Derivation of a Micro-Macro Link for Collective Decision-Making Systems

Uncover Network Features Based on Drift Measurements

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Abstract. Relating microscopic features (individual level) to macroscopic features (swarm level) of self-organizing collective systems is challenging. In this paper, we report the mathematical derivation of a macroscopic model starting from a microscopic one for the example of collective decision-making. The collective system is based on the application of a majority rule over groups of variable size which is modeled by chemical reactions (micro-model). From an approximated master equation we derive the drift term of a stochastic differential equation (macro-model) which is applied to predict the expected swarm behavior. We give a recursive definition of the polynomials defining this drift term. Our results are validated by Gillespie simulations and simulations of the locust alignment.

1 Introduction

Distributed and decentralized systems that rely on self-organization to coordinate a large number of agents are characterized by nonlinear dynamics. These systems rely on positive feedback (amplification), negative feedback (damping), and a multitude of interactions between their components [1]. As a consequence of nonlinearity combined with a large quantity of microscopic details (i.e., features of individual agents), they are generally difficult to analyze and design. Designers may deepen the understanding of these systems by defining appropriate models that reflect specific features of these systems but are "not flooded with microscopic details" [2]. Deriving a macro-model (i.e., a model of swarm features not representing individual agents) mathematically from a micro-model or vice versa is commonly believed infeasible for the general case. In sociology this micro-macro relation is known as the micro-macro link [3, 4] that has applications to biology, physics, and engineering, too [5]. An approximation to an actual micro-macro-model for swarm robotics [6] has been proposed [5, 7, 8] that is capable to represent individual agent trajectories as well as swarm densities. If the considered self-organizing system relies also on inhomogeneous spatial distributions of agents, the modeling task is even more difficult [5, 8]. A promising

approach is to use network models as an abstraction of the interaction patterns that emerge from self-organized behaviors (i.e., which agent interacts with which other agents) as done, for example, by Huepe et al. [9] for the example of locusts.

In this paper, we focus on self-organizing collective decision-making (CDM) systems because they generally consist of few, simple control rules and therefore provide a good subject to approach the micro-macro problem under inhomogeneous spatial distributions of agents. CDM systems are found both in natural and artificial swarms. A prominent example in nature is given by CDM in ant colonies [10, 11]. CDM systems with tight requirements concerning scalability and robustness are also investigated in swarm robotics [12, 13].

In this paper, we build on a recent work by Biancalani et al. [14]. They define a simplistic collective decision-making model of foraging in ants (meetings of two agents followed by a spontaneous switch of one of them) and investigate noise-induced bistability. Above a critical swarm size, their system fails to converge on a valid collective decision (i.e., it converges on states with conflicting opinions); that is, in contrast to swarm intelligence systems, it does not scale. Nonetheless, their derivation of a macro-model based on the microscopic description of the system behavior, namely chemical reactions, is of particular relevance. We investigate systems operating on local majority rules, that is, subgroups of the swarm cooperate temporarily and have a consensus on the local majority decision. These systems scale [12, 13], comply with principles of swarm intelligence, and correspond well to both artificial and natural swarm systems. Following the mathematical approach of Biancalani et al. [14], we derive a stochastic differential equation as a macro-model starting with the microscopic description given by a reaction schema. We focus on the drift terms of these equations that allow to predict the long-term system behavior. Hence, we succeed in establishing a mathematically sound micro-macro link requiring only two minor approximations (Taylor expansion and empirical approach for coefficients in the master equation).

2 Model and derivation of a micro-macro link

We consider a swarm of N agents undergoing a CDM process. Agents are characterized by their current opinion, for example, their direction of motion or a preference for a particular site in the environment. We restrict our investigations to the simplest case of a binary decision scenario. Each agent favors one out of two possible opinions, henceforth referred to as opinion 1 and opinion 2. Agents change their opinion during the CDM process as they apply a majority rule based on their local neighborhood. That is, when taking a decision, an agent perceives the opinions of its neighbors in a limited perception range, it includes in this group its own opinion, and eventually adopts the opinion favored by the majority of this group. The size G of the neighborhood may vary between different applications of the majority rule. We consider only odd neighborhood sizes to simplify the analysis (i.e., no tie-breakers necessary) and therefore $G \in \mathcal{G} = \{3, 5, \dots\}$. Agents take decisions at a rate r. In addition, an

agent may spontaneously change its opinion at a rate ε. With these spontaneous switches we model noise.

We represent the above described microscopic model using a set of chemical reactions that model all possible causes affecting the opinion of an agent—the reaction schema. Generally, chemical reactions are used to model the dynamics of well-mixed compounds; therefore, our model implicitly assumes a spatially well-mixed system and is thus an approximation of the actual system dynamics. The definition of the reaction schema depends on the particular scenario of interest. We give general equations to define the reaction schema:

$$R_{i,j}^{M,m} : mX_i + MX_j \xrightarrow{r} (m-1)X_i + (M+1)X_j, \tag{1}$$

$$R_{i,j} : X_i \xrightarrow{\varepsilon} X_j. \tag{2}$$

Given an agent X_i with opinion $i \in \{1, 2\}$, eq. 1 models the result of the majority rule applied (at a rate r) to a group of $G = m + M$ agents of which a minority mX_i of m agents favor opinion i while a majority MX_j of M agents favor opinion $j \in \{1, 2\} \setminus i$ (thus $m < M$, G odd, transition of one agent from opinion i to j). Eq. 2 describes the spontaneous switch (at a rate ε) of an agent X_i from opinion $i \in \{1, 2\}$ to opinion $j \in \{1, 2\} \setminus i$ and models noise. For clarity, we provide an example of a reaction schema for group size $G = 3$:

$$R_{1,2}^{2,1} : X_1 + 2X_2 \xrightarrow{r} 3X_2, \quad R_{1,2} : X_1 \xrightarrow{\varepsilon} X_2, \tag{3}$$

$$R_{2,1}^{2,1} : 2X_1 + X_2 \xrightarrow{r} 3X_1, \quad R_{2,1} : X_2 \xrightarrow{\varepsilon} X_1. \tag{4}$$

Reaction $R_{1,2}^{2,1}$ describes a situation in which an agent with opinion 1 has two neighbors with opinion 2 and, after applying the majority rule, it switches to opinion 2 (respectively, reaction $R_{2,1}^{2,1}$ for an agent with opinion 2). Besides, reaction $R_{1,2}$ models the spontaneous switch in the opinion of an agent with opinion 1 to opinion 2 (respectively, reaction $R_{2,1}$ for an agent with opinion 2).

If we would know the probability density function $f(x_1, x_2, t)$ that describes the time evolution of the proportions of agents x_1 and x_2 (respectively, with opinion 1 and opinion 2), then we would have a complete understanding of the system dynamics. Following the approach of van Kampen [15], $f(x_1, x_2, t)$ is obtained by writing and solving the corresponding master equation

$$\begin{aligned} \partial_t f(x_1, x_2, t) = \sum \big[& T(x_1, x_2 | x_1', x_2') f(x_1', x_2', t) \\ & - T(x_1', x_2' | x_1, x_2) f(x_1, x_2, t) \big], \end{aligned} \tag{5}$$

where $x_1' = x_1 \pm 1/N$, $x_2' = x_2 \mp 1/N$, and $T(a|b)$ represents the transition rate from state b to state a. However, analytical solutions of master equations are known only for a limited number of cases (cf. van Kampen [15]). Nonetheless, Biancalani et al. [14] derive an approximation to the master equation 5 by means of step operators, which represent the change in the opinion of a single agent,

and a Taylor expansion in $1/N$, which yields

$$\partial_t f(x_1, x_2, t) \approx [\frac{1}{N}(\partial_{x_2} - \partial_{x_1})T_1 + \frac{1}{N}(\partial_{x_1} - \partial_{x_2})T_2 + \frac{1}{2N^2}(\partial_{x_1} - \partial_{x_2})^2(T_1 + T_2)]f(x_1, x_2, t). \quad (6)$$

Biancalani et al. [14] reduce eq. 6 to a Fokker-Planck equation by inserting the expressions of the transitions rates T_1 and T_2 followed by rescaling time: $t/N \to t$. The Fokker-Planck equation is characterized by a drift term, that describes the change in the mean proportions of agents x_1 and x_2, and a diffusion term, which accounts for the variability of the same quantities. Finally, Biancalani et al. show the equivalence of the obtained Fokker-Planck equation to a system of stochastic differential equations (SDE). We focus on the drift term defined in the system of SDEs because it determines the dominant features of the investigated systems.

The transition rates T_1 and T_2 give the rates at which x_1 and x_2 increase over time. The transition rates depend on the particular reaction schema used to describe the original process. As above, we provide general functions for the corresponding reaction rates of a given reaction schema

$$\begin{aligned} T_1 &\equiv T(x_1 + \frac{1}{N}, x_2 - \frac{1}{N}|x_1, x_2) \\ &\approx \varepsilon x_2 + \sum_{G\in\mathcal{G}} \sum_{n=1}^{\lceil G/2\rceil - 1} r\binom{G}{n}^2 x_1^{G-n} x_2^n, \quad (7) \\ T_2 &\equiv T(x_1 - \frac{1}{N}, x_2 + \frac{1}{N}|x_1, x_2) \\ &\approx \varepsilon x_1 + \sum_{G\in\mathcal{G}} \sum_{n=1}^{\lceil G/2\rceil - 1} r\binom{G}{n}^2 x_1^{n} x_2^{G-n}. \quad (8) \end{aligned}$$

In eqs. 7 and 8, $\mathcal{G}$ is the set of all possible group sizes while n represents the number of agents in the group favoring the opinion associated to the minority. The binomial coefficients are included based on a heuristic consideration and account for all possible combinations of agents in the group. For the example reaction schema presented above, eqs. 7 and 8 yield the transition rates

$$T_1 = \varepsilon x_2 + r\binom{3}{1}^2 x_1^2 x_2 \quad \text{and} \quad T_2 = \varepsilon x_1 + r\binom{3}{1}^2 x_1 x_2^2. \quad (9)$$

In both reaction rates, the first term models the effect of noise due to spontaneous switching, while the second term models applications of the majority rule.

The approximation of the master equation in eq. 6 together with the transition rates in eqs. 7 and 8 provides a complete macroscopic model derived from the microscopic process described through the reaction schema. As done by Biancalani et al., we reduce the model to a single variable $z = x_1 - x_2$. The change of z over time is given by

$$\dot{z} = \dot{x_1} - \dot{x_2} = 2(T_1 - T_2) + D'(x_1, x_2) = 2\Delta_z(T_1, T_2) + D(z). \quad (10)$$

In eq. 10, the drift $2\Delta_z(T_1, T_2)$ and the diffusion $D(z)$ summarize the contributions given by all possible combinations of group sizes and according majorities defined by the reaction schema. Henceforth, we focus on averages $\langle \dot{z} \rangle$ and hence omit the treatment of the diffusion term. The drift term defines the system's main features, such as fixed points and negative/positive feedback. The manual derivation of the term $2\Delta_z(T_1, T_2)$ as a function of z is a rather complex task for nontrivial reaction schemas. Nevertheless, $2\Delta_z(T_1, T_2)$ has a regular structure that consists of a fixed term $-\varepsilon z$, which results from noise, plus a linear combination of polynomials $p_G(z)$ spanning over all considered group sizes G resulting from the majority rule. Linear combinations of such polynomials are easily manageable when applied to the analysis of systems.

The explicit derivation of polynomials $p_G(z)$ for all considered group sizes requires an extensive sequence of change, expansion, and collection of variables. It is a strenuous task whose difficulty increases with the size of the group. We propose a set of recursive functions that automatically generate the corresponding polynomial for a given group size G. The first function

$$p_G(z) = \sum_{m=1}^{\lceil G/2 \rceil - 1} \Delta t_G^m \left(r \binom{G}{m}^2 \right) \tag{11}$$

factorizes the polynomial $p_G(z)$ in a sum of simpler terms Δt_G^m which provide the contribution to the overall drift of a particular group size G and minority m. Function Δt_G^m, that is defined as

$$\Delta t_G^m(\rho) = \rho \left[\frac{1}{4} \left(1 - z^2\right) \right]^m h(G - 2m), \tag{12}$$

together with function

$$h(w) = z^w + \sum_{i=1}^{\lceil w/2 \rceil - 1} \Delta t_w^i \left((-1)^{i+1} \binom{w}{i} \right), \tag{13}$$

implement a recursive series of mathematical operations based on the binomial theorem (and related to Pascal's triangle) that are aimed at finalizing the change of variable $z = x_1 - x_2$. The resulting polynomial is characterized by odd powers of z with exponents within $[1, G]$. For the above example, where $G = 3$, the noise term $-\varepsilon z$ plus the recursion of eqs. 11, 12 and 13 yields for the average change

$$\begin{aligned} \langle \dot{z} \rangle = -\varepsilon z + p_3(z) &= -\varepsilon z + \Delta t_3^1 \left(r \binom{3}{1}^2 \right) \\ &= -\varepsilon z + r \binom{3}{1}^2 \left[\frac{1}{4} \left(1 - z^2\right) \right] h(1) \\ &= -\varepsilon z + r \binom{3}{1}^2 \left[\frac{1}{4} \left(1 - z^2\right) \right] z \\ &= -\varepsilon z + \frac{9}{4} r z - \frac{9}{4} r z^3. \end{aligned} \tag{14}$$

3 Simulation of a locust alignment behavior

The desert locust, *Schistocerca gregaria*, exhibits a collective motion behavior ('marching bands') [16] in which a majority of locusts align and move in a same direction. Individual locusts seem to change their direction of motion as a response to neighbors. In locust experiments [16], the complexity of the natural environment is reduced to a pseudo-1-d setting by using a ring-shaped arena. We use the microscopic model of self-propelled particles proposed by Czirók et al. [17] as our reference model (henceforth 'Czirók model').

We study a system of $N = 41$ particles in 1-d space. A particle i has coordinate $y_i \in [0, C)$ (circumference $C = 70$) and discrete, dimensionless velocity $u_i \in [-1, 1]$. We refer to particles with velocity $u_i < 0$ as 'left-goers' (respectively, 'right-goers' for $u_i > 0$). The dynamics of a particle is defined by $y_i(t+1) = y_i(t) + v_0 u_i(t)$, where $v_0 = 0.1$ is the nominal particle velocity and $u_i(t+1) = F(\langle u(t)\rangle_i) + \xi_i$ models the particle interaction with its neighbors (subject to noise ξ_i uniformly distributed over $[-\eta/2, \eta/2]$, $\eta = 2.5$). The local average velocity $\langle u(t)\rangle_i$ for the ith particle is calculated over all neighbors located in the interval $[y_i - \Delta, y_i + \Delta]$ for perception range $\Delta = 1.0$. F describes both propulsion and friction forces

$$F(u) = \begin{cases} (u+1)/2, & \text{for } u > 0 \\ (u-1)/2, & \text{for } u < 0 \end{cases}. \tag{15}$$

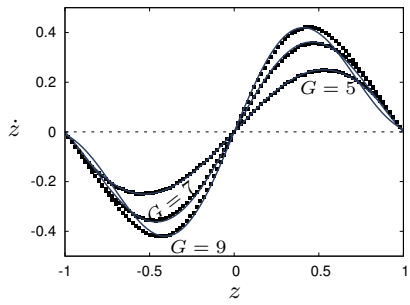
The initial condition is a random uniform distribution for both the particles' coordinates $y_i \in [0, C)$ and their velocities $u_i \in [-1, 1]$.

4 Validation of the model

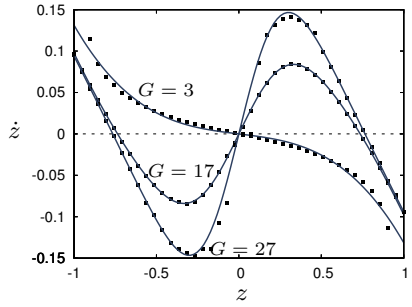
We validate the results presented in Sec. 2 by fitting the drift term defined by linear combinations of polynomials $p_G(z)$ to simulations of two microscopic scenarios. First, for selected group sizes, we fit single polynomials $p_G(z)$ to the average result of simulations of the Gillespie algorithm [18]. Then, we focus on the locust system described in Sec. 3 and we validate our full methodology.

4.1 Gillespie simulations

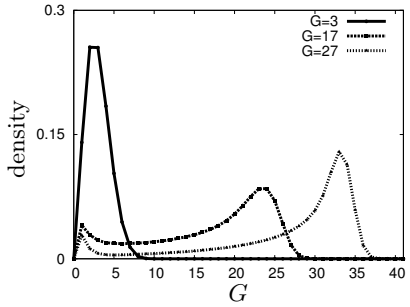
The Gillespie algorithm, also known as Stochastic Simulation Algorithm (SSA), is a Markov chain Monte Carlo method that is proven to generate statistically correct trajectories of a given reaction schema [18]. One of the primary advantages of the Gillespie algorithm is its capability to provide a numerical solution equivalent to that of the master equation by averaging over an ensemble of independent realizations. Given a particular reaction schema, the Gillespie algorithm consists of 3 steps: (i) update the reaction rates for each reaction according to the current state; (ii) randomly determine which and when the next reaction will occur; and (iii) update the system state and jump



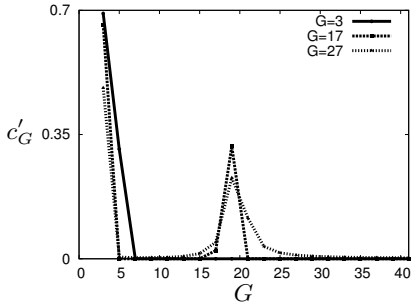
(a) Gillespie, squares: Gillespie simulation (2.5×10^5 samples), line: fitted polynomial



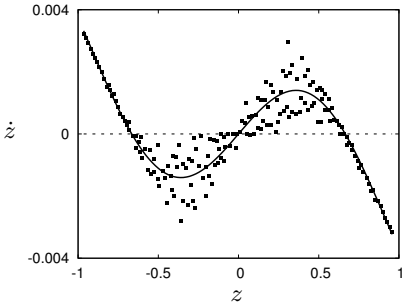
(b) Czirók model, squares: Czirók simulation (10^6 samples), line: fitted polynomials



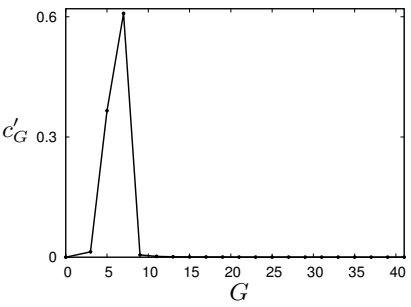
(c) Czirók model, group size distribution measured in simulation



(d) Czirók model, fitted coefficients interpreted as group size distribution



(e) locusts, squares: data from Fig. 3B of [19] (local model of swarm alignment in locusts), line: fitted polynomials



(f) locusts, fitted coefficients interpreted as group size distribution

Fig. 1. Results for fitting the polynomials (eq. 11) to data from Gillespie simulations, simulations of the Czirók model, and to data of swarm alignment in locusts [19].

back to step (i). We run Gillespie simulations for reaction schemas that implement the majority rule for one fixed group size G. In the simulation we measure the probability $P(x_1 + \frac{1}{N}|\text{switch}, x_1)$ that once an agent's switch in opinion is observed it increases x_1. To get an approximation to the drift term we rescale $\dot{z} \approx P(x_1 + \frac{1}{N}|\text{switch}, x_1) - (2x_1 - 1) = P(z + \frac{1}{N}|\text{switch}, z) - z$. By fitting polynomials $p_G(z)$ to the results of Gillespie simulations we can assess the validity of the approximations introduced in Sec. 2. Fig. 1a shows the results for group sizes $G \in \{5, 7, 9\}$ of the fits between polynomials $p_G(z)$ and the average of 2.5×10^5 Gillespie simulations. We achieve good fits for a range of values $-0.4 < z < 0.4$ but observe systematic deviations for $z < -0.4$ and $z > 0.4$.

4.2 Locust simulations

In the simulation of the Czirók model we measure the average change $\langle \dot{L} \rangle$ of the ratio of left-goers (averaged over 10^6 independent simulation runs) as a function of the current ratio of left-goers L and the current average neighborhood size G of agents (i.e., agents within perception range Δ) averaged over all agents. These measurements are easily converted to variable z as introduced in Sec. 2 ($z = L - R = 2L - 1$), for the ratio of right-goers R. The measured values of $\langle \dot{z} \rangle$ are then fitted[3] using a sum over the above polynomials $p_G(z)$

$$\langle \dot{z} \rangle = -\varepsilon z + \sum_{G \in \mathcal{G}} c_G p_G(z), \tag{16}$$

with the additional constraints of $c_G \geq 0$ which allow us to interpret the coefficients c_G as weights of each polynomial $p_G(z)$. The results for $G \in \{3, 17, 27\}$ are shown in Fig. 1b. We achieve good fits. In the simulation of the Czirók model we also measure the distributions of neighborhood sizes for given averages of neighborhood sizes (10^6 simulation runs) as shown in Fig. 1c. In Fig. 1d we plot the coefficients, that were obtained in the fitting process for Fig. 1b, in increasing order of G and normalized to $\sum_G c'_G = 1$. By interpreting the coefficients as weights for each neighborhood size G we can read this plot as an approximation of the neighborhood size distribution. Although there is no quantitative agreement, we notice a qualitative agreement. The neighborhood size distribution for $G = 3$ is unimodal as reflected by the coefficients. For $G \in \{17, 27\}$ we have bimodal distributions as in the coefficients. Furthermore, the mean of the fitted coefficients monotonically increases with increasing neighborhood sizes (data not shown).

Finally, we show results for a different source of data. A publication of Yates et al. [19] shows in Figs. 2B and 3B how the drift coefficient depends on the current alignment of a swarm (average velocity). Because the data obtained from experiments with locusts, Fig. 2B in [19], is too noisy, we use instead data from their model, Fig. 3B in [19], to fit our polynomials. The result is a good fit (see Fig. 1e). Fig. 1f shows the corresponding coefficients which peak for

[3] Nonlinear least-squares Marquardt-Levenberg algorithm [20] using gnuplot 4.6 patchlevel 1 (2012-09-26), see http://www.gnuplot.info/

neighborhood size $G = 7$ and have a second high value for $G = 5$. Unfortunately, Yates et al. do not report neighborhood sizes. Still, our result seems reasonable given their parameters: locust density $1/3$ and interaction radius 5. Assuming a uniform distribution, we get neighborhood sizes of about 3.3. However, we know that locusts align and concurrently tend to cluster. Hence, significantly higher local densities should be expected which supports our finding of neighborhood sizes $G \in [3, 9]$.

5 Discussion and conclusion

In this paper we extend the approach of Biancalani et al. [14] to reaction equations that include more than two reacting molecules. The obtained method allows to model majority-rule decisions and is applied to CDM systems relevant to swarm intelligence. We report a recursive equation to systematically obtain a set of polynomials. With these polynomials we form linear combinations that define functions of candidate drift terms (i.e., the average change $\langle \dot{z} \rangle$ of swarm fractions that are in favor of one of the opinions). We have shown that, starting from a sample of drift measurements of a particular system, it is possible to obtain a qualitative prediction of the underlying group size distribution by fitting these linear combinations. Our method relies only on measurements of the drift term. Such measurements can be easily obtained as we have shown for the Czirók model and as done for locusts by Yates et al. [19]. Our method applies to both directions of micro-macro transitions: from a given average drift term $\langle \dot{z} \rangle$ (e.g., measured or desired) to an approximation of the underlying group sizes (macro to micro); or vice versa, from a given group size distribution to the prediction of the average drift $\langle \dot{z} \rangle$ (micro to macro). Hence, we establish a micro-macro link.

Motivated by these preliminary results, we plan several extensions. We will investigate methods to apply the master equation instead of approximations while keeping the constraint that the model should be concise and manageable. Alternatively, we will investigate the use of different approximations with the goal of decreasing, for larger group sizes, the difference between the predicted drift term and the results obtained with Gillespie simulations (see Fig. 1a). We will also investigate generalizations of this approach that allow for different decision-making strategies (beyond pure majority decisions) and we will validate the model against a wider set of simulations (e.g., varied perception ranges). We plan to search for polynomials describing CDM systems that are orthogonal functions which can be used as basis functions. This would allow for deterministic calculations of coefficients in the form of a discrete transform (similar to a Fourier transform) instead of the proposed fitting approach.

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