

## **PHY 122 LAB : Error Propagation - Volumes**

### **Introduction:**

In this lab, we practice error analysis by measuring volumes of a few regular shapes. You need to carefully read the “DataAna” handout. The aim is to understand how errors propagate through formulas. For example, the volume of a box is  $V=L*W*H$ . If we measure the width  $W$ , height  $H$ , and length  $L$  many times and so have errors for each quantity, what will the error in the Volume be when we multiple the average values of the three quantities together ? The need to answer this type of question arises in all areas of the physical sciences. (See Bevington for more details). In this lab we will learn the single most important formula used in all error analysis, used in quality control throughout industry.

### **Procedure:**

We will determine the volume of various shaped objects using Vernier calipers to measure relevant dimensions. The Vernier is accurate to 0.05 mm. Take several independent readings for each dimension. Make sure everyone learns to use the calipers, and that you understand how a Vernier works - ten divisions on one scale equals nine on the other, etc. Take the mean and standard deviation of these readings for all partners for each dimension.

Measure the relevant dimensions for each of the following objects:

1. A rectangular prism.
2. A triangular prism.
3. A hollow cylinder (find the volume of solid material). Be sure to also measure the “wall thickness” directly with the calipers.
4. A bullet shape object, approximated as a cylinder plus hemisphere of matching diameter.
5. A small plastic cup. Estimate the volume of the cup by finding the volume of water that fills a regular shape (rectangular box).

### **Analysis**

Find the volume of each of the above objects and give the error in the volume. Express your answer in  $\text{cm}^3$ . To do this you need the mean and standard deviation of each of the dimensions on which the volume depends - this is your raw data. Note that the error in the calipers is only 0.05mm, so many of your measurements may give the same result. Take five measurements of each dimension to get the mean and std dev.

Do the error analysis using the expression

$$\sigma_x^2 = \sigma_u^2 \left( \frac{\partial x}{\partial u} \right)^2 + \sigma_v^2 \left( \frac{\partial x}{\partial v} \right)^2 + \dots$$

where  $\sigma_x^2$  is the square of the standard deviation in the volume. This expression is derived in the appendix - you don't need to follow the derivation in detail. This formula contains one term on the right for each dimension. Each  $\sigma^2$  on the right is the square of the standard deviation in the mean value of one of the dimensions. The quantities in parenthesis indicate how strongly this dimension affects the volume, and is called a partial derivative. Partial derivatives are not much different from ordinary derivatives - for example, for the rectangular prism, where volume  $V=XYZ$ , this equation would be written

$$\sigma_V^2 = \sigma_X^2 \left( \frac{\partial V}{\partial X} \right)^2 + \sigma_Y^2 \left( \frac{\partial V}{\partial Y} \right)^2 + \sigma_Z^2 \left( \frac{\partial V}{\partial Z} \right)^2$$

Then, since  $V=XYZ$ , the partial derivative  $\delta V/\delta X = YZ$ . The idea is that we hold the other quantities  $Y$  and  $Z$ , which we are not differentiating with respect to, constant, while we differentiate with respect to  $X$ . Similarly,  $\delta V/\delta Y = XZ$ . So the variance in the volume in this case is given by

$$\sigma_V^2 = \sigma_X^2 Y^2 Z^2 + \sigma_Y^2 X^2 Z^2 + \sigma_Z^2 X^2 Y^2$$

Each of the terms  $\sigma_X^2$  etc. is obtained by calculating the standard deviation from a series of measurements of  $X$ . The values of  $X$ ,  $Y$  and  $Z$  are average values. The program graphical analysis will calculate this for you, or you can use the STDEV function in EXCEL, or the std dev button on many calculators. Finally, the standard error in the volume is  $\sigma_V / N^{1/2}$  with  $N$  the total number of measurements.

### **Report:**

1. State clearly each formula for volume.
2. Organize your report with tables that clearly state the value and error for each dimension and calculated volume.
3. You need not replicate original data in the report if it is reasonably neat.
4. Your abstract need only give full details for the cylinder, but give results for all cases.

## Appendix.

From the data analysis notes and the first lab we learnt that the standard deviation  $\sigma$  is the width of a histogram of our measurements. The square of the standard deviation is called the variance. We normally have a formula giving the quantity we want (e.g. the volume  $V$ ) in terms of the individual quantities we measure (Length  $X$ , Width  $Y$ , Depth  $Z$ , with  $V = XYZ$  for a cube, for example). (The  $X, Y, Z$  are called "parameters"). We want the variance in  $V$ , given the variances in  $X, Y$ , and  $Z$ , which we can find by repeated measurement of these quantities. Then the error in volume will be the square root of the variance in  $V$  divided by the square root of the number of measurements. In more general terms, we want to find the variance in a function  $x = f(u, v, \dots)$  of several variables. This variance is defined (see Bevington) as

$$\sigma_x^2 = \frac{1}{(N-1)} \sum (x_i - x_m)^2 \quad 1$$

where  $x_m$  is the mean. (This is what your calculator evaluates when you hit the std dev button). The way in which small changes (errors, or departures  $du = u_i - u_m$  from the mean  $u_m$ ) in all the variables  $u, v, \dots$  produce corresponding changes  $dx = x_i - x_m$  in  $x$ , can be obtained from the chain-rule for differentiation :

$$x_i - x_m = (u_i - u_m) \left( \frac{\partial x}{\partial u} \right) + (v_i - v_m) \left( \frac{\partial x}{\partial v} \right) + \dots \quad 2$$

Plugging equation 2 into equation 1 we get

$$\sigma_x^2 = \sigma_u^2 \left( \frac{\partial x}{\partial u} \right)^2 + \sigma_v^2 \left( \frac{\partial x}{\partial v} \right)^2 + \dots \quad 3$$

if all the cross-terms are ignored. (These terms represent correlations between the fluctuations in all the variables  $u, v, \dots$ , in which a change in  $u$  causes some degree of change in  $v$ , for example. We will assume this is not so, and that the errors are completely independent, with individual variances  $\sigma_u^2, \sigma_v^2, \dots$  etc. Thus we assume, for example, that your measurement of one side of the prism does not effect the accuracy in your measurement of another dimension.).

Equation 3 is the famous "error propagation" equation. The standard deviation (the width of a histogram)  $\sigma$  is the square root of  $\sigma^2$ , and the error in the average or mean value is  $\sigma / N^{1/2}$  if there are  $N$  measurements. The partial derivatives can be evaluated once the formula relating the measured quantities  $u, v, \dots$  to the

wanted  $x$  is known. For example, for a box of volume  $V = XYZ$ , we have  $dV/dX = YZ$  etc. So the error in the volume is a weighted sum of the errors ( variances) in the separate dimensions, with the weighting factor (the derivative) indicating how strongly the function depends on each variable. Notice how the individual variances involve squares of differences, so that errors of opposite sign cannot cancel out. Equation 3 is the most important equation in error analysis - you should remember it.

In many cases errors in one variable will be much more important than those in others. Then we can neglect errors in all but one. Consider the function  $x = uv^8$ . Use equation 3 to compare the importance of errors in  $u$  and  $v$  on the error in  $x$ . Knowing this, you could decide that it was not worth spending much time measuring errors in some quantities, but more on others.

In everything from the auto industry to the semiconductor industry, the analysis of errors (called "quality control") is vitally important.

**Lab Quiz (You must show your work, to get proper credit!)**

Yourname \_\_\_\_\_ Section day/time \_\_\_\_\_

1. Find the area of a flat piece of metal according to the data in the table below for the length of each side. State the answer as mean  $\pm \sigma_m$  units. eg Area =  $3.2 \pm 0.2 \text{ cm}^2$ . Here 0.2 is the error in the mean, equal to the standard deviation divided by the square root of the number of measurements. Do not estimate the area several times and take the average. Instead, find the mean and standard deviation of each of the dimensions on which the volume depends, and use these as in the example above under "Analysis". (Except that you are doing area, not volume).

| Side X (cm) | Side Y (cm) |
|-------------|-------------|
| 8.4         | 6.3         |
| 8.6         | 5.9         |
| 8.3         | 6.2         |
| 8.7         | 6.5         |
| 8.5         | 6.3         |
| 8.8         | 6.1         |