In a full binary tree of n nodes and k edges, the number of edge bottoms is exactly the number of nodes minus one, since each node except the root lies at the bottom of exactly one edge. Hence,

$$\#\text{ edge bottoms} = n - 1.$$

Also, in a full binary tree each internal node is at the top of exactly two edges, so

$$2 \times (\#\text{ internal nodes}) = \#\text{ edge tops}.$$

We note further that the number of nodes n is the sum of the number of leaves and the number of internal nodes:

$$n = (\#\text{ leaves}) + (\#\text{ internal nodes}).$$

Putting these facts together, we get:

$$\begin{aligned} 2 \times (\#\text{ internal nodes}) &= \#\text{ edge tops} = \#\text{ edge bottoms} \\ &= n - 1 \\ &= (\#\text{ leaves}) + (\#\text{ internal nodes}) - 1. \end{aligned}$$

Solving for the number of leaves in terms of the number of internal nodes gives:

$$\#\text{ leaves} = (\#\text{ internal nodes}) + 1. \tag{1}$$

This relationship is used in numerous places throughout the text.

3.3.2 The number of binary trees with *n* nodes

The objective of this section is to determine for a given n the number b_n of binary trees with distinct shapes that can be formed each containing n nodes. Figure 3.14 illustrates the different binary tree shapes obtainable for each of the n from 1 to 4. If $n = 1$, $b_n = 1$. If $n > 1$, we can pick one of the n nodes as the root of the tree, and we can partition the remaining $n - 1$ nodes into left and right subtrees. Let j be the number of nodes assigned to form the left subtree $(0 \leq j \leq n - 1)$. Then $n - 1 - j$ nodes remain to form the right subtree. In this instance, $b_j b_{n-1-j}$ gives the total number of binary trees that can be formed with j nodes in the left subtree, but to get all possible binary trees on n nodes, we must sum up product terms of this form over all possible values j of the number of nodes in the left subtree. This gives us the relation

$$b_n = b_0 b_{n-1} + b_1 b_{n-2} + \cdots + b_{n-1} b_0. \tag{2}$$

We now solve this recurrence relation using a generating function† of the form

† For a good introduction to generating functions and their uses, see Knuth [1973a].

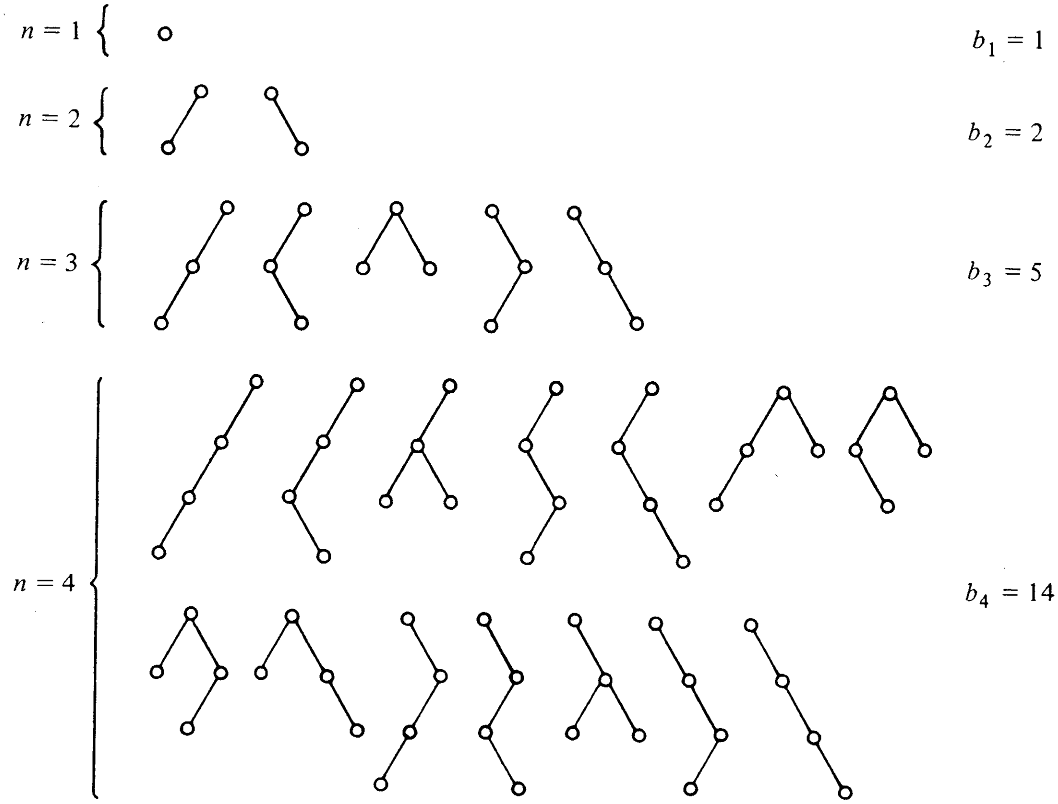

FIGURE 3.14

$$G(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

When we multiply $G(x)$ by itself, we get

$$\begin{aligned} G(x)G(x) &= (b_0 + b_1 x + b_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots) \\ &= (b_0 b_0) + (b_0 b_1 + b_1 b_0)x + (b_0 b_2 + b_1 b_1 + b_2 b_0)x^2 + \cdots \\ &= \sum_{n \geq 0} \left(\sum_{0 \leq j \leq n} b_j b_{n-j} \right) x^n. \end{aligned}$$

Thus, we observe that the coefficient of x^n in this expansion of $G(x)^2$ is exactly the formula for b_{n+1} (or, equivalently, that b_n is the coefficient for x^{n-1}). Hence, if we multiply $G(x)^2$ by x, we can get the coefficients of x^n to match those of $G(x)$, except for the first term, since then $xG(x)^2 = b_1 x + b_2 x^2 + b_3 x^3 + \cdots$. Indeed, we can make $xG(x)^2$ match the expansion of $G(x)$ exactly, by adding a term of the form b_0 to $xG(x)^2$. But since $b_0 = 1$, this implies

$$1 + xG(x)^2 = G(x).$$

The latter equation is quadratic in $G(x)$, and yields two solutions, one of which is

$$G(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \frac{1}{2x}\left(1 - \sqrt{1 - 4x}\right). \tag{3}$$

Using a Taylor series expansion of $(1 + z)^r$, we get a binomial generating function

$$(1 + z)^r = 1 + rz + \frac{r(r-1)}{1 \cdot 2} z^2 + \frac{r(r-1)(r-2)}{1 \cdot 2 \cdot 3} z^3 + \cdots. \tag{4}$$

The coefficients of powers of z in this generating function can be written more succinctly using the following generalized definition of binomial coefficients, where r is any real number and k is an integer:

$$\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{1 \cdot 2 \cdot 3 \cdots k} \qquad \text{for } k > 0;$$
$$\binom{r}{0} = 1, \qquad \text{and} \qquad \binom{r}{k} = 0 \qquad \text{for } k < 0. \tag{5}$$

Thus, the binomial generating function (4) can be rewritten as

$$(1 + z)^r = \sum_{k \geq 0} \binom{r}{k} z^k. \tag{6}$$

Since $\sqrt{1 - 4x} = (1 + (-4x))^{1/2}$, we can replace $\sqrt{1 - 4x}$ in (3) with an appropriately substituted version of (6) to get

$$G(x) = \frac{1}{2x}\left(1 - \sum_{k \geq 0} \binom{1/2}{k} (-4x)^k\right).$$

Using a change of dummy variable in which k is replaced by $n + 1$, and then simplifying, gives

$$\begin{aligned} G(x) &= \frac{1}{2x}\left(1 - \sum_{n+1 \geq 0} \binom{1/2}{n+1} (-4x)^{n+1}\right) \\ &= \frac{1}{2x} + \sum_{n+1 \geq 0} \binom{1/2}{n+1} (-1)^n 2^{2n+1} x^n \\ &= \sum_{n \geq 0} \binom{1/2}{n+1} (-1)^n 2^{2n+1} x^n. \end{aligned} \tag{7}$$

But the latter yields coefficients of x^n which match b_n in the original definition of $G(x)$. Hence,

$$b_n = \binom{1/2}{n+1} (-1)^n 2^{2n+1}. \tag{8}$$

The latter form of b_n is unpleasantly messy so it is worthwhile attempting to tidy it up a bit. This can be done as follows. We first expand $\binom{1/2}{n+1}$ using definition (5):

$$\binom{1/2}{n+1} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n\right)}{1\cdot 2\cdot 3\cdots(n+1)}$$

$$= \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{-2n+1}{2}\right)}{1\cdot 2\cdot 3\cdots(n+1)}$$

$$= \frac{(-1)^n}{2^{n+1}}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 2\cdot 3\cdot 4\cdots(n+1)}.$$

We observe that the numerator of the latter expression contains a product of odd numbers of the form $1\cdot 3\cdot 5\cdots(2n-1)$. Such a product can be expressed by taking the product of all the numbers from 1 to $2n$ and striking out the even numbers

$$1\cdot 3\cdot 5\cdots(2n-1) = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-1)(2n)}{2\cdot 4\cdots(2n)} = \frac{(2n)!}{2^n\cdot n!}.$$

This permits us to rewrite

$$\binom{1/2}{n+1} = \frac{(-1)^n}{2^{n+1}}\cdot\frac{1}{(n+1)!}\cdot\frac{(2n)!}{2^n\cdot n!} = \frac{(-1)^n}{2^{2n+1}}\cdot\frac{1}{(n+1)}\binom{2n}{n}.$$

Substitution of the latter result back into (8) yields, after cancellation, the following final form

$$b_n = \frac{1}{n+1}\binom{2n}{n}. \tag{9}$$

Using Stirling's approximation (see Exercises 3.4 and 3.6), we can show that the latter is $4^n/n\sqrt{\pi n} + O(4^n n^{-5/2})$.

A corollary of importance to our later discussion is the following:

$$b_{n-1} = \text{the number of distinct ordered trees on } n \text{ vertices.} \tag{10}$$

If we take an ordered tree on n vertices and remove its root, together with the edges connecting the root to the subtrees of the root (if any), we obtain an ordered forest on $n-1$ vertices. The construction of Figs. 3.12 and 3.13 then shows a way of establishing a 1–1 correspondence between such ordered forests and appropriately constructed binary trees, and the result in (10) follows.