A Simple Probability Trick for Bounding the Expected Maximum of n Random Variables

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In this note, we introduce a simple probability trick that can be used to obtain good bounds on the expected value of the maximum of n random variables. This trick was discovered when trying to re-derive a well known bound on the expected value of the maximum of n Normal random variables. We present this first and then we demonstrate the utility of the method by generalizing it and applying it to some other random variables.

1 A Bound on the Expected Value of the Maximum of n Gaussian Random Variables

Let $X_1, X_2, \ldots, X_n$ be n (not necessarily independent) random variables that are drawn from a Gaussian distribution $\mathcal{N}(0, \sigma^2)$ and let $M_n := \max_{1 \le i \le n} X_i$. We will now show that

$$\mathbb{E}\left[M_n\right] \le \sigma\sqrt{2\log n}.$$

Observe that for any $s > 0$, we can apply Jensen's inequality to e^{sM_n} and proceed as follows

$$\begin{aligned} e^{s\mathbb{E}[M_n]} &\le \mathbb{E}\left[e^{sM_n}\right] && (1)\\ &= \mathbb{E}\left[\max_i e^{sX_i}\right] && (2)\\ &\le \sum_{i=1}^{n} \mathbb{E}\left[e^{sX_i}\right] && (3)\\ &= ne^{\frac{s^2\sigma^2}{2}} && (4) \end{aligned}$$

where in (4) we have used the fact that the moment generating function of $\mathcal{N}(m, \sigma^2)$ is $e^{ms^2 + \frac{s\sigma^2}{2}}$. Applying logarithms on either side of (4), we get

$$\mathbb{E}\left[M_n\right] \le \frac{\log n}{s} + \frac{s\sigma^2}{2}. \qquad (5)$$

Since the above inequality holds for any choice of $s > 0$, we can pick s to "optimize" the bound. By the AM-GM[1] inequality, we know that $\frac{\log n}{s} + \frac{s\sigma^2}{2} \ge \sigma\sqrt{2\log n}$ which gives us the necessary bound.

[1]The *Arithmetic Mean - Geometric Mean* inequality states that for positive numbers $a_1, a_2, \ldots, a_n$, the following inequality holds $\frac{\sum_{i=1}^n a_i}{n} \ge (a_1 a_2 \cdots a_n)^{1/n}$. Equality occurs iff all the numbers are equal

2 Generalization and Applications

This technique, of course, works in more generality. Let $X_1, X_2, \ldots, X_n$ be n (not necessarily independent) random variables with moment generating functions $m_1(s), m_2(s), \ldots, m_n(s)$ respectively. Further, suppose that there is a function $m(\cdot)$ such that for each s, $m(s)$ upper bounds $m_i(s), i = 1, 2, \ldots, n$. Then, following the steps above, we arrive at the bound

$$\mathbb{E}\left[M_n\right] \le \inf_{s \in Dom(m)} \frac{\log n + \log m(s)}{s}. \tag{6}$$

where $Dom(m)$ is the set of all $s \ge 0$ such that $m(s) \ge 1$. Since the terms involved in the above sum are non-negative, we can obtain the best bound by simply finding an s such that $\log n$ equals $\log m(s)$ (again using the AM-GM inequality).

As an application of the technique, let us suppose that $X_1, X_2, \ldots, X_n$ are drawn *i.i.d* from the **Gamma Distribution** with parameters k and θ (i.e., $\Gamma(k, \theta)$). Using the fact that the moment generating function of the Gamma distribution is $(1 - s\theta)^{-k}$ in (6), we get the following bound

$$\mathbb{E}\left[\max_i X_i\right] \le \frac{2\theta \log n}{1 - n^{-1/k}} \tag{7}$$

Of course, by picking values for k and θ, (7) gives us bounds for the **Exponential, Chi-Squared and Erlang** distributions among others.

As another example, if we suppose $X_1, X_2, \ldots, X_n \sim \text{Laplace}(0, b)$, then we have

$$\mathbb{E}\left[\max_i X_i\right] \le \frac{2b \log n}{\sqrt{1 - n^{-1}}} \tag{8}$$