Micro-Macro Links for Self-Organizing Collective Systems:
From Local State Transition Rules to Global Transition Probabilities and Back

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The design of self-organizing, collective systems is challenging. The major challenge is understanding the complex, non-linear relation between the microscopic level (individual agent) and the macroscopic level (collective behavior). An option to get an understanding of these systems is to derive micro-macro links that define a direct (mathematical) connection between the two levels. Models can help in gaining novel, general insights about collective systems and can also help during the design process of a particular system (Hamann and Wörn, 2008). In the following, we present a modeling technique that has high potential for general applicability in the domain of self-organizing collective systems.

Natural and artificial collective systems typically rely on simple agents. The controllers of simple agents (e.g., reactive control) are easily modeled with finite state automata. The transitions between states depend on the agent's internal state s and its perceptions $\{p_0, \dots, p_n\}$. These perceptions could represent anything from detected objects, walls, neighboring agents and their communicated internal states, etc. We denote the probability of a transition from s_j to s_i as $P(s_i|s_j, p_0, \dots, p_m)$ where we assume the Markov property. We assume ergodicity and that perceptions p_i can be modeled by probabilities depending on the agent's state averaged over time and ensembles. This definition allows the notation of a master equation (van Kampen, 1981)

$$\frac{df}{dt} = \sum T(\mathbf{s}|\mathbf{s}')f(\mathbf{s}', t) - T(\mathbf{s}'|\mathbf{s})f(\mathbf{s}, t). \tag{1}$$

It describes the dynamics of a probability density function f over all agent states and perceptions $\mathbf{s} = (s_0, \dots, s_n, p_0, \dots, p_m)$ for a transition rate T that corresponds to P. Similarly, a collective system can be modeled as a chemical reaction network where agents' state transitions are modeled as chemical reactions (Martinoli et al., 2004). Following this approach, the dynamics of transitions between state pairs (i.e., an isolated analysis of $s_i \to s_j$ and $s_j \to s_i$ for two states s_i and s_j) can be modeled by ordinary differential equations (ODE) based on the corresponding transition rates T (Biancalani et al., 2014) or alternatively be measured in a particular collective system (Hamann, 2013b). These ODE are solutions of the master equation obtained by a Taylor expansion. The resulting ODE may have high degrees depending on the number of features (neighborhood size, internal states, perceptions) that are involved in the definition of the state transitions (Hamann et al., 2014).

In the following we restrict our study to the special case of perceptions limited to the detection of close-by agents and their signaled internal state. The perception is defined by counting an agent's neighbors and these neighbors' states. The transition probability is then $P(s_i|s_j, \mathcal{N}_0, \mathcal{N}_1)$, for the number of neighbors $\mathcal{N}_k$ with state s_k and considering only two states or $P(s_i|s_j, \mathcal{N}_0, \dots, \mathcal{N}_n)$ (similarly for the transition rates T). As mentioned above, it is possible to investigate an isolated pair of states and their transition probabilities. Alternatively, it might be possible to divide the state space $\mathbf{s}$ usefully into two parts (e.g., based on robot positions: robots in left/right half) and the analysis is then based on summations over the respective transition probabilities between these two parts (Hamann, 2013b). The transition rates T can be simplified by notating them as functions of swarm fractions $x_i \in [0, 1]$, $x_i = \mathcal{N}_i/N$, for swarm size N. By restricting the system to one state transition at a time, we get $T(x_i|x_0, \dots, x_j \pm 1/N, \dots)$ (Hamann et al., 2014; Biancalani et al., 2014). For the dynamics of swarm fractions x_i we have (cf. eq. 1)

$$\frac{dx_i}{dt} \propto \sum_{i \neq j} T(x_i + 1/N|x_0, \dots, x_j - 1/N, \dots) - T(x_i - 1/N|x_0, \dots, x_j + 1/N, \dots). \tag{2}$$

It turns out that these functions dx_i/dt have similar properties across a number of different collective systems, such as alignment in locust swarms, aggregating robots, and other collective decision-making systems (Hamann, 2013a,b; Hamann et al., 2014). See Fig. 1 for examples. For systems that have only two states and hence depend only on two swarm fractions x_1 and x_2 (or systems with state spaces that can be divided into two parts), we define $z = x_1 - x_2$ and can approximate their dynamics with polynomials (Hamann et al., 2014) $\frac{dz}{dt} = -\varepsilon z + c_1 z + c_2 z^2 + \dots$, where the term $-\varepsilon z$ describes the influence of noise due to spontaneous state transitions.

Typically the plots of dz/dt allow for an intuitive interpretation. We focus on the right half $z > 0$ (similarly for $z < 0$). The sign of dz/dt directly gives the type of feedback, $dz/dt < 0$ is negative feedback (i.e., driving the system towards $z = 0$) and $dz/dt > 0$ is positive feedback (i.e., driving the system towards $z = 1$). For $z = 1$ we expect $dz/dt|_{z=1} \leq 0$ because $x_1 = 1$. For noise-free systems we have $dz/dt|_{z=1} = 0$ which defines an absorbing system (absorbing states $z = \pm 1$). For an unbiased system

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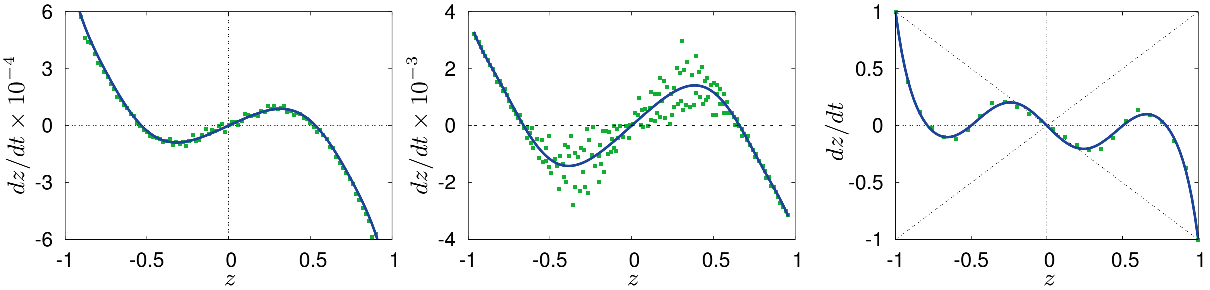

(a) density estimation, $w_1 = 3.97 \times 10^{-4}$, $w_3 = 5.04 \times 10^{-4}$, $w_5 = -2.58 \times 10^{-5}$, $w_7 = 3.47 \times 10^{-5}$, $w_9 = 3.66 \times 10^{-5}$.

(b) locust alignment, $w_1 = 0.001$, $w_3 = 0.0035$, $w_5 = -0.00083$, $w_7 = -5.97 \times 10^{-6}$, $w_9 = 0.000262$.

(c) aggregation (normalized), $w_1 = 0.221$, $w_3 = 0.254$, $w_5 = 0.553$, $w_7 = -0.114$, $w_9 = 0.0789$.

Figure 1: Transition probabilities/dynamics of swarm fractions; (a) 'density estimation', squares: simulation, line: fitted polynomials (Hamann, 2013a), (b) locusts, squares: data from Fig. 3B of Yates et al. (2009) (model of swarm alignment in locusts), line: fit, (c) aggregation (BEECLUST), squares: simulation (Hamann, 2013b), line: fit.

we expect $dz/dt|_{z=0} = 0$. In terms of collective decision-making systems we know that a majority rule (agents follow the opinions of the majority in their neighborhood) results in pure positive feedback (except for noise), whereas negative feedback ($dz/dt < 0$) implies the effectivity of a minority rule (agents follow the minority of their neighbors).

The plots show similarities to Legendre, Chebyshev, Gegenbauer, and Jacobi polynomials. Here we focus on the odd-indexed Legendre polynomials with negative sign:$-P_1(z) = -z$, $-P_3(z) = -\frac{1}{2}(5z^3 - 3z)$, $-P_5(z) = -\frac{1}{8}(63z^5 - 70z^3 + 15z)$, etc. An advantageous property of these polynomials is their orthogonality which helps in establishing a link back from macro to micro. For example, by measuring dz/dt in a collective system it allows to determine a unique fit to the Legendre polynomials by a weighted sum over polynomials: $\sum_{i \text{ odd}} -w_i P_i(z)$. The weights/coefficients, in turn, can be interpreted based on the particular polynomial as the contribution of noise (P_1), pure positive/negative feedback (P_3, depending on the weight's sign), mixed pos./neg. feedback (P_5, switch of feedback sign at $z \approx 0.53$), etc. Fig. 1 shows fits of the first 5 odd Legendre polynomials (solid lines). In the special case of collective decision-making systems, this method can even be exploited to estimate the number of neighbors that an individual agent takes into account when switching its opinion (Hamann et al., 2014).

We have established a link from micro to macro by deriving global system properties (dz/dt) from local interaction rules and a link from macro to micro via fitting Legendre polynomials and interpreting the resulting coefficients. Also note that there are mathematical methods for multivariate orthogonal polynomials (Dumitriu et al., 2007), such as the Zernike polynomials (Zernike, 1934), which might allow for a direct generalization of this approach to investigate the transition probabilities between three, four, and more states. Based on the micro-macro link, desired system properties can be defined via the polynomials and transferred to appropriate microscopic rules.

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