \sqrt{(.31)^2 + (.09)^2} = 0.32\ \text{m/sec}^2. \qquad \text{eq.1.6.6}$$

We can see that the quadratic sum weights the larger value (Δg_t) heavily, so we can essentially neglect the smaller value (Δg_h). This neglect approximation can greatly simplify calculations for more complex functions.

For many cases, the "master equation" can be simplified by considering the ratio ($\Delta f/f$).

<u>Case I: pure product function</u>

$$f(x,y) = x*y \qquad \text{eq.1.6.7}$$

Then we have

$$(\Delta f/f)^2 = \frac{\left(\frac{\partial f}{\partial x}\delta x\right)^2 + \left(\frac{\partial f}{\partial y}\delta y\right)^2}{(xy)^2} = (y\delta x/xy)^2 + (x\delta y/xy)^2 = (\delta x/x)^2 + (\delta y/y)^2.$$

eq.1.6.8

This is easier to say than write:

> *"For product (and divide) functions, the fractional error of the result is the quadratic sum of the fractional errors of the variables".*

Case II: power function

$$f(x) = x^m \qquad \text{eq.1.6.9}$$

Then we have

$$(\Delta f / f) = \frac{1}{f}(\partial f / \partial x)\delta x = m x^{m-1} / x^m = m(\delta x / x) \qquad \text{eq.1.6.10}$$

"For power functions, the fractional error of the result is m times the fractional error of the variable."

This result holds for "m" negative or fractional as well, corresponding to division (x^{-1}) and roots ($x^{1/2}$). Cases I and II can be combined as

$$f(x,y) = C\, x^m y^n \qquad \text{eq.1.6.11}$$

Then we have

$$(\Delta f / f)^2 = (m\delta x / x)^2 + (n\delta y / y)^2 \qquad \text{eq.1.6.12}$$

Case III: pure addition

$$f(x,y) = Ax + B\,y \qquad \text{eq.1.6.13}$$

where A and B are constants. Note that A or B may be negative, corresponding to subtraction. Now the ratio trick is no longer useful and we have

$$(\Delta f)^2 = (A\delta x)^2 + (B\delta y)^2. \qquad \text{eq.1.6.14}$$

"For sum/difference functions, absolute errors (not fractions) add quadratically"

In many cases, error propagation can be figured almost on inspection of input errors, using the cases above plus neglect of small terms. In our ball-drop example, we recognize that the function g(t, h) is a simple product, and see that the fractional error in "h" is small compared with the fractional error in "t". Thus we have $(\Delta g/g) \sim 2(\delta t/t)$. This is apparent in the "table of results" shown above for the 3 Stooges, where we see that Larry had $\delta t/t \sim 1.6\%$ and $\Delta g/g \sim 3.2\%$.

> *It is much more important for students to learn what to neglect and approximate than it is to see them grind out detailed unnecessary calculations of errors.*

It is possible that the model function does not break down into the simple cases shown above. In that case, it is not difficult to tabulate the terms in the Master Equation by directly plugging extremal values into the function. This is trivial in a spreadsheet. It will then be apparent which variables are important sources of error and which are not.

Advanced:

The general result for finding the square of the standard deviation (called the variance) in a function x = f(u, v, ...) of several variables can be derived as follows. We want the variance

$$\sigma_x^2 = \frac{1}{(N-1)} \sum_1 (x_1 - x_m)^2 \qquad 1$$

where x_m is the mean. The way in which small changes (errors, or departures $du = u_i - u_m$ from the mean u_m) in the variables u,v,.... produce changes in x is given by the chain-rule for differentiation

$$x_i - x_m = (u_i - u_m)\left(\frac{\partial x}{\partial u}\right) + (v_i - v_m)\left(\frac{\partial x}{\partial v}\right) + \ldots \qquad 2$$

Plugging equation 2 into equation 1 we get

$$\sigma_x^2 = \sigma_u^2\left(\frac{\partial x}{\partial u}\right)^2 + \sigma_v^2\left(\frac{\partial x}{\partial v}\right)^2 + \ldots \qquad 3$$

if the fluctuations in all the variables u,v,.... are unrelated ("uncorrelated") and independent with individual variances σ_u^2, σ_v^2, ... etc. This is the famous "error propagation" equation. The partial derivatives can be evaluated once the formula relating the measured quantities u,v,.... to the wanted x is known.

7 Counting statistics.

One of the most important results that all experimental scientists use constantly is the following:

"For a Poisson distribution the standard deviation is equal to the square root of the mean".

Most measurements in science involve Poisson distributions for the special case where the mean is large. Then the Poisson distribution becomes a Gaussian or Bell-shaped normal distribution, which is symmetrical (Poisson isn't). A simple common case involves counting statistics, where we measure, for example, the number of photons arriving at a detector per second, for example. If the average number per second was 1000, then we can predict immediately, given Poisson statistics, that the standard deviation of the distribution would be sqrt(1000) = 31.6 . In other words, in successive counts with a steady beam, the count will fluctuate, due to random fluctuations, by about 32 counts about the average of 1000. We call this ratio 1000/32 the Signal to Noise ratio, and it is found that for useful data we need a ratio greater than 5. This sets a limit on the shortest counting time for acceptable signal to noise. For our camera on Mars, we could then figure out the shortest exposure time which would allow us to discern a (very noisy) image. Background sources must also be considered. This is probably the single most useful result in all of statistical theory for scientists.

SUMMARY.

1. There are two kinds of errors, random (precision) and systematic (accuracy). Bombs falling close together in the wrong place have high precision and low accuracy. Bad rulers introduce systematic errors.

2. Random errors are reduced by averaging over more measurements. We can make a histogram of all the measurements. The standard deviation σ is proportional to the width of the histogram. (FWHM = 2.354 σ). This measures the spread of values in the measurements. A distribution is a histogram with very fine increments on the x axis.

3.The error E in the average or mean is the standard deviation (SD) divided by the square root of the number of measurements N.

4. Always write your results for any set of measurements like this...

"The drop time was found to be t = 1.520 ± 0.024 sec (1.6%)."

where 0.024 = E.

5. In counting (and other) experiments the standard deviation is usually equal to the square root of the mean.

6. The effect of errors in the terms ("experimental parameters") of a formula ("model") on the result can be calculated. This could be done by evaluating the formula using the largest values (mean value plus error) and smallest values (mean minus error) of all the parameters. Or use formulae given.

Appendix

Significant Figures.

Some level of uncertainty (errors) is immediately implied in the format of written numbers, in terms of the "number of significant figures". This phrase tells how many digits are known in some number. This can differ from the total digits in the number because zeros are used as place keepers when digits are not known. For example, in the number 123 there are three significant figures. In the number 1230 one might expect that there are four significant figures, but there are only three because the zero is assumed to be merely keeping a place. Similarly the numbers 0.123 and 0.0123 both have only three significant figures. The rules for determining the number of significant figures in a number are:

1. The most significant digit is the left-most nonzero digit. In other words, zeros at the left are never significant.
2. If there is not decimal point explicitly given, the rightmost nonzero digit is the least significant digit.
3. If a decimal point is explicitly given, the rightmost digit is the least significant digit, regardless of whether it is zero or nonzero.
4. The number of significant digits is found by counting the places from the most significant to the least significant digit.

As an example consider the following list of numbers: (a) 3456 (b) 135700 (c) 0.003043 (d) 0.01000 (e) 1030. (f) 1.057 (g) 0.0002307. All the numbers in the list have four significant figures for the following reasons: (a) there are four nonzero digits so they are all significant; (b) the two rightmost zeros are not significant because there is no decimal point; (c) zeros at the left are never significant; (d) the zero at the left is not significant, but the three zeros at the right are significant because there is a decimal point; (e) there is a decimal point so all four numbers are significant; (f) again there is a decimal point so all four are significant; (g) zeros at the left are never significant.

Students must take care to write the appropriate number of significant figures, both in their original data sheets, and especially in their final reports, as detailed further in the section describing how to write reports. Experimental values should always be stated with the same absolute accuracy (same decimal place) as the errors. For example, a result might be stated as $12.3 \pm 0.2 \times 10^2$ cm/sec^2. Calculations should always be carried out with at least one more significant figure than the final answer.

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