The Moral Hazard Problem with High Stakes

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Abstract

We study the moral-hazard problem when the agent's reservation utility is large, but so is the agent's value to the principal. We show that the principal's cost of implementing effort has a very simple limiting form. For large enough outside option, the principal's cost is convex in the action, so the optimally-implemented action is unique, and optimal effort rises with the agent's ability, and falls with the agent's wealth and outside option. In a competitive market setting where heterogenous principals and agents match, positive sorting ensues and effort increases in match quality, despite conflicting forces.

Keywords. Moral Hazard, First-Order Approach, Principal-Agent Problem, Comparative Statics, Matching.

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1 Introduction

The moral-hazard problem (Mirrlees (1975), Holmstrom (1979), Grossman and Hart (1983)) is fundamental to information economics, with numerous applications in contract theory, industrial organization, macroeconomics, and finance. In this paper we explore the principal-agent problem when the *stakes are high*–the agent has an attractive outside option, but is sufficiently important to the principal that she still wishes to employ him.

In the canonical moral-hazard problem, a principal (she) hires an agent (he) to take an action that gives him disutility. The principal cares about the action, but can only condition compensation on a noisy signal of the action. The principal's optimal contract is thus constrained by the agent's incentives to deviate to a different action, and also by the possibility that the agent might take a more attractive outside option.

Grossman and Hart (1983) provide a useful two-step approach to analyze this problem. First, for each action find the compensation scheme that minimizes the principal's expected cost of implementing that action, which yields a cost function C. Second, find the action that maximizes the principal's gross benefit to the action less C.

Beyond the classic Holmstrom-Mirrlees form of the optimal compensation scheme in terms of the likelihood ratio, we know surprisingly little about C. If the agent is more capable, or has a higher outside option or higher wealth, what happens to the action the principal optimally chooses to implement? The literature is largely silent. Is C convex in the action, so that the problem faced by a principal who has a concave gross benefit from effort will necessarily be well-behaved, with a unique maximum that is continuous in the underlying parameters? Jewitt, Kadan, and Swinkels (2008) is the only result of which we are aware.[1]

We make two main contributions. First, as the outside option grows, C converges in a very strong sense to the full information case, with the formidable complications of the problem falling away. Second, we leverage this to make novel but intuitive predictions about the comparative statics of the problem in the underlying economic parameters when the stakes are sufficiently high. As an application of these results, we show that a natural but hitherto little-analyzed matching problem between principals and agents admits a general solution with an unambiguous prediction for sorting and the equilibrium action.

At the heart of our first contribution is that as the outside option rises, both C itself and its relevant first and second derivatives converge to remarkably simple objects with clean economic intuitions. In particular, *each relevant derivative converges to its analog in a setting without moral hazard*–the ratio of each derivative to its full information analog goes to one as the outside option diverges, uniformly in the choice of the other parameters. Thus, the marginal cost of providing

[1] As Jewitt, Kadan, and Swinkels (2008), Example 1 shows, absent convexity of C in the action, it can easily be that the principal is harmed by a constraint such as a minimum wage, but then chooses a contract that never pays according to the constraint.

the agent with utility (the shadow value of the participation constraint) converges simply to the marginal cost of utility when the agent receives a deterministic income that covers his outside option and disutility of effort, and similarly for the other first and second derivatives.

At the heart of our second contribution is that the full-information problem has a very simple structure. The cost of implementing effort is convex in effort, supermodular in effort and ability, submodular in effort and the outside option, and submodular in effort and the wealth of the agent. Hence in the full information problem, the comparative statics are both simple and intuitive. But, given our convergence results, as long as the outside option of the agent exceeds some threshold, these *same comparative statics properties will hold under moral hazard.* Thus, for a sufficiently high outside option (1) the principal's profit-maximization problem is concave and well-behaved; and (2) the optimal effort *increases* in the ability of the agent, *decreases* in his outside option, and *decreases* in his wealth.[2]

The key assumptions driving our results are on the agent's utility for income. Core is that as the agent's compensation grows large, the cost of providing him an extra util changes by an amount that goes to zero when the agent becomes a dollar richer. The assumptions are satisfied for a broad class of *HARA* (hyperbolic absolute risk aversion) utility functions. But, they are not without bite: log utility is excluded, and indeed the convergence results fail in this setting.

To show our results, we begin with a key lemma: even as the agent's reservation utility grows, the difference between the highest and lowest utility provided to the agent stays *bounded.* Using this, we consider first the shadow value of the participation constraint. Since the contract's variation in utility remains bounded, but utility is increasingly high, most of the utility the agent receives is about retention, and a relatively small amout is about motivation. And, we show that the *percentage* difference between the highest and lowest cost of providing an extra util grows small over the range of utilities offered. But then, the expectation of this marginal cost–the shadow value of the participation constraint–converges to the marginal cost of providing the agent utility in the full information setting. Using this, we show that the ratio of C to the full-information cost function, C^{FI}, also converges to one as the outside option increases.

We next consider the limiting behavior of the shadow value of the incentive constraint in the cost minimization problem. It is the product of three objects; the marginal disutility of effort, the rate of change (in absolute terms) of the marginal cost of providing utility, and the reciprocal of the Fisher Information. Intuitively, when the marginal disutility of effort is high, then the contract must already be steep. Moving utility from low to high outcomes is expensive if the contract is steep, and the marginal cost of providing utility varies a lot in income. And the amount of utility one has to move to increase incentives depends on how informative the signal is about small changes in effort.

[2]We view our convexity result on C as complementary to that of Jewitt, Kadan, and Swinkels (2008), who make a strong assumption on the information structure of the problem but do not need high stakes.

Since the cost functions C and C^{FI} converge, one might assume that their derivatives do too. But, the notion of convergence here is *much* too weak for that conclusion. To proceed, we dig more deeply into the behavior of the shadow values of the constraints, building towards a key technical "domination" result: as the agent's reservation utility grows, the shadow value of the participation constraint dominates the shadow value of the incentive constraint and its derivative with respect to effort. Intuitively, relaxing the participation constraint involves adding utility everywhere, which is increasingly expensive, while relaxing the incentive constraint involves adding utility at high outcomes but *removing* it at low outcomes, which limits the relative rate of growth of the associated shadow value.

Thus armed, we prove our main results about the derivatives of C. The strategy is to express each of the relevant first and second derivatives in terms involving the shadow values and their derivatives, and then use the domination result and the bounded-variation-in-utility result to focus attention on a small number of "lead" terms. This allows us to connect the respective derivatives of C and C^{FI}. And, since the limit expressions have the appropriate signs for a sufficiently high outside option, so do the derivatives of C.

The main economic contribution of the paper follows immediately from these results: when the outside option is sufficiently high, the principal's profit maximization problem (choice of optimal action to implement) has simple and intuitive structure. The convexity of C yields a unique optimal action, while its cross-partials pin down the comparative statics with respect to the agent's ability, outside option, and wealth.

These comparative-statics results have, to the best of our knowledge, no counterpart in the literature. Nor are any of these results intuitively obvious when the outside option is small. It is not for example obvious that a higher outside option leads to lower optimal effort: when the agent has a high outside option, then the principal needs to provide high utility somewhere, and perhaps this is optimally in the form of strong incentive payments. Similarly, part of the advantage of a more capable agent (holding fixed the outside option) is that they have a lower direct cost of effort and so a relaxed participation constraint. Thus, if the shadow value of the participation constraint falls with effort, then this advantage falls as well, providing a force against submodularity. Finally, a richer agent is typically less risk averse, and so can perhaps be faced with higher-powered incentives at lower marginal cost.

At the core of our results is that when the outside option is large, the shadow value of the participation constraint rises with the action. In contrast, when the outside option is small, the combination of payments added at high outcomes and subtracted at low outcomes to induce a higher action may on net *relax* the participation constraint, lowering the associated shadow value. For this reason, we conjecture that any analogous comparative-statics results without the high-stakes assumption will depend on additional structure on information.

So far, we have treated the outside option of the agent and his ability as separate, signing the comparative statics in each variable. But in many economic settings, more capable agents have more attractive outside options, creating two countervailing forces. We show that as long as the outside option of the agent rises at a finite rate in his ability, then for sufficiently large outside option, more capable agents are asked a higher action.

Next, in the spirit of endogenizing the outside option of the agent, and as a foray beyond the standard principal-agent model, we embed the principal-agent model in a competitive market setting, where principals who differ in their technology match with agents who differ in their ability. We show that when the agents' utility outside of the market is sufficiently large, then the equilibrium sorting pattern exhibits positive sorting–better principals are matched with higher-ability agents. Further, the action increases in the quality of the principal-agent match. Although there are sorting results under moral hazard (see Legros and Newman (2007) and the references therein), none of them is at anything like the level of generality of this one. And, while our results are in an intuitive direction, they are far from obvious given the conflicting forces present.

Knowing that things become tractable as $\bar{u}$ grows raises an interesting question: how big is big enough for the intuitive comparative statics to hold? In our final section, we examine two classes whose extra structure allows a clear–and fairly reassuring–answer to this question. If utility is square root, and there are two outcomes, with the probability of the good outcome linear in effort, then for C to be convex in a, the outside option needs only to be high enough that "retention" is at least 9/16 of the cost of employing the agent at full effort, and for the principal to nonetheless wish to employ the agent, it is enough that the outside option of the agent is no more than the same 9/16 of the principal's gross benefit from full effort.

The next section describes the model. Section 3 discusses the additional assumptions on the agent's utility for income. Section 4 examines the limiting structure of C. Section 5 turns to the principal's profit-maximization problem and the comparative statics of optimal effort when the stakes are high. Section 5.2 embeds the model in a matching setting and shows positive sorting and increasing actions for large reservation utility. Section 6 looks at how large the outside option needs to be to for the key convexity and comparative statics results to hold. Section 7 concludes.

2 The Model

The model is essentially the standard principal-agent problem with moral hazard. There is a risk neutral principal (she) and a strictly risk-averse agent (he). The agent has utility for income, u, and takes an action $a \in [0,1]$, which entails disutility $c(a,\theta)$, where $\theta \in [0,1]$ is interpreted as the ability of the agent.

Assumption 1 *The agent's utility from effort a and wage w is $u(w) - c(a)$, where $u, c \in \mathcal{C}^3$, with $u' > 0$ and $u'' < 0$, and with $c_a > 0$, and $c_{aa} > 0$. For $a > 0$, $c_\theta < 0$, $c_{a\theta} < 0$, and $c_{aa\theta} \leq 0$.*

Finally, $c_a c_{a\theta}/c_\theta$, $c_\theta/c_{a\theta}$, and $c_a c_{aa\theta}/c_{a\theta}$ are uniformly bounded.[3]

That is, the agent's disutility is increasing and convex in effort, where high-ability agents have lower disutility and lower marginal disutility of effort, and have costs that are less convex in effort. A function c that satisfies Assumption 1 is $c(a, \theta) = (1 - k\theta)a^2/2$ for some $0 < k < 1$.

The agent has an outside option whose utility is $\bar{u}$. We will later consider variations of this model in which $\bar{u}$ depends on θ. The principal does not observe a, but instead observes a signal $x \in [0, 1]$ distributed according to a cumulative distribution function (*cdf*) $F(\cdot|a)$.

Assumption 2 *$F \in \mathcal{C}^4$, with density f and likelihood ratio $l(\cdot|a) \equiv f_a(\cdot|a)/f(\cdot|a)$, where l is bounded, and $l_x > 0$.*

Thus f satisfies the strict monotone likelihood ratio property (*MLRP*). The principal is risk neutral and her expected utility if the agent exerts effort a and she pays a wage w is $B(a, \tau) - w$, where $\tau \in [0, \infty)$ is a parameter. A natural interpretation of B is that $B(a, \tau) = \tau\mathbb{E}[x|a]$, so that x is output and τ its unit value, but other interpretations are subsumed by the general formulation.

Assumption 3 *$B \in \mathcal{C}^2$, where for all τ, $B_a > 0$ and $B_{aa} \leq 0$ and where $\lim_{\tau\to\infty} B(1, \tau) = \infty$ and $\lim_{\tau\to\infty} B_a(1, \tau) = \infty$.*

The timing is as follows: the principal offers a contract, consisting of a compensation scheme $\pi : [0, 1] \to \mathbb{R}$, which pays $\pi(x)$ for each x, and a recommended action a; the agent accepts or rejects; if he accepts then the agent chooses an action; x is observed and the wage is paid.

Let $v(x) \equiv u(\pi(x))$ be the agent's utility from income when the signal is x, and let $\varphi \equiv u^{-1}$ be the inverse of u, which is strictly convex since u is strictly concave. We can equivalently interpret the contract as a function v and a recommended action a, which we do from now on.

Assumption 4 *The First-Order Property (*FOP*) is valid.*

That is, the solution to the relaxed problem in which the full set of incentive constraints is replaced by the agent's first order condition in fact satisfies the full set of incentive constraints.[4]

Following Grossman and Hart (1983), it is standard to analyze the problem in two steps: cost minimization and profit maximization. First, for each a one finds the compensation scheme v that minimizes the cost of inducing the agent to exert effort a. Second, one uses the cost function so defined to find the profit-maximizing action that the principal induces the agent to choose.

[3]We use increasing and decreasing in the weak sense of nondecreasing and nonincreasing, adding 'strictly' when needed, and similarly with positive and negative, and concave and convex. For any function h, we write $(h)_x$ for the total derivative of h with respect to x, and h_x for the partial derivative. We use the symbol $=_s$ to indicate that the objects on either side have strictly the same sign. We will omit arguments from functions and limits of integration where convenient.

[4]See Rogerson (1985) and Jewitt (1988) for seminal contributions giving primitives for this to hold.

Let $C(a, \bar{u}, \theta)$ be the cost of implementing action a for an agent of ability θ who needs to receive utility level $\bar{u}$. That is,

$$C(a, \bar{u}, \theta) = \min_{v} \int \varphi(v(x)) f(x|a) dx \tag{1}$$

$$s.t. \qquad \int v(x) f(x|a) dx - c(a, \theta) \geq \bar{u} \tag{IR}$$

$$\int v(x) f_a(x|a) dx = c_a(a, \theta). \tag{IC}$$

Note that θ appears in both constraints. Following Holmstrom (1979) and Mirrlees (1975), the optimal solution satisfies, for each $x \in [0, 1]$

$$\varphi'(v(x)) = \lambda + \mu l(x|a), \tag{2}$$

where λ and μ, which depend on $(a, \bar{u}, \theta)$, are the Lagrange multipliers of, respectively (IR) and (IC). Given *MLRP* and *FOP*, the optimal v given a is strictly increasing in x.

Armed with the cost function C, we can set up the profit maximization problem

$$\max_{a \in [0,1]} B(a, \tau) - C(a, \bar{u}, \theta). \tag{3}$$

We wish to understand the properties of C when the agent has an attractive outside option, a large $\bar{u}$. This is most interesting when τ is large, so that the principal still wishes to employ him.

3 Additional Structure on Utility

To derive our results, we will need some extra structure on u beyond the very standard assumptions made so far. None of the assumptions we make are particularly demanding, but each plays an essential role, and, as we shall show, they do have bite. For each, we will provide an interpretation in terms of u, but also in terms of the behavior of φ, the inverse utility function.

Our first assumption is that as w diverges, utility diverges, but has slope that goes to zero.

Assumption 5 *φ has domain with least upper bound ∞, and with $\varphi' \to \infty$ as utility goes to ∞. Equivalently, as $w \to \infty$, $u \to \infty$, and $u' \to 0$.*

The equivalence follows from differentiating the identity $\varphi(u(w)) = w$. Recall the coefficient of absolute risk aversion $A = -u''/u'$. Our next assumption asks that in the limit, φ' grows slowly in percentage terms, or, equivalently, that A falls faster than u' as w diverges.

Assumption 6 *$\varphi''/\varphi' \to 0$ as utility goes to ∞. Equivalently, as $w \to \infty$, $A/u' \to 0$.*

Since $A/u' = -u''/(u')^2 = (1/u')'$, and since $1/u'$ is the cost of providing the agent an extra util, this is the condition that, as w become large, it becomes equally costly to provide an extra util to an agent with wealth w versus wealth $w-1$. The equivalence is established by a second differentiation of the identity $\varphi(u(w)) = w$, and rearrangement.

Recall the coefficient of absolute prudence $P = -u'''/u''$. Our final assumption states that φ'' also grows slowly in percentage terms in the limit or, equivalently, that $3A - P$ falls faster than u'.

Assumption 7 *$\varphi'''/\varphi'' \to 0$ as utility goes to ∞. Equivalently, as $w \to \infty$, $(3A - P)/u' \to 0$.*

Given Assumption 6, this assumption is equivalent to P also falling faster than u' as w diverges, for which in turn, is sufficient that P/A is bounded. This last condition is very mild. If $\lim_{w\to\infty} P(w)/A(w) = \infty$, then $-1/(u')^N$ fails to be concave for any N.[5] The equivalence follows from a third differentiation of the identity $\varphi(u(w)) = w$ and manipulation.

Example 1 *Consider the* HARA *utility functions u given by $u(w) = ((1-\gamma)/\gamma)\kappa^\gamma(w)$, where $\kappa(w) = (\alpha w/(1-\gamma)) + \beta$, $\alpha > 0$ and $(\alpha w/(1-\gamma)) + \beta > 0$, and where limit cases are derived as appropriate. Then, all our assumptions hold if $\gamma \in (0,1)$. When $\gamma = 0$, $u = \log w$, $A/u' = 1$, and Assumption 6 fails.*[6,7]

In what follows we will make extensive use of the mapping ρ which maps $1/u'$ into utils, and hence is defined by $1/u'(\varphi(\rho(\tau))) = \tau$, or, equivalently, since $\varphi = u^{-1}$, by $\varphi'(\rho(\tau)) = \tau$.

Our first lemma shows that under Assumption 6, as utility becomes large, φ' becomes essentially constant over any given range of utility. If 7 holds as well, then the same is true of φ'' and ρ'. The proof is direct given Assumptions 6 and 7.

Lemma 1 *If Assumptions 5 and 6 hold, then for each $\Delta > 0$, and for all $\varepsilon > 0$, there is u^* such that for any u_1 and u_2 with $u_1, u_2 \geq u^*$ and $|u_1 - u_2| \leq \Delta$,*

$$1 - \varepsilon \leq \frac{\varphi'(u_1)}{\varphi'(u_2)} \leq 1 + \varepsilon. \tag{4}$$

If Assumption 7 also holds, then u^* can be chosen such that, in addition,

$$1 - \varepsilon \leq \frac{\varphi''(u_1)}{\varphi''(u_2)} = \frac{\rho'(\rho^{-1}(u_2))}{\rho'(\rho^{-1}(u_1))} \leq 1 + \varepsilon, \tag{5}$$

[5] Straightforward calculations show that $(-1/(u')^N)'' =_s P - (N+1)A$

[6] Except when $\gamma = 0$, $u' = \alpha\kappa^{\gamma-1}$, and $u'' = -\alpha^2\kappa^{\gamma-2}$, and so $A = \alpha/\kappa$ is decreasing if and only if $\gamma < 1$. Since $A/u' = \kappa^{-\gamma}$, $A/u' \to 0$ if $\gamma \in (0,1)$, and since $P/A = (2-\gamma)/(1-\gamma)$, it is constant and finite except at $\gamma = 1$.

[7] It is easy to construct non-HARA examples (i.e., that do not have a linear coefficient of risk tolerance) satisfying our assumptions. One such example is $u(w) = w^{0.1}\log(1+w)$.

The key to the proof of (4) is that $\varphi''(\hat{u})/\varphi'(\hat{u})$ goes to zero in $\hat{u}$ by Assumption 6. But then, φ' does not move very far in proportionate terms over an interval of any fixed length. Similarly, $(\log\varphi''(\hat{u}))_{\hat{u}} = \varphi'''(\hat{u})/\varphi''(\hat{u})$ goes to zero in $\hat{u}$ by Assumption 7. Finally, since by definition $\varphi'(\rho(\tau)) = \tau$, it follows that $\varphi''(\hat{v}) = 1/\rho'(\rho^{-1}(\hat{v}))$.

4 Limiting Structure of C

This section contains our first main contribution: the limiting structure of C and its derivatives as $\bar{u}$ diverges. This entails an analysis of the behavior of λ and μ that is of independent interest. We first show that λ diverges, while incentives, in utils, remain bounded. We then derive the limiting behavior of λ and μ. These steps allow a simple proof that C converges to the full information cost C^{FI} for each pair (a, θ). Then we show that in the limit λ dominates μ and the derivatives of λ and μ with respect to a. Using this, we show that the first and second derivatives of C behave in the limit like those of C^{FI}, uniformly in (a, θ).

From here on, we will take Assumptions 1–7 as maintained and will not repeat them in the statements of the results.[8]

4.1 First Connections

We will link C as $\bar{u}$ grows to the cost function with full information (no moral hazard), given by $C^{FI}(a, \bar{u}, \theta) = \varphi(\bar{u} + c(a, \theta))$. Our first lemma establishes that as $\bar{u}$ diverges, so does λ and hence the utility at the "neutral" point where $l = 0$.

Lemma 2 *As $\bar{u} \to \infty$, $\lambda \to \infty$ uniformly in a and θ.*

The intuition is simple. Since $\bar{u}$ diverges, so must the utility given to the agent at $x = 1$, which is given by $v(1) = \lambda + \mu l(1|a)$. But then, if λ did not diverge, μ must, which would imply arbitrarily strong incentives in the limit, contradicting (IC). Note the key role here of $1/u'$ diverging. If it does not, then λ, which is the expectation of $1/u'$, does not diverge, and most of the analysis that follows falls apart.

Next, we show that even when $\bar{u}$ is large, contracts have a finite range of utilities. The idea is that if $v(1) - v(0)$ is too large, so are incentives, contradicting (IC), since c_a is bounded.

Lemma 3 *There exist $\hat{u} < \infty$, $0 < J < \infty$, such that for all $(a, \bar{u}, \theta)$ with $\bar{u} > \hat{u}$, $v(1) - v(0) \leq J$.*

Say that two functions g and $\tilde{g}$ of $(a, \bar{u}, \theta)$ are limit-ratio equal*, denoted by $g =_\ell \tilde{g}$, if the ratio of g and $\tilde{g}$ converges to one as $\bar{u}$ grows, and does so uniformly in (a, θ).*[9] Note that limit-ratio equality is transitive.

[8] In this section, we focus on the cost minimization problem, and Assumption 3 will play no role.

[9] That is, for each $\delta > 1$, there is $u^* < \infty$ such that for all $(a, \bar{u}, \theta)$ with $\bar{u} > u^*$, $1/\delta < g/\tilde{g} < \delta$.

Our next result shows that each of λ and μ has a simple limit form that is easily tied to economic fundamentals.

Proposition 1 *We have* $\lambda =_\ell \varphi'(\bar{u} + c(a,\theta))$ *and* $\mu =_\ell \varphi''(\bar{u} + c(a,\theta))c_a(a,\theta)/\int l^2 f$.

Thus, λ is in the limit simply the marginal cost of providing a util to the agent when his outside option plus disutility of effort is being covered. To see some intuition for the expression for μ, note that one adds incentives by removing utils at low outcomes, and replacing them at high outcomes. How much of this one has to do to relax (IC) by a given amount depends on how fast l is changing, for which the reciprocal of the Fisher Information $\int l^2 f$ turns out to be the right measure. The cost of moving utils from low to high outcomes in turn depends on how much φ' varies between low and high outcomes. This is determined by φ'', and by how fast the utility of the agent is already increasing in the outcome, which is related to c_a.

The shadow value μ will be increasing (decreasing) in $\bar{u}$ if φ' is convex (concave). Indeed, for the case of *HARA* utility, we have $\varphi''(v) = ((\gamma/1-\gamma)\, v)^{(1/\gamma)-2}/\alpha$. Thus, μ diverges for $\gamma \in (0, 1/2)$, is constant for $\gamma = 1/2$, and converges to zero for $\gamma \in (1/2, 1)$.

The first result can be equivalently stated as $C_{\bar{u}} =_\ell C_{\bar{u}}^{FI}$. This follows since $\lambda = C_{\bar{u}}$, and since the full-information cost of implementing effort is $C^{FI}(a, \bar{u}, \theta) = \varphi(\bar{u} + c(a, \bar{u}))$. There is no similar interpretation for μ, since there is no incentive constraint in the full-information problem.

Since it is both simple and illustrates ideas that will be central going forward, we prove Proposition 1 here. Recall that

$$\lambda = \int \varphi'(v(x)) f(x|a) dx,$$

which follows from $\varphi'(v(x)) = \lambda + \mu l(x|a)$ after multiplying both sides by f and integrating. This is intuitive, as φ' is the marginal cost of providing utils to the agent, and one way to relax the participation constraint (IR) without affecting incentives is to add a util at all outcomes. Note also that from (IR), we must have $v(0) < \bar{u} + c(a,\theta) < v(1)$. But then, it follows from Lemma 1 and Lemma 3 that uniformly across a and θ, for $\bar{u}$ large enough,

$$\frac{\lambda}{\varphi'(\bar{u} + c(a,\theta))} = \int \frac{\varphi'(v(x))}{\varphi'(\bar{u} + c(a,\theta))} f(x|a) dx$$

is as close to one as desired.

To see the proof for μ, start from $\varphi'(v(x)) = \lambda + \mu l(x|a)$, multiply both sides by f_a, integrate, and rearrange to arrive at

$$\mu = \frac{\int \varphi'(v(x)) f_a(x|a) dx}{\int (l(x|a))^2 f(x|a) dx}, \tag{6}$$

and so, integrating by parts,

$$\mu = \frac{\int \varphi'' v_x(-F_a)}{\int l^2 f} = \varphi''(\bar{u}+c)\frac{\int \frac{\varphi''}{\varphi''(\bar{u}+c)} v_x(-F_a)}{\int l^2 f}.$$

But, from Lemma 1 and Lemma 3, $\varphi''/\varphi''(\bar{u}+c)$ converges to one in $\bar{u}$ uniformly in x, a and θ over the relevant range of utilities, while $\int v_x(-F_a) = \int v f_a = c_a$, and we are done.

Proposition 1 lets us establish a first connection between C and C^{FI}.

Proposition 2 *For each* (a, θ),

$$\lim_{\bar{u}\to\infty} \frac{C(a,\bar{u},\theta)}{C^{FI}(a,\bar{u},\theta)} = 1.$$

This follows by L'Hospital's rule, since each of $C(a,\bar{u},\theta)$ and $C^{FI}(a,\bar{u},\theta)$ diverges in $\bar{u}$, and so

$$\lim_{\bar{u}\to\infty} \frac{C(a,\bar{u},\theta)}{\varphi(\bar{u}+c(a,\theta))} = \lim_{\bar{u}\to\infty} \frac{\lambda}{\varphi'(\bar{u}+c(a,\theta))} = 1,$$

where the second equality follows from Proposition 1.[10]

Thus, in a specific sense, the costs of implementing effort in the moral hazard setting and the cost in the full information setting converge.[11] Intuitively, as $\bar{u}$ diverges, the participation constraint looms large compared to the incentive constraint, and in the full information problem, only the participation constraint is relevant.

Remark 1 *Fix a and θ, and let $\tilde{v}$ be any contract with finite range that solves IC. Then, under our conditions, for large $\bar{u}$, a parallel upwards shift (in utils) of $\tilde{v}$ has costs within any given percentage of C^{FI}. See the Appendix for details.*[12] *This is perhaps one reason why simple contracts are observed in practice: when the outside option is large, then a linear contract or a step-bonus contract is pretty close to optimal. This also serves as another illustration that this notion of convergence is very weak. For example, convexity of costs in a need not hold in a "simple" contracts environment.*[13]

We close this section with an illustration of the central role of our utility assumptions.

Remark 2 *Recall that when $u = \log w$, $A/u' = 1$, and so Assumption 6 fails. It is a matter of direct calculation (see Section 6 below) that for the case of two outcomes, high and low, where the*

[10]While we will do so later, at this point we are not claiming this convergence is uniform in (a, θ).

[11]Note that the convergence established so far says nothing about the derivatives. For example, $(\bar{u}+\sqrt{a})/(\bar{u}+a^2) \to 1$ uniformly, with the numerator strictly convex in a, and the denominator stricly concave.

[12]We thank Dan Barron for suggesting this line of thought.

[13]Give the principal access to both step-bonus and linear contracts, fix θ, and vary a. The resulting cost function will be the *lower* envelope of the cost function for each type of contract separately, and hence will kink downwards where the principal transitions from one contract type to the other.

probability of a high outcome is a,

$$\frac{C}{C^{FI}} = \frac{C_{\bar{u}}}{C_{\bar{u}}^{FI}} = \frac{ae^{\bar{u}+c+(1-a)c_a} + (1-a)e^{\bar{u}+c-ac_a}}{e^{\bar{u}+c}} = ae^{(1-a)c_a} + (1-a)e^{-ac_a},$$

where the last expression is independent of $\bar{u}$, and, by Jensen's inequality, is strictly bigger than one for $a \notin \{0,1\}$. Thus, the conclusions of Propositions 1 and 2 fail to hold.

4.2 Digging into the Multipliers

To go further in our analysis, we need to more fully understand the multipliers λ and μ. For given $(a, \bar{u}, \theta)$, and associated λ and μ, let ξ be the parameterized density given by

$$\xi(x|a) = \frac{\rho'(\lambda + \mu l(x|a))f(x|a)}{\int \rho'(\lambda + \mu l(s|a))f(s|a)ds}.$$

Critical to our results is that as $\bar{u}$ diverges, ξ converges to f.

Lemma 4 *For all $\varepsilon > 0$, there is $\hat{u} < \infty$ such that for all $(a, \bar{u})$ with $\bar{u} > \hat{u}$,*

$$1 - \varepsilon < \frac{\xi(\cdot|a)}{f(\cdot|a)} < 1 + \varepsilon.$$

To see the proof, note that

$$\frac{\xi(x'|a)}{\xi(x|a)} = \frac{\rho'(\rho^{-1}(v(x')))}{\rho'(\rho^{-1}(v(x)))}\frac{f(x'|a)}{f(x|a)} = \frac{\varphi''(v(x))}{\varphi''(v(x'))}\frac{f(x'|a)}{f(x|a)},$$

where by Lemmas 1 and 3, $\varphi''(v(x))/\varphi''(v(x'))$ is arbitrarily close to 1 for $\bar{u}$ sufficiently large.

Our next result derives expressions for the behavior of λ and μ as a changes. The proof is standard, and hinges on the fact that (IC) and (IR) implicitly define λ and μ. To distinguish the use of ξ from f, we will denote by $var_\xi(\cdot)$ a variance computed using ξ, and similarly for $cov_\xi(\cdot,\cdot)$.

Lemma 5 *We have*

$$\lambda_a = -\mu_a \int l\xi - \mu \int l_a\xi \quad and \quad \mu_a = \frac{1}{var_\xi(l)}\left(\frac{1}{\int \rho' f}\left(c_{aa} - \int \rho f_{aa}\right) - \mu\, cov_\xi(l_a, l)\right). \tag{7}$$

We close this section with a main building block for our results to come. As $\bar{u}$ diverges, λ dominates μ, μ_a, and λ_a, while $\mu \int \rho' f$ and $\mu_a \int \rho' f$ remain finite.

Lemma 6 *As $\bar{u} \to \infty$, $\mu/\lambda \to 0$, $\mu_a/\lambda \to 0$, and $\lambda_a/\lambda \to 0$, uniformly in a and θ while $\mu \int \rho' f$ and $\mu_a \int \rho' f$ are bounded, also uniformly in a and θ.*

The proof that $\mu/\lambda \to 0$ is immediate from Proposition 1, since $\varphi''/\varphi' \to 0$. Intuitively, as $\bar{u}$ increases, the contract moves up, and since u is concave, this involves $1/u'$, the cost of providing an extra util to the agent, being higher everywhere. But then, λ, the shadow value of the participation constraint (IR), becomes increasingly high as well. But μ reflects the cost of providing a little more incentives, something one does by adding a little utility at high outcomes, while removing it at low outcomes. As $\bar{u}$ increases, it is increasingly painful to the principal to add utility at high outcomes, and this raises μ. But, it is increasingly beneficial for her to remove utility at low outcomes, providing an offsetting force so that μ grows more slowly.

The object μ_a is more involved, but again, there is the intuition that when a is raised, utility is both being added at some outcomes, and taken away at others, resulting in slower growth than for λ. The heart of bounding $\mu \int \rho' f$ uses the Mean Value Theorem to derive a bound for μ in terms of the minimum of $1/\rho'$ over the relevant range. But, since $\int \rho' f$ is less than the maximum of ρ' over the relevant range, and since from Lemma 1, $\max \rho' / \min \rho' \to 1$, we are done. The proof for $\mu_a \int \rho' f$ follows since the expression for μ_a that we derived in (7) consists of a term that once multiplied by $\int \rho' f$ is easily bounded, and a term involving $\mu \int \rho' f$.

4.3 Derivatives of C

Armed with the results on the limiting behavior of the Lagrange multipliers, we can now turn to the limiting behavior of the derivatives of C. Note first that, as is standard, by the Envelope Theorem applied to (1) we obtain

$$C_a = \mu \left(c_{aa} - \int v f_{aa} \right) + \int \varphi(v) f_a. \tag{8}$$

The term $c_{aa} - \int v f_{aa}$ measures how fast (IC) goes askew when a is raised, with c_{aa} being the effect through the marginal cost of effort to the agent, and $\int v f_{aa}$ reflecting that as effort changes, so does the impact of the agent's effort on the distribution of outputs. Finally, since $\varphi(v)$ is the cost to the principal of fulfilling the incentive scheme at any given output, $\int \varphi(v) f_a$ is the direct impact to the principal's cost when the agent exerts higher effort.

Proposition 3 *We have,*

$$\begin{aligned} C_{\bar{u}} &=_\ell C^{FI}_{\bar{u}} = \varphi'(\bar{u} + c), \\ C_a &=_\ell C^{FI}_a = \varphi'(\bar{u} + c) c_a, \text{ and} \\ C_\theta &=_\ell C^{FI}_\theta = \varphi'(\bar{u} + c) c_\theta. \end{aligned}$$

Since $C(0, \bar{u}, 0)/C^{FI}(0, \bar{u}, 0)$ converges to 1 by Proposition 2, and the ratios of the first derivatives converge uniformly by Proposition 3, $C =_\ell C^{FI}$ by the Fundamental Theorem of Calculus.

We now turn to the second derivatives of C. We begin with C_{aa}.

Lemma 7 *An expression for C_{aa} is given by*

$$C_{aa} = \lambda c_{aa} + \mu \left(c_{aaa} - \int v f_{aaa} - \int v_x l F_{aa} \right) + \left(\int \rho' f \right) \left(\mu_a^2 var_\xi(l) - \mu^2\, var_\xi(l_a) \right). \qquad (9)$$

The complexity of this expression–noting that λ and μ do not in general have a closed form expression–is presumably why the literature beyond Jewitt, Kadan, and Swinkels (2008) has had little to say on when C_{aa} is well-behaved. Taming all of the forces at play for given $\bar{u}$ is possible only under very stringent assumptions. It is thus useful to know that there is a strong force in the direction of simplicity as $\bar{u}$ grows.

The next result is fundamental both in characterizing the solution to the principal's optimal effort choice problem and in determining its comparative statics properties.

Proposition 4 *We have,*

$$\begin{aligned} C_{aa} &=_\ell C_{aa}^{FI} =_\ell \varphi'(\bar{u}+c)c_{aa} > 0, \\ C_{a\bar{u}} &=_\ell C_{a\bar{u}}^{FI} = \varphi''(\bar{u}+c)c_a > 0, \text{ and} \\ C_{a\theta} &=_\ell C_{a\theta}^{FI} =_\ell \varphi'(\bar{u}+c)c_{a\theta} < 0. \end{aligned}$$

Note that the limit expression for C_{aa}^{FI} is simpler than $(\varphi(\bar{u}+c))_{aa}$, since $\varphi''(\bar{u}+c)c_a$ disappears relative to $\varphi'(\bar{u}+c)c_{aa}$, and similarly for $C_{a\theta}^{FI}$.[14]

5 Economic Applications

In this section we present our second main contribution. We first show that the limiting structure of C affords a comprehensive solution to the profit maximization step of the principal-agent problem, with intuitive comparative statics properties. Then we apply the results beyond the single principal-agent problem by embedding it in a market setting where heteregeneous principals match with heterogeneous agents.

5.1 Properties of Optimal Effort

When $\bar{u}$ is large enough, C is strictly convex in a and hence the first-order condition to the profit maximization problem characterizes the optimal action she implements. Moreover, the limiting behavior of the cross partial derivatives yields clear-cut comparative statics of the solution.

[14]We focus on these second derivatives as they are the ones that are relevant for our comparative statics. For completeness, Appendix A.12 shows that $C_{\bar{u}\bar{u}} =_\ell \varphi''(\bar{u}+c)$, that $C_{\theta\theta} =_\ell \varphi'(\bar{u}+c)c_{\theta\theta}$ as long as $c_{\theta\theta}$ is bounded away from 0 (as for example $c = (1-k\theta^2)a^2/2$), and that $C_{\bar{u}\theta} =_\ell \varphi''(\bar{u}+c)c_\theta$.

Proposition 5 *There exists $u^* < \infty$ such that for all $\bar{u} > u^*$, $C_{aa} > 0$, $C_{a\bar{u}} > 0$, and $C_{a\theta} < 0$. For all such $\bar{u}$, there is $\tau^* < \infty$ such that for any $\tau > \tau^*$, the principal strictly wants to hire the agent, with the optimal effort induced by the principal unique, differentiable in $(\bar{u}, \theta)$, strictly decreasing in $\bar{u}$, and strictly increasing in θ.*

That each second derivative of C eventually has the right sign is immediate from Proposition 4. But then, for given $(\bar{u}, \theta)$, $B - C$ is strictly concave in a, and uniqueness follows. And, since $B_{aa} - C_{aa} < 0$, the Implicit Function Theorem yields the remaining claims conditional on the principal wishing to hire the agent. Finally, fix any $\bar{u}$ such C has the appropriate comparative statics. Then, for any sufficiently large τ, $B(1, \tau) - C(1, \bar{u}, \theta) \geq 0$ for all θ, and so the principal will indeed wish to employ the agent. Depending on the behavior of B_a in τ, optimal effort may be small or large. For example, if τ is the price of output, then as τ diverges, so does B_a and so, for any given a_0 and $\bar{u}$, τ can be chosen large enough that effort is at least a_0. In other economically reasonable settings, B may, for example, move parallel to itself.

We emphasize that continuity of the optimal effort in the parameters has real economic content–small policy changes, for example, will not lead to jumps in behavior. The comparative statics of optimal effort with respect to $\bar{u}$ are notoriously difficult to obtain in general, and have applications beyond the static case.[15]

An Ability-Dependent Outside Option. Proposition 5 does not settle the question of whether a more capable agent will, in equilibrium, exert more effort. This is because the outside option of the agent will typically increase with θ, providing a countervailing force in the direction of less effort. To analyze this, assume the outside option of the agent is $\bar{u}(\theta)$, where with some abuse of notation $\bar{u}$ is now some differentiable function with finite slope. For example, if the agent has available to him some fixed set of outside options (that does not vary with ability) then $\bar{u}'(\theta) = -c_\theta(\alpha(\theta), \theta)$, where $\alpha(\theta)$ is the optimal effort in the chosen outside option (this follows from the Envelope Theorem) and where we note that this is bounded. (In the next section, we consider a contrasting setting where utility is endogenously determined by principals who can observe ability competing for agents.) Then,

$$\frac{d}{d\theta} C_a(a, \bar{u}(\theta), \theta) = C_{a\bar{u}}\bar{u}' + C_{a\theta} =_\ell \varphi''(\bar{u} + c)c_a\bar{u}' + \varphi'(\bar{u} + c)c_{a\theta} =_\ell \varphi'(\bar{u} + c)c_{a\theta} < 0$$

where the second limit ratio equality uses that $\varphi''/\varphi' \to 0$. Thus, for any given $\bar{u}'$ finite, it is indeed the case that if the stakes are high, then the principal chooses to induce more effort from more capable agents.

The Effect of Wealth. As a small extension to our model, assume that instead of agents being indexed by ability, they differ in their initial wealth, denoted by ω, as in Thiele and Wambach

[15] See Spear and Srivastava (1987) for the key role that such a result plays in the repeated version of the principal-agent problem with moral hazard, where $\bar{u}$ is replaced by the agent's promised utility.

(1999). That is, the agent's utility if the observed outcome is x and he gets paid $\pi(x)$ is $u(\omega+\pi(x))$, and his reservation utility is $\bar{u}(\omega)$, which depends on his wealth ω (and also on the financial and other aspects of his best alternative to this relationship), so that $\bar{u}_\omega > 0$. We want to know how the optimal effort level induced by the principal changes with ω.

It is straightforward to show that $C_\omega = -1 + \lambda \bar{u}_\omega$, where C_ω can be either positive or negative (so the principal may prefer poorer or richer agents) depending on the details of how $\bar{u}$ is modelled (see Thiele and Wambach (1999), Kadan and Swinkels (2013), and Chade and Vera de Serio (2014)). Nonetheless, if $\bar{u}$ is sufficiently high (either because the agent is rich, or has an attractive alternative job), then unambiguously it is more costly to induce effort from a richer agent, and hence optimal effort strictly decreases in ω. The point is that $C_{a\omega} = \lambda_a \bar{u}_\omega$, where by Proposition 4, $\lambda_a = C_{\bar{u}a} > 0$ when $\bar{u}$ is sufficiently large.

5.2 Matching Principals and Agents

As a final application of our high-stakes principal-agent problem with moral hazard, we consider a competitive market. A unit-measure continuum of heterogeneous principals differ in their technology parameter τ indexing B. We strengthen Assumption 3 by assuming that $B_{a\tau} > 0$. We assume that τ is distributed according to $\mathcal{C}^1$ *cdf* Q which has support equal to some subinterval $[\underline{\tau}, \bar{\tau}]$ of $[0, \infty)$. The principals match pairwise with a unit-measure continuum of heterogenous agents who differ in ability $\theta \in [0, 1]$, where θ is distributed according to $\mathcal{C}^1$ *cdf* H. Agents who choose not to participate in the market receive outside option $\bar{u}$. Ability is observable to the principals. Our market clearing condition is that no agent and principal strictly prefer each other to their current partner. An important question is to understand the sorting pattern that emerges.

Matching problems of this sort have many applications, as for example the matching of *CEO*s and firms. But due to the moral hazard problem, it is difficult to obtain results of any generality. Most available results consist of highly parameterized cases.[16] The main stumbling block is that the complementarity properties of C are hard to sign. We will show that when the stakes are high (when $\bar{u}$ is sufficiently large), a definite answer emerges under our usual primitives.

Let $a^*(\tau, u_0, \theta) = \arg\max_a B(a, \tau) - C(a, u_0, \theta)$ be the optimal effort to implement for the principal of type τ given that she is hiring the agent of type θ at utility u_0. The following function is crucial for determining the sorting pattern:

$$\Pi(\tau, u_0, \theta) = B(a^*(\tau, u_0, \theta), \tau) - C(a^*(\tau, u_0, \theta), u_0, \theta). \tag{10}$$

For any given τ and θ, $\Pi(\tau, u_0, \theta)$ is the maximum surplus the principal of type τ can earn by hiring a worker of type θ and given him utility u_0. Hence, $\Pi(\tau, \cdot, \theta)$ describes the Pareto frontier

[16] See, for instance, Legros and Newman (2007), Section 5.2, and Serfes (2005).

available to that pair.[17]

If C were quasi-linear in u_0, then positive sorting would emerge if (and only if) Π were supermodular in (τ, θ). But, since utility is imperfectly transferable, the relevant condition for sorting is the one in Legros and Newman (2007). Intuitively, their condition requires that a higher type of principal has a higher willingness to pay for a higher versus a lower type of agent than a lower type of principal does. If this is the case, then principals with higher types outbid those with lower types in the competition for higher types of agents, and positive sorting ensues. The differential version of their condition (see Chade, Eeckhout, and Smith (2017)) is simply the Spence-Mirrlees condition that $-\Pi_\theta/\Pi_{u_0}$, which is the marginal rate of substitution between θ and u_0 for a principal with type τ, is increasing in τ for all (τ, u_0, θ) or, equivalently

$$\frac{\Pi_{\tau\theta}}{\Pi_\theta} \geq \frac{\Pi_{\tau u_0}}{\Pi_{u_0}}. \tag{11}$$

This says that, in percentage terms, the value to the principal of a smarter agent, holding fixed the agent's surplus, goes up faster with τ than does the cost of transfering utility. Note in passing that in the quasi-linear case $\Pi_{\tau u_0} = 0$, and this reduces to supermodularity of Π in (τ, θ).

Using (10) and the Envelope Theorem, (11) is equivalent to

$$\frac{C_{a\theta}}{C_\theta} \geq \frac{C_{au_0}}{C_{u_0}}. \tag{12}$$

where we have cancelled $a^*_\tau = -B_{a\tau}/(B_{aa} - C_{aa})$, which is strictly positive since B is strictly supermodular and C is strictly convex for large enough $\bar{u}$, from each side.

Since without moral hazard, $C^{FI} = \varphi(u_0 + c(a, \theta))$, the profit of the principal reduces under full information to

$$\Pi^{FI}(\tau, u_0, \theta) = B(\hat{a}(\tau, u_0, \theta), \tau) - \varphi(u_0 + c(\hat{a}(\tau, u_0, \theta), \theta)),$$

where $\hat{a}$ is the optimal action function in the observable case. Then (11) reduces to $\varphi' c_{a\theta} \leq 0$, which trivially holds. With moral hazard, however, even when $\bar{u}$ is large enough, we have $C_{a\theta} < 0$, $C_\theta < 0$, and $C_{au_0} > 0$, and $C_{u_0} > 0$, and so both sides of (12) are positive. The issue is that a principal with higher τ is more benefited directly by a smarter agent, but more harmed by that agent's higher reservation utility. The direction of sorting is driven by the balance of these two forces. As a result, the unambiguous sorting pattern and comparative statics of the action in the next proposition are not obvious.

[17] Clearly, in a competitive equilibrium the utility that an agent receives depends on his type, and is at least $\bar{u}$. But to ensure principals of higher types hire agents of higher types (positive sorting), we will provide a sufficient condition that must hold not just for the equilibrium utility that agents receive but for any u_0.

Proposition 6 *For $\bar{u}$ sufficiently large, there is $\underline{\tau}$ sufficiently large such that in any equilibrium, all principals and agents are matched, with matching exhibiting positive sorting, and a^* strictly increasing in match quality.*

So for a large class of principal-agent matching problems, as long as we focus on large enough $\bar{u}$ (which lifts the payoff of the lowest type and thus that of all types of agents), the equilibrium/optimal sorting pattern is to match better principals with higher-ability agents.[18,19] Moreover, principal-agent pairs with better characteristics implement strictly higher actions.

Remark 3 *In the case discussed in Section 5.1, the utility of the agent rises at rate $-c_\theta$. Here, because ability is observable, more capable agents receive better opportunities (see Appendix A.10 for details), and utility rises at rate $-\Pi_\theta/\Pi_{u_0} = -C_\theta/C_{u_0} = -(c_\theta + (\mu/\lambda)\, c_{a\theta})$. Since $-C_\theta/C_{u_0} =_\ell -c_\theta$, the models make similar predictions when $\bar{u}$ is large.*

6 How Big is Big Enough?

Our results show that for $\bar{u}$ large enough, all of the relevant first and second derivatives take on simple and interpretable limit forms with appropriate signs. Even absent a rate of convergence result, knowing that many terms are falling away as $\bar{u}$ grows tells us a lot about the structure of the problem. But, it is also clearly interesting to know how big $\bar{u}$ needs to be for the relevant comparative statics to have the intuitive signs. This section begins the exploration of this important question in two tractable classes of problems.

6.1 Two Outcomes

Consider the case of two outcomes, where for simplicity, the probability of a high output is $a \in [0,1]$.[20] We assume *HARA* utility. It is a straightfoward exercise (See Chade and Swinkels (2019) for a derivation) that $C = (1-a)\varphi(v_l) + a\varphi(v_h)$, where $v_l = \bar{u} + c(a,\theta) - ac_a(a,\theta)$ and $v_h = \bar{u} + c(a,\theta) + (1-a)c_a(a,\theta)$. Differentiating C with respect to a yields

$$C_a = \varphi(v_h) - \varphi(v_l) + (1-a)\varphi'(v_l)v_{l,a} + a\varphi'(v_h)v_{h,a}$$

where $v_{l,a} = -ac_{aa}$ and $v_{h,a} = (1-a)c_{aa}$. To simplify what follows, let us assume throughout this subsection that $c(a,\theta) = (1-k\theta)a^2/2$.

[18] As before, for any given $\bar{u}$, for high enough $\underline{\tau}$, the principal of any given type above $\underline{\tau}$ can strictly profitably employ the agent of type 0, and hence, *a fortiori*, to employ their equilibrium match.

[19] In our model, the number of principals is equal to the number of agents. With more agents than principals, the lowest employed agent still receives $\bar{u}$, and positive sorting ensues. With more principals than agents, the utility of the lowest type of the agent must yield negative profits to excluded principals. When $\underline{\tau}$ is sufficiently high, this utility will be at least $\bar{u}$, and our results again go through.

[20] Note that the likelihood ratio is not bounded as a approaches its endpoints. But, since we will use explicit expressions for the optimal contract, this will not matter. Alternatively, take $a \in [\varepsilon, 1-\varepsilon]$ for some small ε.

Strict Convexity of C. Appendix A.11 shows that for $C_{aa} \geq 0$ for all a and θ, it is necessary and sufficient that[21]

$$\bar{u} \geq \frac{1}{2} + \frac{1}{2^{\frac{\gamma}{1-\gamma}} - 1}.$$

As $\gamma \to 0$ (i.e., as utility approaches the log case), this bound explodes. It follows that for $u(w) = \log w$, C will always fail to be convex for a sufficiently close to one and θ close to zero.

Consider the case $u = \sqrt{w}$ (that is, $\alpha = \gamma = 1/2$, and $\beta = 0$). Our condition reduces to $\bar{u} \geq 3/2$. For scale, $c(1,0) - c(0,0) = 1/2$, and the utility of an agent who is employed but has no income and exerts no effort is zero. In income terms, employing the agent but asking for no effort costs $\bar{u}^2$, while to employ the agent of type 0 and ask for $a = 1$ costs $(\bar{u} + 1/2)^2$, for a ratio of $\bar{u}^2/(\bar{u} + 1/2)^2$. For $\bar{u} \geq 3/2$, this ratio is at least $(3/2)^2/(2)^2 = 9/16$. That is, the outside option needs to be high enough that at full effort "retention" is at least 9/16 of the total cost of employing the agent.

For another sense of the size of $\bar{u}$, note that if $B(1,\tau) \geq C(1,\bar{u},0)$, then the principal certainly wants to employ the agent. Thus, if $\bar{u} = 3/2$, we need $B(1,\tau) \geq 4$, with the required ratio $B(1,\tau)/C(0,\bar{u},0)$ starting at 16/9 and falling with $\bar{u}$. Thus, neither the agent's outside option nor the surplus of the relationship needs to be extreme for the problem to be well-behaved.

Supermodularity of C in $(a, \bar{u})$. A similar exercise (see Appendix A.11) shows that to ensure that $C_{a\bar{u}} \geq 0$, it is enough that $\bar{u} \geq 1/2$. Similar interpretations as the ones given for C_{aa} apply.

Submodularity of C in (a, θ). Finally, Appendix A.11 shows that $C_{a\theta} \leq 0$ for all $\bar{u}$. Indeed, this holds for any strictly concave u, and for any c satisfying the additional condition that $c_\theta(0,\theta) \equiv 0$.

6.2 Square Root Utility

Let us consider the case $u = \sqrt{w}$, and restrict ourselves to the question of convexity of C with respect to a. Then, Jewitt, Kadan, and Swinkels (2008), p. 69, show that

$$C_a = (\bar{u} + c)\, c_a + c_a c_{aa} \hat{I} + \frac{c_a^2 \hat{I}_a}{2},$$

where $\hat{I}(a) = 1/\int l^2 f$ is the reciprocal of the Fisher Information that output carries about a. The approach of Jewitt, Kadan, and Swinkels (2008) is to assume that $c_{aaa} \geq 0$, and then tame the information term, which comes down in the square-root example to assuming that $\hat{I}$ is increasing and convex. Note that both assumptions on $\hat{I}$ fail for the two outcome case above, since $\hat{I} = a(1-a)$, which is neither increasing nor convex.

[21] We thank Henrique Castro-Pires for helping us to substantially improve this bound.

As a complementary approach, note that

$$C_{aa} =_s c_a^2 + (\bar{u} + c)\, c_{aa} + c_{aa}^2 \hat{I} + c_a c_{aaa} \hat{I} + 2 c_a c_{aa} \hat{I}_a + \frac{c_a^2 \hat{I}_{aa}}{2}$$

and so for $C_{aa} \geq 0$, it suffices that

$$\bar{u} \geq -c - \frac{c_a^2}{c_{aa}} - c_{aa} \hat{I} - \frac{c_a c_{aaa}}{c_{aa}} \hat{I} - 2 c_a \hat{I}_a - \frac{c_a^2 \hat{I}_{aa}}{2 c_{aa}},$$

where each term on the right side is a tractable function of primitives. Hence, the required $\bar{u}$ will be moderate as long as $\hat{I}_a$ and $\hat{I}_{aa}$ are not extreme (for example, the two outcome case).

Remark 4 *As a final note, in the full information case, note that $C_a^{FI} = (\bar{u} + c)\, c_a$. So, the difference between C_a and C_a^{FI} does not depend on $\bar{u}$ and has the "wrong" sign any time information improves sufficiently quickly as a increases, as measured by the behavior of the Fisher Information. Thus, independent of $\bar{u}$, a high-stakes agent may be working harder or less hard at the moral-hazard optimum than at the full information optimum.*

7 Concluding Remarks

The principal's cost function C of inducing effort in the moral-hazard problem is inherently extremely complicated. Without strong assumptions, it is not even known if the principal's payoff function is concave in the effort she chooses to implement. And, there are a myriad of forces at play as one varies parameters like the agent's ability, outside option, or wealth.

In this paper we show that C drastically simplifies as one increases the outside option of the agent. The shadow values of the incentive and participation constraints have simple and economically interpretable limits, as does each relevant first and second derivative of C. Further, these limiting objects have intuitive signs. Thus, when the agent's outside option is large enough, the principal's optimal choice of action problem is a concave maximization problem with a unique and continuous solution and with unambiguous and economically interpretable comparative statics properties. We extend the model to a matching setting, where a sufficiently large reservation utility leads to positive sorting and actions that increase in the quality of the match.

A Appendix: Proofs

A.1 Proof of Lemma 1

Since φ is strictly increasing,

$$\frac{\varphi'(u_1)}{\varphi'(u_2)} = \exp(\log\varphi'(u_1) - \log\varphi'(u_2)) = \exp[(u_1 - u_2)(\log\varphi'(\hat{u}))_{\hat{u}}]$$

for some $\hat{u} \in [\min(u_1, u_2), \max(u_1, u_2)]$, using the Mean Value Theorem. But, since $\hat{u} > u^*$, $|u_1 - u_2| \leq \Delta$, and $\exp(0) = 1$, to show (4), it is enough that $(\log\varphi'(t))_t = \varphi''/\varphi' \to 0$,which is immediate from Assumption 6.

Similarly

$$\frac{\varphi''(u_1)}{\varphi''(u_2)} = \exp[(u_1 - u_2)(\log\varphi''(\hat{u}))_{\hat{u}}]$$

for some $\hat{u} \in [\min(u_1, u_2), \max(u_1, u_2)]$, and so it is enough that $\varphi'''/\varphi'' \to 0$, which follows by Assumption 7. □

A.2 Proof of Lemma 2

Choose any sequence $(a^k, \bar{u}^k)$ with $\bar{u}^k \to \infty$, and let λ^k, μ^k be the associated multipliers. Assume by way of a contradiction that $\liminf \lambda^k = \hat{\lambda}$ for some $\hat{\lambda} < \infty$. Then, since λ is bounded below by zero, we can choose a relabelled subsequence along which $\lambda^k \to \hat{\lambda}$, and, since a^k comes from a compact set, we can assume as well that $a^k \to \hat{a}$. Recall that $v(x) = \rho(\lambda + \mu l(x|a))$. Thus, by (IR), we have that $\rho(\lambda^k + \mu^k l(1|a^k)) \geq \bar{u}^k \to \infty$. But, $\lambda^k \to \hat{\lambda}$, and so, by Assumption 5, $\rho(\lambda^k)$ is bounded. It follows that $\mu^k \to \infty$. Let $l(\hat{x}|\hat{a}) = 0$. Then, for $x < \hat{x}$, $\lim \rho(\lambda^k + \mu^k l) \leq \rho(\hat{\lambda})$, while for $x > \hat{x}$, $\lim \rho(\lambda^k + \mu^k l) = \infty$. It follows that $\int \rho(\lambda^k + \mu^k l) f_a(x|\hat{a})$ diverges, contradicting (IC), since $c_a(a, \theta) \leq c_a(\bar{a}, \underline{\theta}) < \infty$. □

A.3 Proof of Lemma 3

Our goal is to find a finite upper bound J on $v(1) - v(0)$ that holds for $\bar{u}$ sufficiently large and uniformly across a and θ. In particular, we will construct a J such that if $\bar{u}$ is sufficiently large, and if $v(1) - v(0) > J$, then $\int v f_a > c_a$, and so (IC) is slack, a contradiction. Thus, J bounds $v(1) - v(0)$ as needed.

For each a, let $x^*(a)$ solve $l(x^*(a)|a) = 0$. Assume $v(1) - v(0) > J$. There are two cases, one where $v(x^*(a)) - v(0) > J/2$ and the other where $v(1) - v(x^*(a)) > J/2$. For the first case, let

$$L_1 = \min_{a \in [0,1]} \int_0^{x^*(a)} \frac{l_x(x|a)}{\int_0^{x^*(a)} l_x(s|a) ds} (-F_a(x|a)) dx.$$

This integral is an expectation of $-F_a$ with respect to a non-degenerate measure (with kernel l_x that is continuous in a), and so is strictly positive and continuous in a, and so $L_1 > 0$. We will show that when $\bar{u}$ is large, any contract that rises by at least $J/2$ to the left of $x^*(a)$ must generate incentives strictly higher than $JL_1/4$.

Similarly, letting

$$L_2 = \min_{a\in[0,1]} \int_{x^*(a)}^{1} \frac{l_x(x|a)}{\int_{x^*(a)}^{1} l_x(s|a)ds} (-F_a(x|a))dx > 0,$$

we will show that any contract that rises by at least $J/2$ to the right of $x^*(a)$ generates incentives strictly higher than $JL_2/4$. But then, if we choose $J = 4c_a(\bar{a}, \underline{\theta}) / \min(L_1, L_2)$, we have the required object: for large $\bar{u}$, any contract that rises by at least J generates incentives strictly higher than

$$\frac{J}{4} \min(L_1, L_2) = c_a(\bar{a}, \underline{\theta}) \geq c(a, \theta),$$

and so any such contract would leave the incentive constraint (IC) slack. Thus any such contract must in fact have $v(1) - v(0) \leq J$.

So, let us show that any contract that rises by at least J must generate incentives strictly higher than $J \min(L_1, L_2)/4$. Using Lemmas 1 and 2, choose $\hat{u}$ such that for all $\bar{u} > \hat{u}$,

(*i*) $\rho'(\rho^{-1}(z'))/\rho'(\rho^{-1}(z)) \leq 2$ for all z and z' in $[\bar{u} - J/2, \bar{u} + J/2]$, and

(*ii*) for each $(a, \bar{u})$ with $\bar{u} > \hat{u}$, $\rho(\lambda) > \tilde{u}$ for the associated λ.

Fix $(a, \bar{u})$ with $\bar{u} > \hat{u}$, let (λ, μ) be the associated multipliers, and v the associated contract. If $v(1) - v(0) > J$, then either $v(x^*(a)) - v(0) > J/2$, or $v(1) - v(x^*(a)) > J/2$. Consider the first case. Let $\hat{x}$ be the point at which $v(\hat{x}) = v(x^*(a)) - J/2$. Then,

$$\int v f_a = \int v_x(-F_a) > \int_{\hat{x}}^{x^*(a)} v_x(-F_a) = \int_{\hat{x}}^{x^*(a)} \mu\rho' l_x(-F_a)$$

where the first equality is by integration by parts, the inequality follows since the integrand is positive using that $-F_a > 0$ on $(0, 1)$ by first-order stochastic dominance (*FOSD*), while $v_x > 0$ by strict *MLRP*, and the second equality follows since $v_x = \mu\rho' l_x$. But then, since

$$\frac{J}{2} = v(x^*(a)) - v(\hat{x}) = \int_{\hat{x}}^{x^*(a)} v_x = \int_{\hat{x}}^{x^*(a)} \mu\rho' l_x,$$

we have

$$\int v f_a > \int_{\hat{x}}^{x^*(a)} \mu\rho' l_x(-F_a) = \frac{J}{2} \int_{\hat{x}}^{x^*(a)} \frac{\rho' l_x}{\int_{\hat{x}}^{x^*(a)} \rho' l_x} (-F_a) \geq \frac{J}{4} \int_{\hat{x}}^{x^*(a)} \frac{l_x}{\int_{\hat{x}}^{x^*(a)} l_x} (-F_a),$$

and the second inequality follows since by definition of $\hat{u}$, ρ' varies by a factor of at most 2 on the relevant interval.

Note next that the measure $l_x / \int_{\hat{x}}^{x^*(a)} l_x$ moves in the sense of *FOSD* in $\hat{x}$, and that $(-F_a)$ is increasing on $[0, x^*(a)]$. Hence,

$$\int v f_a > \frac{J}{4} \min_a \int_0^{x^*(a)} \frac{l_x}{\int_0^{x^*(a)} l_x} (-F_a) = \frac{J}{4} L_1 = c_a(\bar{a}, \underline{\theta}) \geq c_a(a, \theta), \tag{13}$$

where the last inequality follows by the assumed properties of c. But then, the incentive constraint (IC) is slack at v, a contradiction.

The case where $v(1) - v(x^*(a)) > J/2$ is similar, where

$$\int v_x(-F_a) > \int_{x^*(a)}^{\hat{x}} v_x(-F_a) \geq \frac{J}{4} \min_a \int_{x^*(a)}^1 \frac{l_x}{\int_{x^*(a)}^1 l_x} (-F_a) = \frac{J}{4} L_2,$$

and we are done. □

A.4 Proof of Remark 1

Fix a and θ, and let $\tilde{v}$ be any contract that solves $\int \tilde{v} f_a = c_a$. Let $\tilde{u} = \int \tilde{v} f - c$, and let $\tilde{\Delta} \equiv \max_{[0,1]} \tilde{v}(x) - \min_{[0,1]} \tilde{v}(x) < \infty$. Then, by L'Hospital's rule,

$$\lim_{\bar{u} \to \infty} \frac{\int \varphi(\bar{u} - \tilde{u} + \tilde{v}(x)) f(x|a) dx}{\varphi(\bar{u} + c(a, \theta))} = \lim_{\bar{u} \to \infty} \frac{\int \varphi'(\bar{u} - \tilde{u} + \tilde{v}(x)) f(x|a) dx}{\varphi'(\bar{u} + c(a, \theta))} = 1,$$

where the second equality is by Lemma 1, since the contract $\bar{u} - \tilde{u} + \tilde{v}(\cdot)$ has expected utitility $\bar{u} + c(a, \theta)$, and hence has range in $[\bar{u} + c(a, \theta) - \tilde{\Delta}, \bar{u} + c(a, \theta) + \tilde{\Delta}]$. □

A.5 Proof of Lemma 5

Note that λ and μ are implicitly defined by

$$\int \rho(\lambda + \mu l) f_a = c_a, \text{ and } \int \rho(\lambda + \mu l) f = \bar{u} + c,$$

and so,

$$\int (\lambda_a + \mu_a l + \mu l_a) \rho' f_a + \int \rho f_{aa} = c_{aa}, \text{ and } \int (\lambda_a + \mu_a l + \mu l_a) \rho' f + \int \rho f_a = c_a.$$

Using (IC) to eliminate $\int \rho f_a$ and c_a in the second equation, $\lambda_a = -\mu_a \int l\xi - \mu \int l_a \xi$. But then, rearranging the first equation, and substituting for λ_a,

$$\int \left(-\int (\mu_a l + \mu l_a) \xi + \mu_a l + \mu l_a \right) l\xi = \frac{1}{\int \rho' f} \left(c_{aa} - \int \rho f_{aa} \right),$$

which, with some manipulation, yields (7). □

A.6 Proof of Lemma 6

The proof that $\mu/\lambda \to 0$ is in the text. By (6), $\mu = \hat{I}(a)\int \varphi' f_a$, where $\hat{I} = 1/I$. But then

$$\mu_a = \hat{I}_a(a)\int \varphi' f_a + \hat{I}(a)\int \varphi'' v_a f_a + \hat{I}(a)\int \varphi' f_{aa},$$

and so

$$\begin{aligned}\frac{\mu_a}{\lambda} &=_\ell \frac{\hat{I}_a(a)\int \varphi' f_a + \hat{I}(a)\int \varphi'' v_a f_a + \hat{I}(a)\int \varphi' f_{aa}}{\varphi'\left(\bar{u}+c\right)} \\ &= \hat{I}_a(a)\int \frac{\varphi'}{\varphi'(\bar{u}+c)} f_a + \hat{I}(a)\frac{\varphi''(\bar{u}+c)}{\varphi'(\bar{u}+c)}\int \frac{\varphi''}{\varphi''(\bar{u}+c)} v_a f_a + \hat{I}(a)\int \frac{\varphi'}{\varphi'(\bar{u}+c)} f_{aa}.\end{aligned}$$

Since $\varphi'/\varphi'(\bar{u}+c) \to 1$, by Lemmas 1 and 3, while $\int f_a = \int f_{aa} = 0$, and $\hat{I}(a)$ is bounded, the first and third terms have limit zero. Consider the second term. by By Assumption 6, $\varphi''\left(\bar{u}+c\right)/\varphi'(\bar{u}+c) \to 0$, while $\varphi''/\varphi''(\bar{u}+c) \to 1$ by Assumption 7 and Lemma 3. Thus, to show that the second term has limit zero, it is enough to show that $\int v_a f_a$ is finite. But, differentiating (IC) with respect to a yields $\int v_a f_a = c_{aa} - \int v f_{aa}$, and in turn,

$$\left|\int v f_{aa}\right| = \left|\int v_x F_{aa}\right| \le J \max_{a,x} |F_{aa}| < \infty, \tag{14}$$

and we are done. Since $\mu/\lambda \to 0$ and $\mu_a/\lambda \to 0$, it follows from (7) that $\lambda_a/\lambda \to 0$.

Let us show next that $\lim \mu \int \rho' f$ is finite. Note that for any given $\bar{u}$ and a,

$$\varphi'(v(1)) - \varphi'(v(0)) = \mu(l(1|a) - l(0|a)),$$

and hence, by the Mean Value Theorem, for some $\hat{u} \in [v(0), v(1)]$,

$$\varphi''(\hat{u})(v(1) - v(0)) = \mu(l(1|a) - l(0|a)).$$

Thus,

$$\mu = \frac{v(1)-v(0)}{l(1|a)-l(0|a)}\varphi''\left(\hat{u}\right) \le \frac{v(1)-v(0)}{\min_a(l(1|a)-l(0|a))} \max_{\tilde{u}\in[v(0),v(1)]} \varphi''\left(\tilde{u}\right).$$

But, also,

$$\int \rho' f = \int_0^1 \frac{1}{\varphi''\left(v\left(x\right)\right)} f(x|a)dx \le \frac{1}{\min_{\tilde{u}\in[v(0),v(1)]}\varphi''\left(\tilde{u}\right)},$$

and so

$$\mu\int \rho' f \le \frac{v(1)-v(0)}{\min_a(l(1|a)-l(0|a))}\frac{\max_{\tilde{u}\in[v(0),v(1)]}\varphi''\left(\tilde{u}\right)}{\min_{\tilde{u}\in[v(0),v(1)]}\varphi''\left(\tilde{u}\right)}.$$

But, by Lemma 3 for $\hat{u}$ sufficiently large, $v(1) - v(0) \leq J$, and so, since $\min_a(l(1|a) - l(0|a)) > 0$ by assumption, the first fraction on the rhs is finite. Similarly by Lemma 1 and Lemma 3, the second fraction on the rhs converges to one, and we are done.

Consider finally $\mu_a \int \rho' f$. Note that

$$\mu_a \int \rho' f = \frac{1}{var_\xi(l)} \left(c_{aa} - \int v f_{aa} \right) - \mu \left(\int \rho' f \right) \frac{cov_\xi(l_a, l)}{var_\xi(l)},$$

where $\int v f_{aa} = \int v_x(-F_{aa})$ is bounded uniformly in a as above, where $var_\xi(l) \to var_f(l) > 0$, where $cov_\xi(l_a, l)/var_\xi(l)$ is bounded from above as before, and where $\mu \int \rho' f$ is bounded from above, and so we are done. □

A.7 Proof of Proposition 3

The equalities for $C_{\bar{u}}$ are from Proposition 1. Let us turn to C_a. By (8),

$$\begin{aligned}\frac{C_a}{C_a^{FI}} &= \frac{\mu(c_{aa} - \int v f_{aa}) + \int \varphi(v) f_a}{\varphi'(\bar{u}+c)c_a} \\ &= \frac{\mu}{\varphi'(\bar{u}+c)} \frac{c_{aa} - \int v f_{aa}}{c_a} + \frac{\int \frac{\varphi'(v)}{\varphi'(\bar{u}+c)} v_x(-F_a)}{c_a}.\end{aligned}$$

Note that the second term converges to $\int v_x(-F_a)/c_a = 1$, since $\varphi'/\varphi'(\bar{u}+c)$ converges to 1 uniformly by Lemmas 1 and 3. So, it is enough to show that the first term converges to zero. But, we can start from (6), integrate by parts, and multiply and divide by $\varphi''(\bar{u}+c)$, and then take a maximum over $\varphi''/\varphi''(\bar{u}+c)$ to arrive at

$$\begin{aligned}\mu &= \hat{I}(a) \int \varphi' f_a = \hat{I}(a) \int \varphi'' v_x(-F_a) = \hat{I}(a)\varphi''(\bar{u}+c) \int \frac{\varphi''}{\varphi''(\bar{u}+c)} v_x(-F_a) \\ &\leq \hat{I}(a)\varphi''(\bar{u}+c) \left(\max_{\hat{u} \in [v(0), v(1)]} \frac{\varphi''(\hat{u})}{\varphi''(\bar{u}+c)} \right) \int v_x(-F_a) \\ &\leq 2\hat{I}(a)\varphi''(\bar{u}+c)c_a,\end{aligned}$$

where the last inequality is valid for all $\bar{u}$ sufficiently large by Lemmas 1 and 3, and uses that $c_a = \int v f_a = \int v_x(-F_a)$. But then, the first (positive) term is bounded above by

$$\frac{\varphi''(\bar{u}+c)}{\varphi'(\bar{u}+c)} 2\hat{I}(a) \left(c_{aa} - \int v f_{aa} \right).$$

Now, $c_{aa} - \int v f_{aa}$ is uniformly bounded as in the proof of Lemma 6, while $\hat{I}$ is bounded above by Assumption 2, since l_x is uniformly bounded away from zero and hence the Fisher Information is as well. We are thus done, since $\varphi''/\varphi' \to 0$ by Assumption 6.

It remains to show $C_\theta =_\ell C_\theta^{FI} =_\ell \varphi'(\bar{u}+c)c_\theta$. By the Envelope Theorem $C_\theta = \lambda c_\theta + \mu c_{a\theta}$. Thus,

$$\frac{C_\theta}{\varphi'(\bar{u}+c)c_\theta} = \frac{\lambda c_\theta + \mu c_{a\theta}}{\varphi'(\bar{u}+c)c_\theta} = \frac{\lambda}{\varphi'(\bar{u}+c)}\frac{c_\theta + \frac{\mu}{\lambda}c_{a\theta}}{c_\theta} =_\ell 1 + \frac{\mu}{\lambda}\frac{c_{a\theta}}{c_\theta} \tag{15}$$

and we would be done if the last term converges to zero. But,

$$\begin{aligned} 0 \le \frac{\mu}{\lambda}\frac{c_{a\theta}}{c_\theta} &= \frac{\hat{I}\,(a)\int \varphi'' v_x(-F_a)}{\lambda}\frac{c_{a\theta}}{c_\theta} \\ &= \frac{\hat{I}(a)\int \varphi'' v_x(-F_a)}{\lambda}\frac{c_{a\theta}}{c_\theta} \\ &= \frac{\hat{I}(a)\varphi''(\bar{u}+c)\int \frac{\varphi''}{\varphi''(\bar{u}+c)} v_x(-F_a)}{\lambda}\frac{c_{a\theta}}{c_\theta} \\ &\le \frac{\hat{I}(a)\varphi''(\bar{u}+c)2\int v_x(-F_a)}{\lambda}\frac{c_{a\theta}}{c_\theta} \\ &= \frac{\varphi''(\bar{u}+c)}{\lambda}2\hat{I}(a)\frac{c_a c_{a\theta}}{c_\theta}, \end{aligned}$$

as before, and we are done by Assumption 1 and Lemma 6.[22] □

A.8 Proof of Lemma 7

Since $C_a = \mu(c_{aa} - \int v f_{aa}) + \int \varphi(v) f_a$, and since $\varphi' = 1/u' = \lambda + \mu l$,

$$C_{aa} = \mu_a\left(c_{aa} - \int v f_{aa}\right) + \mu\left(c_{aaa} - \int v_a f_{aa} - \int v f_{aaa}\right) + \lambda\int v_a f_a + \mu\int v_a l f_a + \int \varphi(v) f_{aa}.$$

Now, $l f_a = f_{aa} - l_a f$, while differentiating the identity IC gives $\int v_a f_a = c_{aa} - \int v f_{aa}$, and hence, substituting and collecting terms,

$$C_{aa} = (\mu_a + \lambda)\left(c_{aa} - \int v f_{aa}\right) + \mu\left(c_{aaa} - \int v f_{aaa}\right) - \mu\int v_a l_a f + \int \varphi(v) f_{aa}.$$

Integrating by parts, and again using $\varphi' = \lambda + \mu l$,

$$\int \varphi(v) f_{aa} = -\int \varphi'(v) v_x F_{aa} = -\lambda\int v_x F_{aa} - \mu\int v_x l F_{aa}.$$

Substituting and cancelling $-\lambda\int v_x F_{aa}$ with $-\lambda\int v f_{aa}$, we arrive at

$$C_{aa} = \lambda c_{aa} + \mu_a\left(c_{aa} - \int v f_{aa}\right) + \mu\left(c_{aaa} - \int v f_{aaa} - \int v_x l F_{aa}\right) - \mu\int v_a l_a f.$$

[22] Note that the extra work is required: $c_{a\theta}/c_\theta$ is not bounded for $c = (1-\theta)\,a^2/2$, while $c_a c_{a\theta}/c_\theta$ is.

But,

$$\begin{aligned}
\int v_a l_a f &= \int (\lambda_a + \mu_a l + \mu l_a) l_a \rho' f \\
&= \left(\int \rho' f \right) \int (\lambda_a + \mu_a l + \mu l_a) l_a \xi \\
&= \left(\int \rho' f \right) \int \left(\left(- \int (\mu_a l + \mu l_a) \xi \right) + \mu_a l + \mu l_a \right) l_a \xi \\
&= \left(\int \rho' f \right) \left(\mu_a \left(\int l l_a \xi - \int l \xi \int l_a \xi \right) + \mu \left(\int (l_a)^2 \xi - \left(\int l_a \xi \right)^2 \right) \right) \\
&= \left(\int \rho' f \right) \left(\mu_a \, cov_\xi(l, l_a) + \mu \, var_\xi(l_a) \right),
\end{aligned}$$

using (7). Also, rearranging (7), $c_{aa} - \int \rho f_{aa} = (\int \rho' f)(\mu_a var_\xi(l) + \mu cov_\xi(l, l_a))$. Substituting, and cancelling $\mu \mu_a cov_\xi(l, l_a)$ yields

$$C_{aa} = \lambda c_{aa} + \mu \left(c_{aaa} - \int v f_{aaa} - \int v_x l F_{aa} \right) + \left(\int \rho' f \right) \left(\mu_a^2 var_\xi(l) - \mu^2 \, var_\xi(l_a) \right),$$

as claimed. □

A.9 Proof of Proposition 4

Let us start with C_{aa}. We have

$$\begin{aligned}
\frac{C_{aa}}{C_{aa}^{FI}} &= \frac{\lambda c_{aa} + \mu \left(c_{aaa} - \int v f_{aaa} - \int v_x l F_{aa} \right) + \left(\int \rho' f \right) \left(\mu_a^2 var_\xi(l) - \mu^2 \, var_\xi(l_a) \right)}{\varphi' c_{aa} + \varphi'' c_a^2} \\
&= \frac{\lambda}{\varphi'} \frac{c_{aa} + \frac{\mu}{\lambda} \left(c_{aaa} - \int v f_{aaa} - \int v_x l F_{aa} \right) + \frac{\mu_a}{\lambda} \left(\mu_a \int \rho' f \right) var_\xi(l) - \frac{\mu}{\lambda} \left(\mu \int \rho' f \right) var_\xi(l_a)}{c_{aa} + \frac{\varphi''}{\varphi'} c_a^2}
\end{aligned}$$

The term λ/φ' goes to one uniformly by Lemma 1. From (14), for $\bar{u}$ large enough, $|\int v f_{aaa}| \leq J \max_{a,x} |F_{aaa}| < \infty$. Similarly, $|\int v_x l F_{aa}| \leq J \max_{a.x} |l F_{aa}|$ is finite. But then, since $\mu/\lambda \to 0$ uniformly as $\bar{u} \to \infty$,

$$\frac{\mu}{\lambda} \left(c_{aaa} - \int v f_{aaa} - \int v_x l F_{aa} \right)$$

goes to zero uniformly. Next, by Lemma 6,

$$\frac{\mu_a}{\lambda} \left(\mu_a \int \rho' f \right) var_\xi(l)$$

consists of a first term that goes to zero, and two terms that are uniformly bounded in a as $\bar{u}$

diverges. Similarly,

$$\frac{\mu}{\lambda}\left(\mu\int\rho' f\right) var_\xi(l_a)\to 0$$

uniformly. Finally, by Assumption 1 $\varphi''/\varphi'\to 0$, and we are done.

Let us next consider $C_{a\bar{u}}$. Begin by noting that $C_{\bar{u}}=\lambda=\int\varphi'(v)f$, and so

$$C_{\bar{u}a}=\lambda_a=\int\varphi''(v)v_a f+\int\varphi'(v)f_a=\int\varphi''(v)v_a f+\int\varphi''(v)v_x(-F_a),$$

But then,

$$\frac{C_{\bar{u}a}}{\varphi''(\bar{u}+c)c_a}=\frac{1}{c_a}\int\frac{\varphi''(v)}{\varphi''(\bar{u}+c)}v_a f+\frac{1}{c_a}\int\frac{\varphi''(v)}{\varphi''(\bar{u}+c)}v_x(-F_a)$$

which, given Lemma 1 converges uniformly to

$$\frac{1}{c_a}\int v_a f+\frac{1}{c_a}\int v_x(-F_a).$$

Now, note that since $\int vf=\bar{u}+c$, we have $\int v_a f+\int vf_a=c_a$, and hence by (IC), $\int v_a f=0$. And, integrating by parts, $\int v_x(-F_a)=c_a$, and we are done.

Finally, let us consider $C_{a\theta}$. Differentiating $C_\theta=\lambda c_\theta+\mu c_{a\theta}$, and using Proposition 1,

$$\frac{C_{a\theta}}{\varphi'(\bar{u}+c)c_{a\theta}}=\frac{\lambda_a c_\theta+(\lambda+\mu_a)c_{a\theta}+\mu c_{aa\theta}}{\varphi'(\bar{u}+c)c_{a\theta}}=_\ell\frac{\lambda_a}{\lambda}\frac{c_\theta}{c_{a\theta}}+1+\frac{\mu_a}{\lambda}+\frac{\varphi''(\bar{u}+c)}{\varphi'(\bar{u}+c)}\hat{I}(a)\frac{c_a c_{aa\theta}}{c_{a\theta}}=_\ell 1$$

where the last equality uses Lemma 6, and that $c_\theta/c_{a\theta}$ and $c_a c_{aa\theta}/c_{a\theta}$ are bounded by Assumption 1. Similarly

$$\frac{C^{FI}_{a\theta}}{\varphi'(\bar{u}+c)c_{a\theta}}=\frac{\varphi''(\bar{u}+c)c_\theta c_a+\varphi'(\bar{u}+c)c_{a\theta}}{\varphi'(\bar{u}+c)c_{a\theta}}=\frac{\varphi''(\bar{u}+c)}{\varphi'(\bar{u}+c)}\frac{c_\theta c_a}{c_{a\theta}}+1=_\ell 1,$$

since $\varphi''/\varphi'\to 0$ by Lemma 1, and since both $c_\theta/c_{a\theta}$ and c_a are bounded by Assumption 1. □

A.10 Proof of Proposition 6

Inserting the expressions $C_{a\theta}=\lambda_a c_\theta+\lambda c_{a\theta}+\mu_a c_{a\theta}+\mu c_{aa\theta}$, $C_{a\bar{u}}=\lambda_a>0$, $C_\theta=\lambda c_\theta+\mu c_{a\theta}<0$, and $C_{\bar{u}}=\lambda$ in (12), simplifying, and dividing by λ on both sides yields the equivalent expression

$$c_{a\theta}\le\frac{\mu}{\lambda}\frac{\lambda_a}{\lambda}c_{a\theta}-\frac{\mu_a}{\lambda}c_{a\theta}-\frac{\mu}{\lambda}c_{aa\theta}.$$

where left side is strictly negative, while by Lemma 6, the right hand goes to zero as $\bar{u}$ goes to infinity. So for $\bar{u}$ sufficiently large (11) holds strictly, and positive sorting ensues.

Given that we have established positive sorting, define the (increasing) matching function η by $Q(\tau)=H(\eta(\tau))$ for all τ. For the second part of the proposition, let $\tilde{u}(\theta)$ be the utility that agent

θ obtains in equilibrium. One can interpret this as the "price function" for agents that principals take as given in a competitive market. Note that $\tilde{u}$ must satisfy the first-order condition of a principal with characteristic τ when choosing an agent, $\Pi_\theta + \Pi_{\bar{u}}\tilde{u}'(\theta) = 0$, or

$$\tilde{u}'(\theta) = -\frac{\Pi_\theta}{\Pi_{\bar{u}}} = -\frac{C_\theta}{C_{\bar{u}}} = -(c_\theta + \frac{\mu}{\lambda}c_{a\theta}) > 0.$$

Now, in equilibrium, the following identity characterizes the optimal action in each principal-agent pair:

$$B_a(a^*(\tau, \eta(\tau), \tilde{u}(\eta(\tau))), \tau) - C_a(a^*(\tau, \eta(\tau), \tilde{u}(\eta(\tau))), \eta(\tau), \tilde{u}(\eta(\tau))) = 0.$$

Differentiating with respect to τ and simplifying yields

$$(a^*)_\tau =_s B_{a\tau} - \eta'(C_{a\theta} + C_{a\bar{u}}\tilde{u}') > 0,$$

where the inequality follows from $B_{a\tau} > 0$, $\eta' > 0$, and

$$C_{a\theta} + C_{a\bar{u}}\tilde{u}' = C_{a\theta} - \frac{C_\theta}{C_{\bar{u}}}C_{a\bar{u}},$$

which is, for $\bar{u}$ sufficiently large, negative by (12). □

A.11 Derivations For Section 6.1

STRICT CONVEXITY OF C. Differentiating C_a once more,

$$\begin{aligned} C_{aa} &= (1-k\theta)\left(\varphi'(v_h)(2-3a) - \varphi'(v_l)(1-3a)\right) \\ &\quad + (a-a^2)(1-k\theta)^2\left(\varphi''(v_h)(1-a) + \varphi''(v_l)a\right) \\ &\geq (1-k\theta)\varphi'(v_l)\left(2 - \frac{\varphi'(v_h)}{\varphi'(v_l)}\right), \end{aligned}$$

where the inequality arises by discarding the positive term involving φ'', and then taking $a = 1$ in $(2-3a)$ and $(1-3a)$. Hence , for $C_{aa} \geq 0$ everywhere, it is sufficient that $2 - \varphi'(v_h)/\varphi'(v_l) \geq 0$ for all a and θ, while, since $a - a^2$ is zero when $a = 1$, it is necessary that $2 - \varphi'(v_h)/\varphi'(v_l) \geq 0$ for $a = 1$ and $\theta = 0$.

Now, since for *HARA*,

$$\varphi(v) = \left(\left(\frac{\gamma}{1-\gamma}v\right)^{\frac{1}{\gamma}} - \beta\right)\frac{1-\gamma}{\alpha},$$

we have

$$\frac{\varphi''(v)}{\varphi'(v)} = \frac{1-\gamma}{\gamma}\frac{1}{v}.$$

But then,

$$\frac{\varphi'(v_h)}{\varphi'(v_l)} = e^{\int_{v_l}^{v_h} \frac{\varphi''(s)}{\varphi'(s)} ds} = e^{\frac{1-\gamma}{\gamma} \int_{v_l}^{v_h} \frac{1}{s} ds} = e^{\frac{1-\gamma}{\gamma} \log \frac{v_h}{v_l}} = \left(\frac{v_h}{v_l}\right)^{\frac{1-\gamma}{\gamma}}.$$

But, since $v_h = c_a + v_l$,

$$\frac{v_h}{v_l} = \frac{c_a}{v_l} + 1 = \frac{c_a}{\bar{u} + c - ac_a} + 1 = \frac{(1-k\theta)a}{\bar{u} - \frac{(1-k\theta)a^2}{2}} + 1.$$

This is increasing in a, and decreasing in θ, and so is largest at $a = 1$ and $\theta = 0$. Hence, necessary and sufficient for C_{aa} to be everywhere convex is that

$$\left(\frac{1}{\bar{u} - \frac{1}{2}} + 1\right)^{\frac{1-\gamma}{\gamma}} \leq 2,$$

or

$$\bar{u} \geq \frac{1}{2} + \frac{1}{2^{\frac{\gamma}{1-\gamma}} - 1},$$

as claimed.

Supermodularity of C in $(a, \bar{u})$. Note that

$$\begin{aligned} C_{a\bar{u}} &= \varphi'(v_h) - \varphi'(v_l) + (1-a)\varphi''(v_l)v_{l,a} + a\varphi''(v_h)v_{h,a} \\ &= \varphi'(v_h) - \varphi'(v_l) - (1-a)\varphi''(v_l)ac_{aa} + a\varphi''(v_h)(1-a)c_{aa} \\ &= \varphi'(v_h) - \varphi'(v_l) + (1-a)ac_{aa}\left(\varphi''(v_h) - \varphi''(v_l)\right). \end{aligned}$$

But,

$$\varphi'(v) = \frac{1}{\alpha}\left(\frac{\gamma}{1-\gamma}v\right)^{\frac{1}{\gamma}-1}, \text{ and } \varphi''(v) = \frac{1}{\alpha}\left(\frac{\gamma}{1-\gamma}v\right)^{\frac{1}{\gamma}-2},$$

and so

$$\begin{aligned} C_{a\bar{u}} &= \frac{1}{\alpha}\left(\frac{\gamma}{1-\gamma}\right)^{\frac{1}{\gamma}-2}\left(\frac{\gamma}{1-\gamma}\left((v_h)^{\frac{1-\gamma}{\gamma}} - (v_l)^{\frac{1-\gamma}{\gamma}}\right) + (1-a)ac_{aa}\left((v_h)^{\frac{1-2\gamma}{\gamma}} - (v_l)^{\frac{1-2\gamma}{\gamma}}\right)\right) \\ &=_s \frac{\gamma}{1-\gamma}\left((v_h)^{\frac{1-\gamma}{\gamma}} - (v_l)^{\frac{1-\gamma}{\gamma}}\right) + (1-a)ac_{aa}\left((v_h)^{\frac{1-2\gamma}{\gamma}} - (v_l)^{\frac{1-2\gamma}{\gamma}}\right) \\ &=_s \frac{\gamma}{1-\gamma}\frac{1}{(1-a)ac_{aa}} + \frac{1}{v_l}j(\bar{u}), \end{aligned}$$

where, since $v_h = v_l + c_a$,

$$j = \frac{\left(\frac{v_l+c_a}{v_l}\right)^{\frac{1-2\gamma}{\gamma}} - 1}{\left(\frac{v_l+c_a}{v_l}\right)^{\frac{1-\gamma}{\gamma}} - 1}.$$

Note first that when $\gamma \in (0, 1/2]$, j is positive, and hence $C_{a\bar{u}}$ is positive independent of $\bar{u}$. So, in what follows, assume $\gamma \in (1/2, 1)$. As $\bar{u} \to \infty$, $(v_l + c_a)/v_l \to 1$. Hence by L'Hospital's rule, j has limit $(1-2\gamma)/(1-\gamma)$,which is bounded uniformly for all a. Since v_l diverges, the first term dominates, and $C_{a\bar{u}} \geq 0$.

To get a handle on the needed magnitude of $\bar{u}$, note that $\partial((v_l + c_a)/v_l)/\partial\bar{u} < 0$, and thus

$$\frac{\partial j}{\partial \bar{u}} = \frac{\frac{1-2\gamma}{\gamma}\left(\frac{v_l+c_a}{v_l}\right)^{\frac{1-3\gamma}{\gamma}}\left(\left(\frac{v_l+c_a}{v_l}\right)^{\frac{1-\gamma}{\gamma}}-1\right)-\left(\left(\frac{v_l+c_a}{v_l}\right)^{\frac{1-2\gamma}{\gamma}}-1\right)\frac{1-\gamma}{\gamma}\left(\frac{v_l+c_a}{v_l}\right)^{\frac{1-2\gamma}{\gamma}}}{\left(\left(\frac{v_l+c_a}{v_l}\right)^{\frac{1-\gamma}{\gamma}}-1\right)^2}\frac{\partial\left(\frac{v_l+c_a}{v_l}\right)}{\partial\bar{u}}$$

$$=_s \left(\left(\frac{v_l+c_a}{v_l}\right)^{\frac{1-\gamma}{\gamma}}-\frac{v_l+c_a}{v_l}\right)(1-\gamma)-(1-2\gamma)\left(\left(\frac{v_l+c_a}{v_l}\right)^{\frac{1-\gamma}{\gamma}}-1\right)$$

$$=_s \left(\frac{v_l+c_a}{v_l}\right)^{\frac{1-\gamma}{\gamma}}\gamma-\frac{v_l+c_a}{v_l}(1-\gamma)+(1-2\gamma),$$

where, since $\gamma \in (1/2, 1)$, the last expression is decreasing in c_a. Since it is zero at $c_a = 0$, it follows that for $c_a > 0$, $\partial j/\partial\bar{u} < 0$. It follows that

$$\begin{aligned} C_{a\bar{u}} &=_s \frac{\gamma}{1-\gamma}\frac{1}{(1-a)ac_{aa}}+\frac{1}{v_l}j(\bar{u}) \\ &\geq \frac{\gamma}{1-\gamma}\frac{1}{(1-a)ac_{aa}}+\frac{1}{v_l}\lim_{\bar{u}\to\infty}j(\bar{u}) \\ &= \frac{\gamma}{1-\gamma}\frac{1}{(1-a)ac_{aa}}+\frac{1}{v_l}\frac{1-2\gamma}{1-\gamma}, \end{aligned}$$

and so, substituting for v_l and rearranging yields that a sufficient condition for $C_{a\bar{u}} \geq 0$ is that

$$\bar{u} \geq -c + ac_a + \frac{(1-a)ac_{aa}}{\gamma}(2\gamma - 1).$$

For $c(a, \theta) = (1 - k\theta)a^2/2$ this reduces to

$$\bar{u} \geq (1-k\theta)a\left(2-\frac{3}{2}a-\frac{1-a}{\gamma}\right).$$

But,

$$(1-k\theta)a\left(2-\frac{3}{2}a-\frac{1-a}{\gamma}\right) \leq a\left(1-\frac{1}{2}a\right) \leq \frac{1}{2}$$

and so, sufficient for $C_{a\bar{u}} \geq 0$ is that $\bar{u} \geq 1/2$.

Submodularity of C in (a,θ). Differentiating C_a with respect to θ gives

$$\begin{aligned}C_{a\theta} &= \varphi_h'(c_\theta + (1-a)c_{a\theta}) - \varphi_l'(c_\theta - ac_{a\theta}) + a(1-a)c_{aa\theta}(\varphi_h' - \varphi_l') \\ &\quad + a(1-a)c_{aa}(\varphi_h''(c_\theta + (1-a)c_{a\theta}) - \varphi_l''(c_\theta - ac_{a\theta})) \\ &\quad \le -(\varphi_l' + \varphi_l'')(c_\theta - ac_{a\theta}),\end{aligned}$$

using that φ is increasing and convex, with $c_\theta < 0$, $c_{a\theta} \le 0$, and $c_{aa\theta} \le 0$. So, it is enough to show that $c_\theta - ac_{a\theta} \ge 0$, or equivalently, that $c_\theta/a - c_{a\theta} \ge 0$. But, holding fixed θ, and considering c_a as a function of a, c_θ/a is the slope of a ray from the origin to a point on c_θ, while $c_{a\theta}$ is the slope of c_θ at this point. Hence, since c_θ is concave in a and since $c_\theta(0,\theta) \equiv 0$ for the cost function $c(a,\theta) = (1-k\theta)a^2/2$, we have $C_{a\theta} \le 0$ for all $\bar{u}$.

A.12 Limit Behavior of $C_{\bar{u}\bar{u}}$, $C_{\theta\theta}$, and $C_{\bar{u}\theta}$

We first show that $C_{\bar{u}\bar{u}} =_\ell \varphi''(\bar{u}+c)$. We have $C_{\bar{u}\bar{u}} = \lambda_{\bar{u}}$, where, similar to Lemma 5,

$$\lambda_{\bar{u}} = \frac{\int l^2\xi}{var_\xi(l)\int \rho' f}.$$

Hence,

$$\frac{C_{\bar{u}\bar{u}}}{\varphi''(\bar{u}+c)} = \frac{\frac{\int l^2\xi}{var_\xi(l)\int\rho' f}}{\varphi''(\bar{u}+c)} = \frac{\int l^2\xi}{var_\xi(l)}\frac{1}{\int\frac{\varphi''(\bar{u}+c)}{\varphi''}} =_\ell 1,$$

where the second equality follows by rearrangement, and the last equality follows from ξ converging uniformly to f and $\varphi''(\bar{u}+c)/\varphi''$ converging uniformly to one.

We now show that $C_{\theta\theta} =_\ell \varphi'(\bar{u}+c)c_{\theta\theta}$. Note first that $C_{\theta\theta}^{FI} = \varphi''(\bar{u}+c)c_\theta^2 + \varphi'(\bar{u}+c)c_{\theta\theta} =_\ell \varphi'(\bar{u}+c)c_{\theta\theta}$ since φ''/φ' converges to zero uniformly, and since $c_{\theta\theta}$ is assumed bounded away from zero. Second, from $C_\theta = \lambda c_\theta + \mu c_{a\theta}$ we obtain

$$C_{\theta\theta} = \lambda_\theta c_\theta + \lambda c_{\theta\theta} + \mu_\theta c_{a\theta} + \mu c_{aa\theta}.$$

Hence,

$$\frac{C_{\theta\theta}}{\varphi'(\bar{u}+c)c_{\theta\theta}} = \frac{\lambda}{\varphi'(\bar{u}+c)}\frac{\frac{\lambda_\theta}{\lambda}c_\theta + c_{\theta\theta} + \frac{\mu_\theta}{\lambda}c_{a\theta} + \frac{\mu}{\lambda}c_{aa\theta}}{c_{\theta\theta}},$$

and since the first term goes to one uniformly and μ/λ goes to zero uniformly, it suffices to show that λ_θ/λ and μ_θ/λ converge to zero uniformly. Similar steps as in Lemma 5 yield

$$\lambda_\theta = \frac{c_\theta\int l^2\xi - c_{a\theta}\int l\xi}{var_\xi(l)\int\rho' f}, \text{ and } \mu_\theta = \frac{c_{a\theta} - c_\theta\int l\xi}{var_\xi(l)\int\rho' f}.$$

Consider

$$\frac{\lambda_\theta}{\lambda} = \frac{c_\theta \int l^2\xi - c_{a\theta}\int l\xi}{\lambda var_\xi(l)\int \rho' f}.$$

Since, as $\bar{u}$ diverges, $\int l\xi$ converges uniformly to zero and $var_\xi(l)$ coverges uniformly to $\int l^2 f$, it suffices to show that $c_\theta/(\lambda\int\rho' f)$ converges to zero uniformly. But since $\lambda =_\ell \varphi'(\bar{u}+c)$, $\rho' = 1/\varphi''$, φ''/φ' goes to zero uniformly and $\varphi''(\bar{u}+c)/\varphi''$ goes to one uniformly, we have that

$$\lambda\int\rho' f = \frac{\lambda}{\varphi''(\bar{u}+c)}\int\frac{\varphi''(\bar{u}+c)}{\varphi''}f$$

diverges to infinity uniformly as $\bar{u}$ diverges. Hence, $c_\theta/(\lambda\int\rho' f)$ converges to zero uniformly.

The proof of μ_θ/μ converges to zero uniformly as $\bar{u}$ diverges is analogous.

Finally, we show that $C_{\bar{u}\theta} =_\ell \varphi''(\bar{u}+c)c_\theta$. We have $C_{\bar{u}\theta} = \lambda_\theta$, and hence

$$\begin{aligned}\frac{C_{\bar{u}\theta}}{\varphi''(\bar{u}+c)\,c_\theta} &= \frac{1}{\left(\int\rho' f\right)var_\xi(l)}\left(\frac{c_\theta\int l^2\xi - c_{a\theta}\int l\xi}{\varphi''(\bar{u}+c)\,c_\theta}\right)\\ &= \left(\frac{1}{\left(\int\rho' f\right)var_\xi(l)}\frac{c_\theta\int l^2\xi}{\varphi''(\bar{u}+c)\,c_\theta}\right)\left(1-\frac{c_a c_{a\theta}}{c_\theta\int l^2\xi}\frac{\int l\xi}{c_a}\right).\end{aligned}$$

But then, using that the first bracketed term is limit ratio equal to 1, and since $c_a c_{a\theta}/\left(c_\theta\int l^2\xi\right)$ is bounded by Assumption 1, it is enough to show that $\int l\xi/c_a \to 0$ uniformly. Note that

$$\begin{aligned}\int l\xi = \frac{\int\rho' lf}{\int\rho' f} &= \frac{\int_0^{x^*}\rho' lf + \int_{x^*}^1\rho' lf}{\int\rho' f} \leq \frac{\int_0^{x^*}(\min\rho')\,lf + \int_{x^*}^1(\max\rho')\,lf}{\int(\min\rho')\,f}\\ &= \frac{\int_0^{x^*} lf + \int_{x^*}^1\frac{(\max\rho')}{(\min\rho')}lf}{\int f} = \int_{x^*}^1\left(\frac{\max\rho'}{\min\rho'}-1\right)lf,\end{aligned}$$

and similarly,

$$\begin{aligned}\int l\xi = \frac{\int\rho' lf}{\int\rho' f} &= \frac{\int_0^{x^*}\rho' lf + \int_{x^*}^1\rho' lf}{\int\rho' f} \geq \frac{\int_0^{x^*}(\max\rho')\,lf + \int_{x^*}^1(\min\rho')\,lf}{\int(\max\rho')\,f}\\ &= \frac{\int_0^{x^*} lf + \int_{x^*}^1\frac{\min\rho'}{\max\rho'}lf}{\int f} = \int_{x^*}^1\left(\frac{\min\rho'}{\max\rho'}-1\right)lf\end{aligned}$$

Now, reasoning as in the proof of Lemma 3, and in particular using (13), we have that for any given $(a,\bar{u},\theta)$,

$$v(1)-v(0) \leq J(a,\theta) \equiv \frac{4}{\min(L_1,L_2)}c_a(a,\theta),$$

and so,

$$\frac{\max \frac{1}{\varphi''}}{\min \frac{1}{\varphi''}} - 1 = \exp \int_{\arg\min_{\hat{u}\in[v(0),v(1)]} \frac{1}{\varphi''}}^{\arg\max_{\hat{u}\in[v(0),v(1)]} \frac{1}{\varphi''}} \frac{\varphi'''(\tilde{u})}{\varphi''(\tilde{u})} d\tilde{u} - 1$$
$$\leq \exp\left(\left(\max_{\hat{u}\in[v(0),v(1)]} \frac{\varphi'''(\hat{u})}{\varphi''(\hat{u})}\right)(v(1) - v(0))\right) - 1$$
$$\leq \exp\left(\left(\max_{\hat{u}\in[v(0),v(1)]} \frac{\varphi'''(\hat{u})}{\varphi''(\hat{u})}\right)\left(\frac{4}{\min(L_1, L_2)} c_a(a,\theta)\right)\right) - 1.$$

Note that $4c_a(a,\theta)/\min(L_1, L_2)$ is uniformly bounded in a and θ, and that $\max_{\hat{u}\in[v(0),v(1)]} \varphi'''/\varphi''$ goes to zero using Lemmas 3 and Assumption 7. Thus, using that $e^x = (e^x)_x = 1$ at $x = 0$, for $\bar{u}$ large enough

$$\frac{\max \frac{1}{\varphi''}}{\min \frac{1}{\varphi''}} - 1 \leq \left(\max_{\hat{u}\in[v(0),v(1)]} \frac{\varphi'''(\hat{u})}{\varphi''(\hat{u})}\right) \frac{8}{\min(L_1, L_2)} c_a(a,\theta)$$

for all a and θ. Similarly,

$$\frac{\min \frac{1}{\varphi''}}{\max \frac{1}{\varphi''}} - 1 \geq -\left(\max_{\hat{u}\in[v(0),v(1)]} \frac{\varphi'''}{\varphi''}\right) \frac{8}{\min(L_1, L_2)} c_a.$$

But then,

$$\frac{\left|\int l\xi\right|}{c_a} \leq \left(\max_{\hat{u}\in[v(0),v(1)]} \frac{\varphi'''}{\varphi''}\right)\left(\frac{8}{\min(L_1, L_2)} \int_{x^*}^{1} lf\right),$$

and we are done, since $\varphi'''/\varphi'' \to 0$ and the last bracketed term is uniformly bounded in (a,θ). □

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