Disentangling Moral Hazard and Adverse Selection

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Abstract

We analyze a canonical principal-agent problem with both moral hazard and adverse selection. We derive sufficient conditions for a menu of contracts to be feasible. We then provide a method of solution, which we call decoupling. It consists of first minimizing the cost of implementing any given action at any given surplus for any given type in a pure moral hazard problem, and then use the resulting cost function as an input to a pure adverse-selection problem. We show broad classes of primitives under which the solution to this radically simplified program is indeed optimal in the full problem. Decoupling has powerful implications for the structure of optimal menus. We illustrate our results in the context of an insurance market.

Keywords. Moral Hazard, Adverse Selection, First-Order Approach, Incentive Compatibility, Principal-Agent Problem.
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1 Introduction

In many settings of interest, both screening and moral hazard are at play. A firm wants workers to self identify as more or less able, and to tailor incentives and effort accordingly. An insurance company wants to tailor the trade-off between risk sharing and the incentive to take care to the privately known riskiness of the customer. An investor wants an entrepreneur to both reveal what she knows about the quality of the project and choose an appropriate level of effort. In each of these settings, an optimal contract needs to both elicit the agent's type and provide him with incentives to take a suitable action for that type.

Although the design of contracts under either pure adverse selection or pure moral hazard is well understood, little is known about the case where the two of them are present simultaneously, especially when the agent is risk averse.[1] This is because the problem is innately complicated. In each of the pure cases, the set of deviations for the agent is one dimensional. But here, an agent can "double deviate" by first misrepresenting his type and then choosing an action level other than the one recommended for the type announced. Hence, unlike the adverse selection case, where a sweeping incentive compatibility characterization exists (Mirrlees (1975), Myerson (1981)), or the moral hazard case, where the first-order approach (Rogerson (1985), Jewitt (1988)) drastically simplifies the incentive constraints, there is no known analogous simplification in the combined case that handles the myriad of deviations available. The central purpose of this paper is to provide and explore one such simplification.

We analyze optimal contracts with adverse selection and moral hazard in a canonical principal-agent model. A risk-averse agent has a type reflecting his disutility of taking an unobservable action. A signal is generated that depends stochastically on the action of the agent. Both the type of the agent and his action lie in a continuum. A mechanism recommends an action for each announced type, and compensates the agent based on his announced type and the realization of the signal. Thus, the agent, by announcing his type, is effectively choosing over a menu of incentive schemes. The problem facing the principal is to maximize expected profit subject to the willingness of the agent to participate, reveal his type, and take the recommended action.

We first study the necessary local conditions, showing that the local second order conditions reduce to a single condition capturing that in a specific sense, incentives are stronger for more able agents. We then turn to sufficient conditions for feasibility–i.e., for global incentive compatibility. We derive two such sets of conditions, which we view as the first main contribution of the paper. Each is a stronger version of the condition that incentives are stronger for more capable agents.[2]

[1] For adverse selection, see, e.g. Guesnerie and Laffont (1984) and the textbook treatment in Chapter 7 of Fudenberg and Tirole (1991). For moral hazard, see the seminal papers by Holmstrom (1979) and Grossman and Hart (1983). See also the textbook treatments in Laffont and Martimort (2001), and Bolton and Dewatripont (2005) which present examples with both moral hazard and adverse selection with either two types or risk neutrality.

[2] We also require that, holding fixed the type announced, the expected utility from income of the agent is concave in effort, analagous to the first-order approach in the pure moral hazard setting.

These conditions are simple, easy to interpret economically, and form the basis for our later analysis, in which we exhibit primitives that guarantee that one or the other condition holds. Each condition is easily verified in numerical examples, and can plausibly be checked empirically.

Our first condition centers around the action schedule alone. The marginal cost of effort to the agent falls with his type but rises with his action. We show that a sufficient condition for feasibility is that recommended action rises fast enough so that on net a more capable agent faces a higher marginal cost of effort. We refer to this as the *increasing marginal cost* condition (*IMC*).

We show that if an optimal menu satisfies *IMC* then for each type, the compensation scheme is of the standard Holmstrom-Mirrlees (henceforth *HM*) form, as in the pure moral hazard case. That is, the inverse of marginal utility is equal to a Lagrange multiplier on the participation constraint plus a Lagrange multiplier on the (local) incentive constraint times the likelihood ratio. This is somewhat remarkable, since in principle, the solution to the problem–which contains all three of moral hazard, adverse selection and risk aversion–could have been wildly complicated, with the precise structure of incentives provided to one type crucial to whether other types wish to deviate. Instead, when *IMC* holds, compensation consists of objects that are well understood from the moral-hazard setting, and everything we know from that setting (Holmstrom (1979), Mirrlees (1975), and the enormous literature that follows) continues to be relevant.

We then turn to the *linear probability* case in which the cumulative distribution of output below a given threshold is linear in effort. We show that *IMC*–which we have shown is sufficient for feasibility–is also necessary. But then, a dramatic simplification of the problem presents itself. We use the knowledge that an optimal solution will involve only *HM* contracts to construct a cost function over effort for any given type and required surplus. We then insert this cost function into a simple screening problem where *IMC* is imposed as a constraint. While this screening problem is not quasi-linear, we show that it *is* amenable to analysis. We then take the solution to this screening problem, and substitute back in the appropriate *HM* contract for each type.

The problem in the linear setting thus effectively *decouples* into one where we first solve a moral hazard problem and we then solve an adverse selection problem that uses the solution to the moral hazard problem as an input. This makes the problem both technically and computationally tractable.[3] Rather than double deviations making the problem exponentially harder, the problem is instead solvable by simple extensions to tools we already know for the two pure cases.

In a standard screening problem, things are especially simple if one can ignore the monotonicity constraint and solve the relaxed program pointwise. But, *IMC* is just a strengthened monotonicity condition. Hence it is of paramount interest to know primitives under which the solution to the relaxed program is not only monotone, but satisfies *IMC*. Then, because no ironing is involved, the solution is both more easily derived and has a particularly simple structure. Indeed, we

[3]See Section 5 of Kadan and Swinkels (2013) for a numerically efficient algorithm to solve the moral hazard problem. Our existence proof for the adverse selection problem points the way to a simple numerical solution of the second step.

provide a new result showing existence of a solution for this (non quasi-linear) relaxed screening problem, one that shares many of the standard properties that we understand from the screening literature, e.g., Myerson (1981), Maskin and Riley (1984), and Guesnerie and Laffont (1984). The optimal action for any given type reflects the well-understood trade-off between creating surplus on that type and creating rents for more able types. Each type except the least able receives an information rent, and the slope of the information rent at any given type is given by the rate of change of the agent's disutility of effort with respect to his type at the equilibrium effort level.

Motivated by this, we show permissive primitive conditions under which *IMC* is guaranteed to hold at the solution to the relaxed problem. We also examine the case with two outcomes, which can be taken as a special case of the linear case. Here we exhibit even simpler primitives that guarantee *IMC*. Altogether, these results make the linear case a useful testbed for applications of contract theory involving adverse selection and moral hazard.

When one moves beyond the linear case, *IMC* is no longer necessary for feasibility, but it remains *sufficient*. It thus remains of fundamental interest to know when the same basic three step process we followed for the linear case continues to work. That is, we examine when we can *decouple* the problem by (i) solving a relaxed *HM* problem for any given type, surplus and action, (ii) using the cost function generated as an input to a screening problem, and then (iii) substituting the appropriate *HM* contract in for each type at the solution to the screening problem.

The decoupled solution is the solution to a substantially relaxed program: among the incentive constraints in the original problem are that the agent should not want to perturb his recommended action having announced his type honestly, or perturb his announcement of type conditional on having stated his true type. Our decoupled solution is by construction optimal subject only to these two constraints. So, if the decoupled solution is feasible, it is optimal.

We thus look for primitive conditions under which the solution to the second-stage screening problem is guaranteed to satisfy *IMC* and hence be feasible. Because the cost function arises from the *HM* problem, we can leverage its considerable structure. We provide a very general result that imposes only mild assumptions on utility and no assumptions beyond our ambient ones on the distribution over output. But, for any given specification of the fundamentals, our result holds only if the outside option of the agent is sufficiently high. The results is thus most relevant when having the agent participate is sufficiently important to the principal, as for example if the price of output is sufficiently high. To better understanding how large an outside option is required, we examine the more structured case of square-root utility. Under suitable conditions, the outside option need only be large enough that the agent prefers his outside option to working for free.

Decoupling is *not* always valid. We present an example where decoupling fails, and provide intuition for its failure. So, while our results establish that in many interesting settings decoupling works, with all of its implications, one cannot simply start from the presumption that a model with both moral hazard and adverse selection is effectively one where the solution has to satisfy the conditions of both, but no more. Results of the form we derive are needed.

The two-outcome case forms the basis for a central application of our results. We consider an insurance market in which a monopolist provider faces customers who differ in their innate riskiness but *also* take a hidden action to lower the probability of loss. In this setting, there is a common values aspect that we have not previously allowed for since the type of the agent enters *directly* into the probability of loss, and thus the utility of the principal. We show how to reparameterize this problem so that it fits our model, and then compare this setting to ones with full information, only adverse selection, and only moral hazard.

Condition *IMC* is not the only path to sufficiency. Say that the *single crossing condition* (*SCC*) holds if the compensation scheme of a more able agent, as a function of the signal, single-crosses that of a less able agent from below, which is again a condition that more capable agents are matched with higher powered incentive schemes. We provide primitives for *SCC*. The key is to find a class of distributions where any two *HM* contracts are guaranteed to cross at most once. We show that the exponential families are exactly that class.

We close with three extensions of our results. The first, foreshadowed by our analysis of the insurance problem, is to the question of optimal exclusion. Because of the decoupled structure we have exposed, this problem becomes tractable, and we provide economically interpretable necessary and sufficient conditions for optimal exclusion. These share some key properties of optimal exclusion in a pure adverse-selection setting.

Our second extension, again foreshadowed by the insurance setting, is to a model with common values. While very useful when it works, the reparameterization used in our analysis of the insurance market is not always possible. Our last result explores a setting where the type of the agent directly enters into the utility of the principal in an irreducible way. We show conditions under which *SCC* remains a sufficient condition for feasibility of a menu, and discuss why a generalization of *IMC* is harder.

Finally, we examine the question of whether it is without loss of generality that mechanisms are deterministic. We show that under any primitives that guarantee decoupling, a deterministic contract is optimal even if one allows the principal to randomize.

We proceed as follows. After describing the model in Section 2, we analyze three simpler benchmarks in Section 3. We then turn to the central case with both moral hazard and adverse selection. Section 4 describes the necessary conditions for feasibility, while Section 5 presents our first sufficient conditions for global incentive compatibility. Section 6 explores the case of linear output, and exposes the idea of decoupling. Section 7 describes the decoupling program for general environments, and Section 8 analyzes a very general class of primitives that guarantee *IMC*, examines the square-root utility case, and shows an example where decoupling fails. Section 9 applies our tools to an insurance market. Section 10 examines *SCC*, and shows primitives for it to hold. Section 11 examines optimal exclusion, the common-values setting, and randomization.

Section 12 concludes. Omitted proofs are in Appendix A. Appendix B shows existence of a solution to the relaxed pure adverse selection problem. Appendix C shows existence in the pure moral hazard problem, and some differentiability results used in the analysis.

1.1 Literature

The literature on optimal contracts under adverse selection and moral hazard with a risk-averse agent at the level of generality that we pursue is small. Faynzilberg and Kumar (1997) analyze a related model where the agent's type enters solely into the distribution of the signal, and shed light on the solution to a relaxed problem that only considers the local incentive constraints plus a separability condition on the signal distribution. Under that condition our model subsumes theirs by the reparameterization in Section 9. In a procurement setting, Baron and Besanko (1987) analyze a purchaser and a supplier that has private information about cost and takes an unobservable action. Using a parameterized model, they shed light on some properties of optimal contracts subject to local constraints. Neither of these papers takes advantage of the clarity and tractability that decoupling brings to the analysis. In an unpublished paper, Fagart (2002) studies the same combination of moral hazard and adverse selection as us, and discusses decoupling and some its implications. None of these papers tackle the crucial issue of when a solution to the first-order conditions is globally incentive compatible, or of what primitives ensure that the decoupled program yields a solution that satisfies these conditions, which are central contributions of this paper.[4] Also related is Gottlieb and Moreira (2013), who analyze a principal-agent problem with moral hazard and adverse selection but where the agent's private information is about the effect of effort on the distribution of a binary signal. In their setup, both type and effort take on only two possible values, while in our case, types and actions are continuous, and private information is (except when we turn to our analysis of the full common-values model) on the disutility of effort. Gottlieb and Moreira (2013) provide several insights about optimal menus, including distortion, pooling, and exclusion. Given the difference in environments, ours and their paper are best viewed as complementary. Finally, Laffont and Tirole (1986) derives the optimality of linear contracts under moral hazard and adverse selection with a risk neutral agent, which also decouples in a straightforward way.[5]

There is a well-established literature on insurance under adverse selection or moral hazard. Indeed, one of the first papers on screening is Stiglitz (1977), who analyzes a monopolistic insurer whose consumers have private information about their exogenous probability of a loss.[6] Shavell

[4] In the pure moral hazard literature, a different form of decoupling is the two-step procedure of Grossman and Hart (1983), who first cost minimize for each effort level and then profit maximize using the cost function derived. See Section 3.3 below for a description of this two-step methodology.

[5] There is an emerging literature on dynamic contracts thatcombine adverse s election and moral hazard. Recent examples are Strulovici (2011), Williams (2015), and Halac, Kartik, and Liu (2016). For tractability, they impose more restrictive assumptions on primitives than we do in our more general static model.

[6] See Chade and Schlee (2012) for further properties of the profit-maximizing menu of contracts in this setting.

(1979) and Holmstrom (1979) provide substantial insight into the optimal contract when instead the consumer's effort in reducing the probability of a loss is unobservable. We are unaware of any general analysis of the realistic problem in which the consumer is privately informed about his riskiness and can *also* exert care to reduce it, which is the application we study.[7] It can be thought of as a natural extension of Stiglitz (1977) to incorporate moral hazard.

2 The Model

We analyze the following principal-agent problem with moral hazard and adverse selection. The agent has a type $\theta \in [\underline{\theta}, \bar{\theta}]$, with θ distributed according to a cumulative distribution (cdf) H with strictly positive and continuously differentiable density h. The agent exerts effort $a \geq 0$, possibly with upper bound $\overline{a} < \infty$, where effort has disutility given by $c(a, \theta)$ for every (a, θ). The function c is three times continuously differentiable, where $c(0, \theta) = 0$, and where for all (a, θ) with $a > 0$, we have $c_a > 0$, $c_{aa} > 0$, $c_\theta < 0$, $c_{a\theta} < 0$, $c_{aa\theta} \leq 0$, and $c_{a\theta\theta} \geq 0$. That is, cost is strictly increasing and convex in effort, total and marginal costs strictly decrease with ability, cost is less convex in effort when ability is higher, and as effort increases, cost becomes more convex in ability.[8]

The agent is risk averse with strictly increasing, strictly concave, and thrice continuously differentiable utility function u over income. If the agent has type θ, exerts effort a, and obtains wage w, then his total utility is $u(w) - c(a, \theta)$. He has an outside option that yields utility $\bar{u}$.[9]

Neither a nor θ are contractible, since we assume that neither is observable. The principal only observes a signal x, distributed according to cdf $F(\cdot|a)$ when the agent exerts effort a. For the most part we focus on the case where x is continuously distributed on a compact interval $[\underline{x}, \bar{x}]$ and the cdf has a positive density $f(\cdot|a)$ that is twice (and at one point thrice) continuously differentiable. But we will also consider later the case where x has a Bernoulli distribution. We assume that f satisfies the monotone likelihood ratio property (*MLRP*), so that $l(\cdot|a) \equiv f_a(\cdot|a)/f(\cdot|a)$ is increasing in x. To avoid a nonexistence issue, we assume that l is bounded.

The principal is risk neutral and her expected utility if the agent exerts effort a and she pays a wage w is $B(a) - w$, where B, the expected benefit the principal derives from the agent's effort, is twice continuously differentiable, increasing, and concave in a. In some settings, the signal is output, and thus it is natural to assume that $B(a)$ is the expectation of x given a. But, in others, x is a signal distinct from the eventual profits that the principal will realize from the agent's effort, and so $B(a)$ need not be tied to the expectation of x.

[7] An exception is Laffont and Martimort (2001) (Section 7.1.3), which we discuss below.

[8] We use increasing and decreasing in the weak sense of nondecreasing and nonincreasing, adding 'strictly' when needed, and similarly with positive and negative, and concave and convex. For any function f, we write $(f)_x$ for the total derivative of f with respect to x, and f_x for the partial derivative. We use the symbol $=_s$ to indicate that the objects on either side have strictly the same sign.

[9] In Section 9 we consider an insurance setting with type-dependent reservation utility.

The contracting problem unfolds as follows. The principal offers a menu of contracts that consists of a pair of functions (π, α), where $\pi : [\underline{x}, \overline{x}] \times [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ specifies the compensation the agent receives if he announces type θ, and signal x is observed, and $\alpha : [\underline{\theta}, \overline{\theta}] \to [0, \overline{a}]$ recommends an effort level to each type θ. Given the menu, and knowing his type, the agent decides whether to accept or reject. If he accepts, then he reports a type θ' to the principal and chooses an effort level a'. The realization of x is then observed and the agent is paid $\pi(x, \theta')$. If the agent rejects the menu, then he takes his outside option, which delivers $\bar{u}$.

Let $v(x, \theta') \equiv u(\pi(x, \theta'))$ be the agent's utility from income when he reports θ' and the observed signal is x, and let $\varphi \equiv u^{-1}$ be the inverse of u, which is strictly convex since u is strictly concave. As is standard in the moral hazard literature, it will be convenient to work with the utility of the compensation scheme instead of the wages. The principal is restricted to offer measurable functions v such that for each (θ, a), $\int v(x, \theta) f(x|a) dx$ is well defined. We focus on deterministic menus, i.e., for each type θ, the menu specifies an action $a = \alpha(\theta)$ and a function $v(\cdot, \theta)$.[10] We say that v has the *First Order Property (FOP)* if for each θ, $\int v(x, \theta) f(x|\cdot) dx$ is concave.

By the extended revelation principle (Myerson (1982)), it is without loss of generality for the principal to restrict attention to menus of contracts that are incentive compatible (the agent reports his true type), and where the agent chooses the recommended effort level. For the bulk of the paper, we simplify the exposition by assuming that the principal wishes all types of the agent to participate. We examine optimal exclusion in Section 11.

The principal's problem is thus the following one:

$$\max_{(\alpha, v)} \int_{\underline{\theta}}^{\overline{\theta}} \left(B(\alpha(\theta)) - \int_{\underline{x}}^{\overline{x}} \varphi(v(x, \theta)) f(x|\alpha(\theta)) dx \right) h(\theta) d\theta \tag{P}$$

$$s.t. \qquad \int_{\underline{x}}^{\overline{x}} v(x, \theta) f(x|\alpha(\theta)) dx - c(\alpha(\theta), \theta) \geq \bar{u} \qquad \forall \ \theta \tag{1}$$

$$(\theta, \alpha(\theta)) \in \text{argmax}_{(\theta', a')} \int_{\underline{x}}^{\overline{x}} v(x, \theta') f(x|a') dx - c(a', \theta) \quad \forall \ \theta. \tag{2}$$

That is, the principal chooses α and v to maximize her expected profit subject to the participation and incentive compatibility constraints. For each type, the agent must be willing to accept the menu, report truthfully, and follow the recommended action. Any menu (α, v) that satisfies (1)–(2) is a *feasible* menu.

Problem (P) is, in general, quite intractable. One must deal with the double continuum of deviations available to the agent, since he can both lie about his type, and then choose any action, recommended or otherwise.

[10] In many settings, this seems the economically relevant case. Moreover, in Section 11, we show that when decoupling works, then the optimal deterministic menu remains optimal even when the principal can randomize.

3 Benchmark Cases

We begin with three simpler cases, which will serve as benchmarks for comparison and are also the building blocks of our decoupling program. The first is the complete information setting without adverse selection or moral hazard. The second is the pure adverse selection case. Since the agent is risk averse, there are a few differences with the standard screening problem that we point out. We provide an existence result for a relaxed problem that omits some incentive constraints that also applies to our decoupling program, plus conditions on primitives for the solution of the relaxed problem to be a global optimum. Finally, we go over the pure moral hazard case, and derive some properties of the associated cost minimization problem.

3.1 The First Best

Consider first the case where both θ and a are observable and thus we do not have the incentive constraints (2). Then it is immediate that v is independent of x (since the agent is risk averse), and that (1) binds for all types (the principal extracts all the surplus). Hence, setting $v(x,\theta) \equiv \tilde{v}(\theta)$ for all θ, we have that $\tilde{v}(\theta) = \bar{u} + c(\alpha(\theta),\theta)$. In turn, α is uniquely determined at each θ by $B_a(\alpha(\theta)) = \varphi'(\bar{u} + c(\alpha(\theta),\theta))c_a(\alpha(\theta),\theta)$. Since $c_{a\theta} < 0$ and $\varphi'' > 0$, it follows that the optimal α is strictly increasing in θ. In turn, $\tilde{v}'(\theta) = (c(\alpha(\theta),\theta))_\theta = c_a(\alpha(\theta),\theta)\alpha'(\theta) + c_\theta(\alpha(\theta),\theta)$, which has ambiguous sign, since as θ rises, the agent has lower disutility of effort for any given effort, but is required to exert higher effort. In fact, one can show that $\tilde{v}$ is *decreasing* if c_a/c_θ decreases in a.[11]

3.2 Pure Adverse Selection

Consider now a variation on the pure adverse selection case. To rewrite this problem in a way that will be convenient when we turn to the combined problem with both moral hazard and adverse selection, define $\hat{C}(a,u_0,\theta)$ to be the cost of implementing action a when the agent's type is θ and he has to be given a utility level u_0. When there is no moral hazard, $\tilde{v}$ is independent of x as in the first-best case, and so $\hat{C}(a,u_0,\theta)$ is simply $\varphi(u_0 + c(a,\theta))$. When we turn to the combined problem, $\hat{C}$ will reflect the cost, in the context of a *HM* style relaxed moral hazard problem, of inducing effort level a from type θ when surplus u_0 must be provided. One could also imagine $\hat{C}$ as coming from a situation in which the principal faces a restricted contract space, as for example, linear contracts or simple option contracts.

Let $S(\theta) = \tilde{v}(\theta) - c(\alpha(\theta),\theta)$ be the surplus of type θ when he reports his type truthfully. Then, since $c_{a\theta} < 0$, a standard argument shows that incentive compatibility is equivalent to α

[11] To see this, differentiate $B_a = \varphi' c_a$ to solve for α', and then insert into the expression for $\tilde{v}'$

increasing and the following integral representation of S:

$$S(\theta) = S(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} c_\theta(\alpha(s), s) ds, \tag{3}$$

where $S(\underline{\theta}) = \bar{u}$ by the participation constraint and optimality, and where we recall that $c_\theta < 0$.

When the agent is risk neutral (e.g., Guesnerie and Laffont (1984)), and so $\hat{C}(a, u_0, \theta) = c(a, \theta) + u_0$ is linear in u_0, one can rewrite the objective function to eliminate $S(\theta)$, and then maximize pointwise with respect to the effort level. But here, because the agent is risk averse, $\hat{C}$ is not linear in u_0. As such, the trade-off involved in asking extra effort from a particular type θ depends on $S(\theta)$, which by (3), depends on the effort levels of all types *lower* than θ.

To proceed, we formulate the principal's problem as the following optimal control problem:

$$\max_{\alpha} \int_{\underline{\theta}}^{\bar{\theta}} \left(B(\alpha(\theta)) - \hat{C}(\alpha(\theta), S(\theta), \theta) \right) h(\theta) d\theta \tag{P$_{\text{AS}}$}$$

$$\text{s.t.} \quad \alpha \ \text{increasing} \tag{4}$$

$$S(\theta) = \bar{u} - \int_{\underline{\theta}}^{\theta} c_\theta(\alpha(\tau), \tau) d\tau \ \forall \theta. \tag{5}$$

Ignore for now constraint (4). Then the optimality condition at any given θ is

$$B_a(\alpha(\theta)) - \hat{C}_a(\alpha(\theta), S(\theta), \theta) + \frac{c_{a\theta}(\alpha(\theta), \theta)}{h(\theta)} \int_{\theta}^{\bar{\theta}} \hat{C}_{u_0}(\alpha(t), S(t), t) h(t) dt = 0, \tag{6}$$

which reflects the standard efficiency versus information rents trade-off.[12] In particular, the cost of providing an extra util to all types above θ is $\int_\theta^{\bar{\theta}} \hat{C}_{u_0}(\alpha(t), S(t), t) h(t) dt$, which depends on the surplus and action of all types *higher* than θ. Assume $\hat{C}_{aa} > 0$. In the case of pure adverse selection, this is trivially satisfied, since $\hat{C}_{aa} = \varphi'' c_a^2 + \varphi' c_{aa}$, where φ is strictly increasing and strictly convex. We discuss this further below. Given $\hat{C}_{aa} > 0$, $B_{aa} \leq 0$, and $c_{aa\theta} \leq 0$, the first term of (6) is strictly decreasing in a, and the second term is decreasing in a, and so, if (6) yields a solution $\alpha(\theta)$, then this solution is unique.

A standard property of screening problems is that effort equals its first best level for the most capable agent $\bar{\theta}$. Here, however, $\hat{C}_a$ will in general depend on u_0. In particular, if $\hat{C}_{au_0} > 0$, as when $\hat{C}(a, u_0, \theta) = \varphi(u_0 + c(a, \theta))$, then even the effort of the most capable agent, $\bar{\theta}$, will be lower than the complete information case (although still efficient) due to the information rent $\bar{\theta}$ obtains. The effort level of less capable agents will be distorted downwards from efficiency both because $S(\theta) > \overline{u}$ and because the second term of (6) is strictly negative.

[12]Formally, rewrite (5) as $S' = -c_\theta$ for almost all θ and $S(\underline{\theta}) = \bar{u}$, and let $\eta(\theta)$ be the co-state variable associated with $S' = -c_\theta$. Then the Hamiltonian of the relaxed problem that ignores (4) is $\mathcal{H} = (B - \hat{C})h - \eta c_\theta$, the optimality conditions are $\partial \mathcal{H} / \partial a = 0$ and $\eta'(\theta) = -\partial \mathcal{H} / \partial S$, and the transversality condition is $\eta(\bar{\theta}) = 0$. Algebra yields (6).

The validity of this solution is predicated on two properties. First, we need to know that a solution to (5)–(6) exists, and that such a solution does indeed solve the relaxed problem. In Appendix B we show that it is sufficient that $\hat{C}$ is convex in (a, u_0) for each θ and satisfies an appropriate boundary condition at $a = 0$ and $a = \overline{a}$. Our method of proof is constructive and hence points the way to numerical analysis when (5)–(6) do not admit a closed form solution.[13]

Second, for the case of pure adverse selection, we need the omitted monotonicity constraint (4) to be satisfied (in the decoupling program, we will see that monotonicity is not enough, and indeed a major focus of our analysis will be to find a tractable replacement for (4)). We now search for sufficient conditions under which this is the case. In fact, we will provide conditions under which α is *strictly* increasing, and thus the optimal menu completely sorts types.

Totally differentiating (6) with respect to θ yields, after some algebra,

$$\alpha' = \frac{\left(-c_{a\theta\theta} + c_{a\theta}\frac{h'}{h}\right)\int_{\theta}^{\bar{\theta}} \hat{C}_{u_0} h + c_{a\theta}\hat{C}_{u_0} h - \hat{C}_{au_0} c_{\theta} h + \hat{C}_{a\theta} h}{(B_{aa} - \hat{C}_{aa})h + c_{aa\theta}\int_{\theta}^{\bar{\theta}} \hat{C}_{u_0} h}. \tag{7}$$

Given our assumptions, the denominator is strictly negative, and so $\alpha' > 0$ if and only if the numerator is strictly negative. We show in the Appendix that this is the case if h is log-concave and $-c_{a\theta}$ is log-convex in θ.[14]

3.3 Pure Moral Hazard

The last benchmark case is the one where θ is observable but the action is not, i.e., there is moral hazard but no adverse selection. Begin by defining $C(a, u_0, \theta)$ as the cost of implementing action a for type θ given that he needs to receive utility level u_0. That is,

$$\begin{aligned} C(a, u_0, \theta) = \min_{\hat{v}} & \int_{\underline{x}}^{\overline{x}} \varphi(\hat{v}(x)) f(x|a) dx \\ s.t. \quad & \int_{\underline{x}}^{\overline{x}} \hat{v}(x) f(x|a) dx - c(a, \theta) \geq u_0 \\ & a \in \text{argmax}_{a'} \int_{\underline{x}}^{\overline{x}} \hat{v}(x) f(x|a') dx - c(a', \theta). \end{aligned} \tag{8}$$

Consider the relaxed version of (8) in which the incentive constraint is replaced by the first-order condition $\int_{\underline{x}}^{\overline{x}} \hat{v}(x) f_a(x|a) dx - c_a(a, \theta) = 0$. Under well understood primitives, the solution to this problem will satisfy *FOP*, and hence be feasible and thus optimal.[15] We will assume this

[13] Because our analysis in Appendix B covers a general $\hat{C}$, it subsumes the case of pure adverse selection where the agent's utility is not additively separable in income and effort.

[14] There is always complete sorting on an interval including $\bar{\theta}$, even without the extra structure on h and c.

[15] See Rogerson (1985), Jewitt (1988), and a large literature that follows thereon.

going forward. We will also maintain the assumption that such a solution exists.[16] Following Holmstrom (1979) and Mirrlees (1975), if we let λ and μ be the Lagrange multipliers for the participation constraint and for the agent's first-order condition in the relaxed problem, then the cost-minimizing compensation scheme $\hat{v}$ solves $\varphi'(\hat{v}(x)) = \lambda + \mu l(x|a)$ for all x, where λ and μ are functions of a, u_0, and θ, and where, as usual, since l is increasing in x, so is $\hat{v}$.[17] We denote the compensation scheme that solves this program by $\hat{v}(\cdot, a, u_0, \theta)$.

Having solved for C, the solution to the principal's problem is, for each type θ, to choose a to maximize $B(a) - C(a, \bar{u}, \theta)$. Since the type of the agent is observable (no adverse selection), two properties are true. First, there are no information rents, and so each type of agent receives $\bar{u}$. Second, the problem *decouples* in the sense that there is no interaction between the moral hazard problem the principal faces at one type versus another. The thrust of much of what follows is to explore when a version of this sort of decoupling is valid even with adverse selection.

From now on we will assume that $C_{aa} > 0$. It is intuitive that this should be true, but primitives are not trivial to find. One such set is provided by Jewitt, Kadan, and Swinkels (2008), who focus on the behavior of a measure of the local informativeness of the output about effort. Given *MLRP*, a useful case where their information condition is always satisfied is if F_{aa} is everywhere zero (see Jewitt, Kadan, and Swinkels (2008), Example 2). Another condition for $C_{aa} > 0$ is provided by Chade and Swinkels (2019) (henceforth *CS*), who show that under mild conditions on the utility function, $C_{aa} > 0$ will always hold if $\bar{u}$ is sufficiently large.

Given $C_{aa} > 0$, a is implicitly defined as a function of $\bar{u}$ and θ by $B_a - C_a = 0$, and so, if $C_{a\theta} \leq 0$, then the optimal action the principal implements is increasing in θ, and if $C_{au_0} \geq 0$, then it decreases in $\bar{u}$. The second result is significant, because as we will see, given decoupling, the principal will indeed treat each type θ as facing a pure moral hazard problem, but one in which the agent, except the lowest type, has information rents, and hence has to be given utility higher than $\bar{u}$. Thus, when moral hazard is combined with screening, the principal will distort effort *down* from the pure moral hazard case both because C_a is higher when facing the agent who is gathering rents, and because higher effort for one agent implies higher rents for higher agents.

Given this, let us dive for a moment into when C_{au_0} is positive and $C_{a\theta}$ is negative. The Envelope Theorem yields $C_{u_0} = \lambda$, and hence, since the Lagrange multipliers are continuously differentiable (see Jewitt, Kadan, and Swinkels (2008)), $C_{au_0} = \lambda_a$. Similarly, $C_\theta = \lambda c_\theta + \mu c_{a\theta}$, and hence $C_{a\theta} = \lambda_a c_\theta + \lambda c_{a\theta} + \mu_a c_{a\theta} + \mu c_{aa\theta}$. Given our assumptions, it is thus sufficient to show

[16] Mirrlees (1975) and Moroni and Swinkels (2014) each point out problems regarding the existence of a solution to the relaxed problem. Boundedness of l rules out the first. We tackle the other in Appendix C, providing an existence result. This appendix also justifies interchanges of differentiation and integration that we use repeatedly. We henceforth ignore these issues.

[17] For some utility functions such as $u(w) = \sqrt{w}$, there is the implicit constraint $w \geq 0$, and for some specifications of $\bar{u}$ and c, this constraint will bind at the optimal contract. While our analysis could be extended to this case, this adds complexity which is distracting for the purposes of this paper. For $\bar{u}$ sufficiently large, this constraint will not bind, and we will henceforth assume that this is so.

that each of λ_a and μ_a are positive. We show in Appendix A.1 (see Lemma 6) that this is the case if l is submodular, f is log-concave in a, and ρ is concave, where ρ is the well-known mapping introduced by Jewitt (1988), which maps $1/u'$ into utility. (Formally, let ψ map $1/u'$ into money, i.e., ψ solves $1/u'(\psi(\tau)) = \tau$. Then ρ is given by $\rho(\tau) = u(\psi(\tau))$.)

4 Necessary Conditions

Let us now turn to the central topic of the paper, the setting with both moral hazard and adverse selection. We begin by deriving a set of necessary conditions for a menu (α, v) to be feasible. To do so, we will need to assume that a variety of objects are well-defined.

Definition 1 *A menu (α, v) is regular if (i) everywhere that α is differentiable in θ, so is v; (ii) for all θ, there are $\bar{v}(\cdot, \theta)$ and $\underline{v}(\cdot, \theta)$ such that as $\varepsilon \downarrow 0$, $v(\cdot, \theta + \varepsilon)$ converges uniformly to $\bar{v}$ and $v(\cdot, \theta - \varepsilon)$ converges uniformly to $\underline{v}$.*

At many places regularity is more than we need, but imposing it simplifies and clarifies the exposition. Since regularity is imposed on the endogenous object v, when we turn to the decoupling program, we will show that the solution to that program is indeed regular. And since we will also provide *primitives* for the candidate (α, v) generated by the decoupling program to be a global optimum, it will follow that optimal menus are regular in all such cases. For the sake of simplicity, we will also assume that anywhere that α is differentiable each of $\int v f_a$, $\int v_\theta f$, $\int v_\theta f_a$, $\int v_{\theta\theta} f$ and $\int v f_{aa}$ are well defined and finite. Again, we will show later that any solution to the decoupling program will satisfy these properties.

Let us begin by collecting the necessary conditions for feasibility. Let $S(\theta) = \int v(x, \theta) f(x|\alpha(\theta))dx - c(\alpha(\theta), \theta)$ be the surplus of type θ.

Definition 2 *Menu (α, v) satisfies the basic feasibility conditions (*BFC*) if α is increasing, with $S(\underline{\theta}) \geq \bar{u}$,*

$$\int v(x, \theta) f_a(x|\alpha(\theta))dx = c_a(\alpha(\theta), \theta), \text{ and} \tag{9}$$

$$S(\theta) = S(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} c_\theta(\alpha(s), s)ds, \tag{10}$$

and where anywhere that α is differentiable, we have $\alpha' > 0$,

$$\int v_\theta(x, \theta) f(x|\alpha(\theta))dx = 0, \tag{11}$$

and

$$\int v_\theta(x, \theta) f_a(x|\alpha(\theta))dx \geq 0. \tag{12}$$

Proposition 1 *Let (α, v) be regular and feasible. Then, (α, v) satisfies BFC.*

That α is strictly increasing reflects basic incentive compatibility. Intuitively, if a menu recommends the same action to two different types, it must yield the same expected utility of income to both (otherwise both would strictly prefer the contract with the highest expected utility of income). But, since c is strictly submodular and (9) holds, the contract for the lower type must offer strictly higher marginal incentives for effort. Thus the higher type would strictly profit from announcing the lower type and choosing a strictly higher action than the recommended one. A strictly increasing α is necessary to prevent these double deviations.

The first three displayed equations in order ensure that locally, the agent does not want to report truthfully but deviate from the recommended action, change both his announcement and action along the locus $(\theta, \alpha(\theta))$, or misreport his type while following the recommended action.[18]

Equation 12 is a second order necessary condition. It says that if type θ were to raise his announcement by a little bit, then the contract he would face would change in such a way as to strengthen his marginal incentive to work. If this condition was not met, then a double deviation of announcing a lower than truthful type and then taking a higher than recommended action would be attractive. Interestingly, (12) captures *all* of the conditions for the relevant Hessian to be negative semi-definite.

Differentiating (9), one arrives at

$$0 \leq \int v_\theta (x, \theta) f_a (x|\alpha(\theta)) dx = (c_a (\alpha(\theta), \theta))_\theta - \alpha'(\theta) \int v(x, \theta) f_{aa} (x|\alpha(\theta)) dx.$$

using Proposition 1. But then, where α is differentiable, (12) holds if and only if

$$\alpha'(\theta) \geq \frac{c_{a\theta}(\alpha(\theta), \theta)}{\int v(x, \theta) f_{aa}(x|\alpha(\theta)) dx - c_{aa}(\alpha(\theta), \theta)} > 0, \tag{13}$$

where the denominator is negative by *FOP*, and the numerator is strictly negative. Hence, unlike the pure adverse selection case, not only must α be strictly increasing, but also α' must be *bounded away from 0.* Note that (13) must hold even if there are only a finite number of compensation schemes offered, as would for example be the case if there was a menu cost to the principal per compensation scheme. Indeed, if a single compensation scheme is offered to all types, then $\int v_\theta (x, \theta) f_a (x|\alpha(\theta)) dx = 0$, but different agents would indeed choose their actions in such a way that (13) held.

[18] Wherever α is differentiable, any two of (9)–(11) imply the third. This is intuitive. The agent chooses his report and action within the two dimensional space of reports cross actions and (using that $\alpha' > 0$) each of the three local deviations can be expressed as a linear combination of the other two.

5 A First Sufficient Condition: Increasing Marginal Costs

Let menu (α, v) be regular and satisfy *FOP* and *BFC*.[19] In this section, we provide a sufficient condition on (α, v) under which (α, v) is feasible. The condition has a meaningful economic interpretation and interesting economic implications. We will shortly provide broad classes of primitives under which it is satisfied.

We begin with a preliminary result. Let the graph of α be denoted by $L \equiv \{(\theta, \alpha(\theta)) : \theta \in [\underline{\theta}, \overline{\theta}]\}$. We now that show that 'on locus,' type θ is strictly harmed by misstating his type.

Lemma 1 *If the menu (α, v) satisfies BFC, then the agent strictly prefers prefers to report his true type and take the recommended action to any other report-action pair on L.*

Our sufficient condition for feasibility focuses on $c_a(\alpha(\theta), \theta)$, the marginal cost of effort of a compliant agent as a function of his type θ. Say that the menu (α, v) satisfies the *increasing marginal cost condition (IMC)* if $c_a(\alpha(\cdot), \cdot)$ is increasing. That is, as θ increases, α increases fast enough that the increase in c_a driven by c_{aa} overwhelms the decrease in c_a driven by $c_{a\theta}$. By (13) and given *FOP*, *IMC* is stronger than (12), and hence captures a stronger sense in which more capable agents face stronger incentives for effort. Our first theorem states that *IMC* is sufficient for feasibility.

Theorem 1 *If the menu (α, v) is regular and satisfies BFC, FOP, and IMC, then it is feasible.*

The force of *IMC* is to show that "double" deviations are suboptimal. Fix a true type θ_T. Now, starting from any point $(\theta_A, \alpha(\theta_A))$ on the locus, consider increasing the action alone (see Figure 1). This generates an initial return that is less than $c_a(\theta_A, \alpha(\theta_A))$, the marginal cost on locus of the agent whose type is being announced. By *FOP*, this return deteriorates as one moves vertically further up. But, moving up and to the right generates returns equal to the marginal cost of the agent *who is active* at that point on the locus, and by *IMC*, this is better than the returns to vertical movement. In particular, any movement from left to right along the locus is irrelevant by the first order condition with respect to the announced type (11). Thus, for any double deviation, there is an even better on-locus devation available, and so, by Lemma 1, the agent is better off still to state his true type and take the recommended action.

The proof of the proposition extends this argument to the case where v and α can change discontinuously, with α jumping at some points. This is of much more than just technical interest: if for any reason, the principal can offer only a finite number of compensation schemes (as for example due to a menu cost), then jumps in α will be an integral part of the solution. Nonetheless, *IMC* remains a sufficient condition for feasibility.

[19] Recall that a menu (α, v) satisfies *FOP* if $\int v(x, \theta) f_a(x|\cdot)dx$ decreases in a for all θ.

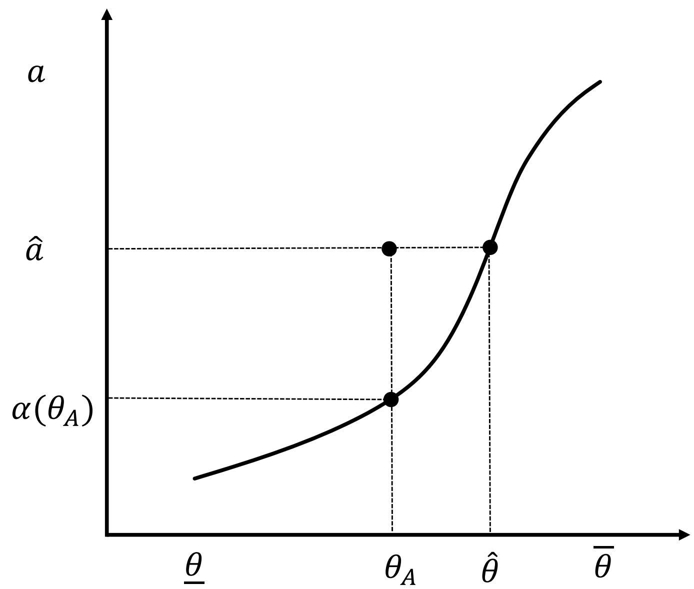

Figure 1: ***IMC.*** Under *IMC*, a deviation to $(\theta_A, \hat{a})$ is dominated by the on-locus deviation $(\hat{\theta}, \hat{a})$, which in turn is dominated by telling the truth and taking the recommended action.

5.1 *IMC* and The Holmstrom-Mirrlees Property

Say that a menu (α, v) has the *Holmstrom-Mirrlees Property* (*HMP*) if, for almost all θ, $v(\cdot, \theta)$ is the *HM* contract given $\alpha(\theta)$. That is, there are λ and μ such that $\varphi'(\tilde{v}(\cdot)) = \lambda + \mu l(\cdot|a)$. Our next proposition shows that if an optimal menu satisfies *IMC*, then it must also satisfy *HMP*.

Proposition 2 *Let* (α, v) *be an optimal menu that satisfies IMC. Then,* (α, v) *has HMP.*

Proof Let (α, v) be optimal and satisfy *IMC*, but not *HMP*. For each θ, let $\tilde{v}(\cdot, \theta)$ be the *HM* contract implementing $\alpha(\theta)$ at surplus $S(\theta)$. Since the *HM* solution is unique, $\tilde{v}$ is strictly cheaper than v where $\tilde{v}(\cdot, \theta) \neq v(\cdot, \theta)$. But, then, since *IMC* depends only on α, and by Theorem 1, $(\alpha, \tilde{v})$ is both feasible and strictly more profitable than (α, v), a contradiction. □

Proposition 2 adds to the empirical content of the results. If one can observe enough to confirm *IMC* (e.g., if one can see either f and the menu of contracts, or c and α), then Proposition 2 gives the (potentially testable) implication that the underlying menu has *HMP*.

6 Linear Output

Consider the *linear probability* case, $f_{aa} = 0$, commonly used in applications, especially when output is Bernoulli, or is continuously distributed with effort linearly mixing between two distributions. This section makes two main contributions. First, we show that the combined problem decouples and becomes essentially as tractable as a (non-quasilinear) screening problem, with the sole difference being that the (necessary and sufficient) monotonicity constraint on the action in the pure screening problem is replaced by a stronger (necessary and sufficient) monotonicity constraint in the combined problem. Second, we provide simple and easy to check primitives under which the solution to the relaxed screening problem in which this constraint is omitted in fact satisfies it, and so is in fact optimal. Together, this provides a complete "plug and play" environment to study simultaneous moral hazard and screening.

As the following proposition shows, *BFC* and *IMC* together characterize feasibility.

Proposition 3 *If (α, v) is regular and $f_{aa} = 0$, then (α, v) is feasible iff BFC and IMC hold.*

The key to the proof is that (9) is an identity, and so, since $f_{aa} = 0$, $(c_a(\alpha(\theta), \theta))_\theta = \int v_\theta f_a$, where the last expression must be positive by (12) so as to prevent double deviations. Hence *IMC*, which together with *BFC* is sufficient by Proposition 1, is also necessary.

It follows by Proposition 2 that any optimal menu in the linear case has *HMP*. This is somewhat remarkable. Despite the double continuum of incentive constraints, in any optimal menu, the contract facing an agent of any given type is of the *HM* form. There is thus a very strong prediction for the form of optimal menus. The fact that *HMP* must hold in the linear setting also points to a massive simplification of (P). Recall that $C(a, u_0, \theta)$ is the cost of implementing effort level a at utility u_0 for type θ in the relaxed moral-hazard problem. Then, we assert, (P) can be written simply as

$$\max_{\alpha} \int_{\underline{\theta}}^{\overline{\theta}} \left(B(\alpha(\theta)) - C(\alpha(\theta), S(\theta), \theta)\right) h(\theta) d\theta \qquad (\mathrm{P_L})$$

$$\text{s.t. } \textit{IMC} \text{ and } S(\theta) = \bar{u} - \int_{\underline{\theta}}^{\theta} c_\theta(\alpha(\tau), \tau) d\tau \ \forall \theta,$$

where one then constructs the requisite v using the relevant *HM* contract at each θ. To see this, note first that, by *HMP*, we can replace the integral $\int \varphi(v(x, \theta)) f(x|\alpha(\theta)) dx$ in the objective function by $C(\alpha(\theta), S(\theta), \theta)$. But, in the linear case, the participation and incentive constraints, (1)–(2), are equivalent to *BFC* and *IMC*. Next, as argued in Footnote 18, to establish *BFC*, it suffices to include two of conditions (9)–(11), and in this case the first-order condition (9) holds since $v(\cdot, \theta)$ solves the *HM* problem, and condition (10) is explicitly given. Finally, since $v(\cdot, \theta)$ is *HM*, *FOP* holds by assumption.[20]

Note that problem ($\mathrm{P_L}$) differs from ($\mathrm{P_{AS}}$), the problem in the pure adverse-selection setting, only in that monotonicity of α has been strengthened to *IMC*. Thus, this is a problem to which standard ironing techniques can be applied. Further, the version of this problem in which *IMC* is omitted is exactly the same as the version ($\mathrm{P_{AS}}$) in which monotonicity is omitted. This problem,

[20] We will establish below that (α, v) constructed from ($\mathrm{P_L}$) is in fact regular. See Lemma 3.

which we analyzed in Section 3.2, is both tractable and of a familiar form that gives considerable economic insight. Hence, exactly as it is interesting to know when the solution to the relaxed screening problem in fact satisfies monotonicity, it is interesting to know when that solution also satisfies *IMC*. When this is true, then the solution to the rather forbidding problem (P) *decouples* into two well-understood problems, a pure screening problem and a pure moral hazard problem.

Our next two results exhibit a broad class of primitives under which *IMC* holds at a solution to the relaxed problem (we will exhibit more primitives which also apply to the linear case in the next section). We begin with an assumption to control one set of forces.

Assumption 1 *For each a, the function $-c_{a\theta}(a,\cdot)/(c_{aa}(a,\cdot)h(\cdot))$ is decreasing.*

This trivially holds if $c(a,\theta) = (1-\theta)a + a^2$ and h is increasing. For many natural cost functions, $c_{aa\theta} < 0$, which is a force in the wrong direction. Thus, if $c(a,\theta) = (2-\theta)a^2/2$, where $\theta \in [0,1]$, then the assumption holds only if h increases fast enough.

Recall that ρ maps $1/u'$ into utility. Our next result will use that ρ is concave, which holds for the most commonly used utility functions in applications.

Proposition 4 *Let Assumption 1 hold, $c_{aa}/c_{a\theta}$ be increasing in a, $F_{aa} = B_{aa} = 0$, $skew_F(l) \leq 0$, and ρ be concave. Then, any solution to the relaxed version of (P_L) in which IMC is omitted in fact satisfies IMC and hence is optimal.*

The proof depends on a result in *CS*, and a covariance inequality. Negative skewness of l is not innocuous, but, as the following Lemma shows, holds for many simple examples.

Lemma 2 *Let f_L and f_H be densities on $[0,1]$, with f_L increasing, and f_H/f_L increasing and concave. Let $f(x|a) = af_H + (1-a)f_L$ be the linear combination of f_L and f_H. Then, for all a, $skew_F(l) \leq 0$.*

6.1 Two Outcomes

A special case in which there is linear output is the model with two outcomes, where we can assume that the probability of the high outcome is simply a, folding any 'curvature' assumptions into $c(a,\theta)$. This setting is of considerable economic interest–in particular, it will be the foundation of our later analysis of an insurance application.

Because the probability of a good outcome is linear in effort, it remains the case that (α, v) is feasible if and only if *IMC* holds. We will use that fact to derive primitives for the solution to the relaxed program (P_L) to be feasible and hence optimal. In this section we use the standard interpretation for B as the expected value of output, so that $B_{aa} = 0$. Denote by v_h and v_ℓ the utility levels that the agent receives after the high and low outcomes, respectively. These are uniquely tied down by the participation and incentive constraints and given by $v_h = u_0 + c(a,\theta) + (1-a)c_a(a,\theta)$, and $v_l = u_0 + c(a,\theta) - ac_a(a,\theta)$. Hence, $C(a,u_0,\theta) = (1-a)\varphi(v_l) + a\varphi(v_h)$.

Proposition 5 *Sufficient for C to be strictly convex in a is*

$$\frac{c_{aaa}}{c_{aa}} \geq \frac{3a-2}{a(1-a)}, \tag{14}$$

while if Assumption 1 holds, then sufficient for IMC to hold strictly at any solution to the second-stage screening problem is

$$\frac{c_{aa\theta}}{c_{a\theta}} + \frac{c_{aaa}}{c_{aa}} \geq \frac{1-3a}{a(1-a)}. \tag{15}$$

Note that (14) holds trivially if $c_{aaa} \geq 0$, and $a \leq 2/3$, and holds for $a > 2/3$ as long as c_{aa} grows quickly enough in a. Similarly, (15) holds trivially if $a \geq 1/3$ and $c_{aaa} \geq 0$, and otherwise holds if $c_{a\theta}$ grows sufficiently quickly in proportionate terms ($c_{a\theta}$ must grow very quickly near zero in the natural case where $c_{a\theta}$ is identically zero at $a = 0$). Recall also that by Theorem 2, if c_{aa} and c_{aaa} are finite for all $a \in [0, 1]$, then making u_0 large enough yields $C_{aa} > 0$ and strict *IMC* without either (14) or (15) holding.

Example 1 *Let $a \in [0, 1]$, with $c(a, \theta) = (1 - \theta)((1/(1 - a)) - a)$. Then, $c_{aaa}/c_{aa} = 3/(1 - a)$, which is greater than $(3a - 2)/(a(1 - a))$ for all a. Algebra shows that (15) is satisfied as well.*

Example 2 *Let $c = (1 - \theta)\, a^2/2$. Then, (15) is satisfied for all a. But, (14) fails for all $a \geq 2/3$, and so C is guaranteed to be convex only on $[0, 2/3]$.*

7 Decoupling

In the linear case, the fundamentally intractable problem (P) reduces to a much simpler problem. First one solves for a cost function in the relaxed moral-hazard problem for any given action, surplus and type. Then, one uses that cost function as an input to a pure screening problem. Each problem, as we have already seen, is fundamentally tractable. In more general problems, *IMC* is no longer necessary for feasibility. But, from Theorem 1, it remains sufficient. Thus, it is of fundamental interest to know under what conditions the solution to the relaxed version of problem ($\mathrm{P_L}$) satisfies *IMC*. In this section, we explore when this *decoupling* process works more generally. When decoupling works, not only does it make the analysis tractable but it also yields a number of insights about optimal menus.

Consider the principal's problem (P). Denote by G the set of feasible mechanisms, i.e., those that satisfy the global incentive constraints (1)–(2). Then, the principal's optimized profits satisfy

$$\max_{(\alpha,v)\in G} \int \left(B(\alpha(\theta)) - \int \varphi(v(x,\theta)) f(x|\alpha(\theta)) dx \right) h(\theta) d\theta \leq \max_{\alpha} \int (B(\alpha(\theta)) - C(\alpha(\theta), S(\theta), \theta)) h(\theta) d\theta,$$

where $S(\theta) = \bar{u} - \int_{\underline{\theta}}^{\theta} c_\theta(\alpha(s), s) ds$. To see the inequality, note we have already shown that any feasible menu satisfies (9), so that the agent does not want to locally change his action having honestly reported his type, and that $C(\alpha(\theta), S(\theta), \theta)$ is by definition the cost of the least-cost compensation scheme subject only to (9) and giving expected utility equal to $S(\theta)$. So, the right-hand side (*rhs*) of the above inequality is the principal's expected profit in a substantially relaxed version of the original problem.

The *rhs* expression effectively nests a moral hazard problem within an adverse selection one. As such, it suggests the following three-step *decoupling* scheme.

Step 1(Moral Hazard): Cost Minimization for Each (a, u_0, θ). For each type θ, action a, and utility u_0, solve the moral-hazard problem as described in Section 3.3, thus constructing the cost function C, and associated contract $\hat{v}$.

Step 2 (Adverse Selection): Profit Maximization with Respect to α. Having constructed the cost function C, solve the relaxed pure adverse selection problem

$$\max_{\alpha} \int (B(\alpha(\theta)) - C(\alpha(\theta), S(\theta), \theta))h(\theta)d\theta,$$

subject to the condition (5) on the surplus function S, but where we initially ignore any condition on α. Hence we are back to the problem analyzed in Section 3.2.

Step 3 (Verification): Candidate Solution and Feasibility. For each θ, let $v(\cdot, \theta) = \hat{v}(\cdot, \alpha(\theta), S(\theta, \alpha), \theta)$. That is, for each θ, $v(\cdot, \theta)$ is the contract solving the relaxed moral hazard problem given the action and surplus generated for θ by the adverse-selection problem solved in Step 2. Since (α, v) solves a relaxed problem, if it is feasible, then it is optimal. But, our candidate solution satisfies *BFC* and *FOP* by construction. Hence in particular, it will be feasible (and thus optimal) if (α, v) satisfies *IMC*.

Decoupled menus exhibit a number of interesting properties. Most centrally, for each θ, $v(\cdot, \theta)$ is an *HM* contract, and so reflects the standard risk sharing versus incentives trade-off, with the likelihood ratio playing a prominent role. This provides both economic insight and empirical content. It is also very useful on a technical level–at multiple points in our construction, we leverage technical properties of the moral hazard problem to facilitate our analysis.

At the same time, because the second stage is a screening problem, all types except for the lowest one obtain an information rent, which is pinned down exactly as in the pure adverse selection case but reflects C. Since every type except for the lowest obtains utility higher than $\bar{u}$, if $C_{au_0} \geq 0$ (see Lemma 6 in Appendix A.1 for sufficient condtions), then as in the discussion of the pure screening problem the action of each type implemented by the principal is *lower* than that in the pure moral hazard case (where all types obtain $\bar{u}$). Once again, the fact that a screening problem is being solved is useful both in terms of economic insight, and at a technical level, as, for example, in allowing us to prove existence of a candidate decoupled menu.

The sufficiency result for feasibility in Theorem 1 assumes that (α, v) is regular, as defined in

Section 5. We close this section with a technical result showing that decoupled menus are indeed *regular*, as long as α is strictly increasing.[21]

Lemma 3 *The functions $\hat{v}$, and C are twice continuously differentiable in their arguments. Let (α, v) be a solution to the decoupling problem. If $C_{aa} > 0$, and if α is strictly increasing, then α is continuously differentiable, and (α, v) is regular.*

8 A General Case where Decoupling Works

We now exhibit a very general setting where decoupling works. With no extra structure on c or F, and only relatively mild conditions on u, we show that if the reservation utility of the agent, $\bar{u}$, is sufficiently high, then *IMC* must hold at any decoupled menu. Of course, when $\bar{u}$ is *very* large, the principal may simply prefer not to hire the agent. Our main result is thus most relevant in settings where the stakes are high on both sides: the agent has a high outside option, but the principal very much needs the agent. For example, if x is output, then our results will be relevant if the price of that output is sufficiently high.

We require first a regularity condition on u.

Assumption 2 *As $w \to \infty$, $u \to \infty$, and $u' \to 0$.*

Recall that the coefficient of absolute prudence, P, is $u'''/-u''$, and of absolute risk aversion, A, is $-u''/u'$. We require that A/u' converges to zero as w grows.

Assumption 3 *As $w \to \infty$, $A/u' \to 0$.*

As Chade and Swinkels (2019) (*CS*) show, since $A/u' = -u''/[u']^2 = (1/u')'$, this says that as $w \to \infty$, the cost of providing an extra util to the agent converges to a constant over intervals of wealth of any fixed length.

Our final assumption on u states that prudence stays within some finite ratio of absolute risk aversion as w diverges. For convenience, we assume that P/A has a well-defined limit.

Assumption 4 $\lim P(w)/A(w) \in [0, \infty)$.

As *CS* show, this assumption is implied by the very weak condition that for some large N, $-1/(u')^N$ is concave in w.

Example 3 *Consider the HARA utility functions $u = (1-\gamma)\kappa^{\gamma}/\gamma$, where $\kappa = aw/(1-\gamma) + b$. These include the linear, log, CARA, and CRRA cases. CS show that Assumptions 2-4 hold for $\gamma \in (0, 1)$. When $\gamma = 0$, $u = \log w$, so that $A/u' = 1$ and Assumption 3 fails.*

[21] That α is strictly increasing is immediate if *IMC* holds, and will also hold under *SCC*, defined below.

Under these assumptions, Lemma 6 in *CS* establishes the following result.

Lemma 4 *As $\bar{u} \to \infty$, μ/λ, μ_a/λ, and λ_a/λ go to 0, while $\mu \int \rho' f$ and $\mu_a \int \rho' f$ remain bounded.*

The intuition for the dominance of λ over other terms is that, in the moral hazard problem, relaxing the participation constraint involves adding utility everywhere, which is increasingly expensive as the outside option rises. But, relaxing the incentive constraint involves both adding utility at high outcomes, and removing it at low outcomes, and so there is an offsetting force, and objects involving μ grow more slowly than λ. One key point in proving this lemma is that when $\bar{u}$ is large, $v(\bar{x}) - v(\underline{x})$ remains finite (*CS* Lemma 3), with the intuition being that otherwise the incentive constraint will be violated (on the side of excessive incentives). A second key point in proving this lemma uses Assumptions 2-4 to show that when $\bar{u}$ is large, ρ' becomes effectively constant over the relevant range. The intuition for $\mu \int \rho' f$ being finite involves the result that $v(\bar{x}) - v(\underline{x})$ remains finite. See *CS* for details.

We can now state the central theorem of this section.

Theorem 2 *Let $F \in \mathcal{C}^4$, let Assumptions 1–4 hold, and let $\overline{a} < \infty$. Then for all $\bar{u}$ sufficiently large, the decoupling problem delivers a solution that satisfies IMC, and is thus optimal.*

The proof starts from Lemma 8 in the Appendix, which shows that the relevant condition involves the term λc_{aa}, and a variety of other terms, that by Lemma 4 are dominated.

We stress that Theorem 2 relies on a broad class of primitives that significantly enlarges the applicability of decoupling to contracting problems with moral hazard and adverse selection.

8.1 Square Root Utility

It is in general difficult to tie down how large $\bar{u}$ must be for IMC to apply, partly because it is difficult to obtain a closed form expression for C from the cost minimization problem. One case that is tractable is when the agent has square-root utility $u(w) = \sqrt{2w}$, so that $1/u' = u$ and ρ is the identity function. Then, letting $\hat{I}(a) = 1/\int l^2 f$ be the reciprocal of the Fisher Information that the output carries about a, we have the following result.

Proposition 6 *If Assumption 1 holds and $u = \sqrt{2w}$, $B_{aa} = 0$, $\hat{I}_a \geq 0$, $\hat{I}_{aa} \geq -2$, and $c_{aa}/c_{a\theta}$ decreasing in a for each θ, then a sufficient condition for IMC is that $\bar{u} + c(0, \underline{\theta}) \geq 0$, which is simply to say that the agent prefers to stay home than to work at zero income.*

It is not hard to satisfy the premise about $\hat{I}$. Indeed, if $f(x|a) = e^{-x/a}/a$, $a \in [0, 1]$, $x \geq 0$, then $\hat{I} = a^2$. The same holds if one truncates the exponential distribution at some $\hat{x} > 0$ (to bound the likelihood ratio).

Proposition 6 shows that if $\hat{I}$ is well-behaved, then the required $\bar{u}$ is anything but large![22] More generally, we suspect that the form of Proposition 2 has much more to do with the absence of closed-form solutions than it does with $\bar{u}$ needing to be particulary large in practical examples.

8.2 Failures of Decoupling

Decoupling is very convenient when it works. But, it need not always do so. Hence, conditions like *IMC* (and *SCC*, analyzed in Section 10), and results establishing primitives for them, are crucial. To see that decoupling can fail, let us return to the two-outcome case analyzed above, where *IMC* is necessary and sufficient for feasibility.

Example 4 *Let $h = 1$ on $[0,1]$, $c(a,\theta) = (1-\theta)a^2/2$, $a \in [0, 2/3]$ (to ensure $C_{aa} > 0$), and $\phi(v) = v^2/2$ (square-root utility). The reciprocal of the Fisher Information is $\hat{I} = a(1-a)$, and IMC fails in the solution to the decoupling program for an interval of types near $\theta = 1$.*

For some intuition, $\mu_a > 0$ is a force towards *IMC* (see (26) in the Appendix). But, because the Fisher Information is increasing for $a > 1/2$, μ_a is negative for such a. Thus, as a increases, information improves in a way that lowers the shadow value of the incentive constraint on effort, and this works against *IMC*. And since in this case *IMC* characterizes the second-order condition, in the decoupled solution types close to $\theta = 1$ can strictly profit by double deviations.

9 Insurance under Moral Hazard and Adverse Selection

We now turn to a major application of our results. We examine a monopolistic insurance market with both moral hazard and adverse selection. Although insurance markets with both moral hazard and adverse selection are of vast real-world importance, little is known about the properties of optimal menus. Unlike the model analyzed so far, here the type of the agent *directly* enters into the risk of a loss, and thus into the principal's profits. Despite this common-value aspect, we show how to reparameterize the model so that it fits into our framework. This allows us to derive a number of interesting properties of optimal menus.

9.1 The Model and Reparameterization

Let e be the level of care an agent can take to increase the probability $p(\cdot, \theta)$ of avoiding a loss of size $0 < \ell \leq \omega$, where ω is the agent's initial wealth. We assume $p_e > 0$, $p_\theta > 0$, $p_{e\theta} \geq 0$, and $p_{ee} \leq 0$. Thus, higher types and higher effort leads to a lower probability of loss, and type and

[22] The square root case is subsumed by Proposition 2. Hence, if the conditions on e.g., $\hat{I}$ are not satisfied, then *IMC* continues to hold, but we must assume a larger $\bar{u}$ to do so.

effort are complements. Let κ be the strictly increasing and convex cost of effort. The agent's outside option–which is to bear the risk themselves–is *type-dependent*.

There is a monopolist insurance firm. If the firm collects premium t from type θ, and provides coverage x, and the agent exercises care e, then the profit of the firm is $t-(1-p(e,\theta))x$. Thus, the setting exhibits *common values*, as θ directly enters into the firm's profit. The firm offers a menu $(\varepsilon(\cdot),t(\cdot),x(\cdot))$, where $\varepsilon(\theta)$ is the recommended effort level for type θ, subject to the usual incentive compatibility and (type-dependent) reservation utility constraints.

To reparameterize this problem as a *private values* setting, think of the agent's choice as the probability of a good outcome, with disutility dependent on his type. Formally, let $a=p(e,\theta)\in[0,1]$. Define $z(a,\theta)$ by $p(z(a,\theta),\theta)=a$, and c by $c(a,\theta)\equiv\kappa(z(a,\theta))$ for all (a,θ).[23] The profit is now $t-(1-a)x$, and we are in a private values setting, where the menu is $(\alpha(\cdot),t(\cdot),x(\cdot))$, with $\alpha(\theta)$ the recommended action level (probability of avoiding a loss) for type θ.

Let us also replace $(t(\cdot),x(\cdot))$ by $v(\cdot)=(v_l(\cdot),v_h(\cdot))$, where $v_h(\theta)=u(\omega-t(\theta))$ is the agent's utility with no loss, and $v_l(\theta)=u(\omega-l+x(\theta)-t(\theta))$ is the agent's utility with a loss. Note that $t(\theta)=\omega-\varphi(v_h(\theta))$ and $x(\theta)=l-(\varphi(v_h(\theta))-\varphi(v_l(\theta)))$, and so the profit on any given type is

$$B(\alpha(\theta))-((1-\alpha(\theta))\varphi(v_l(\theta))+\alpha(\theta)\varphi(v_h(\theta))),$$

where $B(a)=\omega-(1-a)l$. The principal effectively takes ω, pays out the expected loss $(1-\alpha(\theta))l$, and then provides enough income in each state to give (v_l,v_h). The principal then maximizes $\int_{\underline{\theta}}^{\bar{\theta}}[B(\alpha(\theta))-C(\alpha(\theta),S(\theta),\theta)]h(\theta)d\theta$, where C is defined in Section 6.1, subject to incentive compatibility and the type-dependent outside option. For simplicity, we assume that for all u_0 and θ, $B(\cdot)-C(\cdot,u_0,\theta)$ has an interior optimum.[24]

Since we are in the linear probability case, *IMC* is necessary and sufficient for incentive compatibility, where, since $c_a=v_h-v_l$ from the first-order condition on effort, *IMC* is equivalent to $v_h'-v_l'\geq 0$. Thus, the risk faced by the agent is increasing in his type.

We begin by showing that, at any optimal menu, the premium, coverage, and coverage less premium are each *decreasing* in θ. To see this, differentiate t and x with respect to θ and recall that $v_h\equiv S+c+(1-\alpha)c_a$ and $v_l\equiv S+c-\alpha c_a$. Since $t(\theta)=\omega-\varphi_h(\theta)$ and $x(\theta)=l-(\varphi_h(\theta)-\varphi_l(\theta))$, we obtain that $t'=-\varphi_h'(1-\alpha)(c_a)_\theta$, $x'=-((1-\alpha)\varphi_h'+\alpha\varphi_l')(c_a)_\theta$, and $x'-t'=-\varphi_l'\alpha(c_a)_\theta$, each of which is negative from *IMC*.

Let $\alpha_{NI}(\theta)$ be the optimal effort level when θ has no insurance. That is, $c_a(\alpha_{NI}(\theta),\theta)=u(\omega)-u(\omega-\ell)$. Then, the type-dependent outside option is $\bar{U}(\theta)=(1-\alpha_{NI}(\theta))u(\omega-\ell)+\alpha_{NI}(\theta)u(\omega)-c(\alpha_{NI}(\theta),\theta)$, where $\bar{U}'(\theta)=-c_\theta(\alpha_{NI}(\theta),\theta)$ by the Envelope Theorem. It remains the case that by incentive compatibility, $S'(\theta)=-c_\theta(\alpha(\theta),\theta)$ for all types served. Hence, since

[23] If $p(e,\theta)>a$ for all e, then set $c(a,\theta)=0$, and if $p(e,\theta)<a$ for all e, then set $c(a,\theta)=\infty$.

[24] Sufficient for this is that for each θ, $c_a(0,\theta)=0$ and $\lim_{a\to 1}c_a(a,\theta)=\infty$, since then by (31), $C_a(0,u_0,\theta)=0$ and $\lim_{a\to 1}C_a(a,u_0,\theta)=\infty$ as well.

$c_{a\theta} < 0$, if the agent takes less care in the relationship than outside it–something we will establish–then their surplus in the relationship will be shallower than their surplus outside the relationship. Using this, we show that the interval of types served is an interval, with participation binding only for the highest served type.[25] This motivates the following result.

Theorem 3 *In any optimal menu, (i) coverage is positive for all types, and strictly positive for all types in some interval* $[\underline{\theta}, \theta^*)$*,* $\theta^* > \underline{\theta}$*; (ii) the outside option is binding at* θ^**, but for no lower type, and the information rent (the difference between the utility with insurance and without) decreases with type; (iii) effort is distorted upwards from the constrained efficient level* ($B_a - C_a < 0$) *and strictly so for all types except* $\underline{\theta}$*; and (iv) coverage is strictly less than full for all types.*

To see that coverage is positive and decreasing, if there is any type on whom coverage is strictly negative (i.e., the agent pays the principal in the event of a loss) then the set of such types is an interval including $\bar{\theta}$. But, because the agent is risk averse, the principal loses money on any such type, and so excluding them is directly profitable, and–because it reduces the set of feasible deviations–lowers the amount of utility that needs to be offered to types still served.

That a positive measure set of agents are always served is less obvious here than when there is only adverse selection. For example, offering full insurance to the lowest type at an actuarially fair rate–given the zero level of care that will result–may well be strictly worse for the agent than autarky, where care will be exercised. We show that starting from no insurance, the drop in effort inherent in offering a little more insurance is irrelevant, since coverage started at zero, while there is a first order gain (that the principal can capture) from reducing the risk faced by the agent.[26]

To see that effort is distorted strictly *upwards*, note that since the binding participation constraint is on the most capable type served, the way to lower the surplus of inframarginal types is to make the surplus function *steeper*, which is accomplished by raising effort beyond its constrained efficient level. Less-than-full coverage then follows since the constrained efficient effort level is strictly positive, and thus occurs where $c_a > 0$, and thus $v_h - v_l > 0$ by the incentive constraint.

9.2 Distortions

To better understand the distortions that the combined problem entail, let us compare our model with three benchmarks: full information, pure moral hazard, and pure adverse selection.

[25] In general, problems with a type-dependent outside option are amenable to our tools as long as one can establish that the outside action–and hence the slope of the outside option–has a useful relationship to the action and hence the slope of the utility function inside the relationship. If so, then one can pin down that participation binds for example just for the lowest type, just for the highest type, or just at each end of the range of types served.

[26] In a pure adverse selection setting, Hendren (2013) shows that if one allows for the presence of agents who suffer a loss with probability one, then under some conditions the principal excludes all types. Our setting in principle could have an agent who chooses probability one of a loss, but this does not arise at an optimal menu.

Full Information. With full information, the firm will optimally offer full insurance to each type, and require care that solves $B_a(\alpha_{FI}(\theta)) - \varphi'\left(\bar{U}(\theta) + c(\alpha_{FI}(\theta),\theta)\right) c_a(\alpha_{FI}(\theta),\theta) = 0$. Compared to this benchmark even $\underline{\theta}$'s contract is distorted under moral hazard and adverse selection, since he (along with everyone else) obtains strictly less than full coverage.

Pure Moral Hazard. Next, consider pure moral hazard. Optimal care level will solve $B_a(\alpha_{MH}(\theta)) - C_a(\alpha_{MH}(\theta), \bar{U}(\theta), \theta) = 0$, with insurance partial for all types. To the extent that one might typically expect $C_a > \varphi' c_a$, this will result in a lower level of care than in the full information case. Intuitively, inducing more care from the agent requires putting the agent under more risk that the principal must compensate.[27]

In the combined problem, $B_a - C_a < 0$ for $\theta > \underline{\theta}$, pushing towards more care and less insurance than with pure moral hazard. But, all types below $\bar{\theta}$ obtain an information rent, so $S(\theta) > \bar{U}(\theta)$. If φ' is convex (that is, $P \leq 3A$), then $C_{au_0} = \varphi_h' - \varphi_l' + (\varphi_h'' - \varphi_l'')a(1-a)c_{aa} > 0$, pushing towards lower care and more insurance. The lowest type has $B_a - C_a = 0$ and strictly positive information rents, so effort is lower and insurance higher than with pure moral hazard. At the highest type there are no information rents and hence care is distorted upwards. In between, the forces battle. Our conjecture is that care is distorted downward for low types and upwards for high types.

Laffont and Martimort (2001), Chapter 7, analyze a two-type and two-effort case, assuming, *inter alia*, that the principal wants both types to exert high effort. They show that the two informational problems reinforce each other when it comes to distortions. In our setting, where effort varies *endogenously*, things are more subtle. For low types (those with the highest risk of an accident) both the move from full information to pure moral hazard and then the subsequent move to both moral hazard and adverse selection reduce effort and increase coverage. But, at higher types, the forces are countervailing.

Pure Adverse Selection. Finally, consider a setting of pure adverse selection. If only a can be observed, then this problem is simple, falling into our analysis of Section 3.2. Insurance is full, and effort is determined by $B_a - \varphi' c_a = c_{a\theta} \int_{\underline{\theta}}^{\theta} \varphi' h$. Effort is distorted down from full information by the rents of the agent, but up to extract rents from lower types.

But, in most settings, it is e that is observable, not a–the principal can see if the agent's car has modern safety features and is garaged in a safe neighborhood, but not whether the agent is innately attentive. This problem is hard, because the principal can now use two variables–the effort required and the amount of insurance offered–to screen the agent. One can show that beginning from the optimal solution to the combined problem with moral hazard and adverse selection, the principal in the pure adverse selection problem gains by incrementally increasing effort and decreasing insurance (which in the combined problem are tied together via the incentive

[27] Note that $C_a < \varphi' c_a$ can occur–in particular when at high levels of effort, further increases in effort improve information, and hence decrease the risk the agent faces. For example, let $f(0|1) = 0$. Then, as $a \to 1$, $C \to \varphi(u_0 + c)$, since near-forcing contracts become feasible, and so, for some region near $a = 1$, $C_a < \varphi' c_a$.

constraint, but now can be moved independently). Each change reduces information rents to lower types. A fuller characterization and comparison of the two solutions is open.

Remark 1 *The reparameterization that we used in the insurance application is considerably more general. It works any time that the action and the type of the agent enter the distribution of the outcome via an "index." In particular, assume that $\hat{f}(x|e,\theta) \equiv f(x|\eta(e,\theta))$, where η is strictly increasing in e, and where the agent has cost of effort $\kappa(e)$. Then, define $a = \eta(e,\theta)$, and define c by $c(\eta(e,\theta),\theta) = \kappa(e)$. We can then apply the full weight of the machinery developed so far.*

10 Single Crossing and Sufficiency

So far, we have worked with the sufficient condition *IMC*. In this section, we explore an alternative sufficient condition that hinges on a single crossing property of v as θ changes. We will find this condition especially useful below when we turn to the decoupling program and exponential families, a class of distributions for which contracts can only cross in simple ways as θ varies. It will also be central to our further exploration of common values.

Say that v satisfies the *single crossing condition* (*SCC*) if for all $\theta > \theta'$, $v(\cdot,\theta)$ single-crosses $v(\cdot,\theta')$ from below. This can again be interpreted as incentives getting stronger as θ increases.[28]

Our next theorem establishes that *SCC* also implies feasibility.

Theorem 4 *If the menu (α, v) is regular and satisfies BFC, FOP, and SCC, then it is feasible.*

For intuition, consider a deviation $(\theta_A, \hat{a})$, where $\hat{a} = \alpha(\hat{\theta})$ for some $\hat{\theta} > \theta_A$, and so we are above the locus. We will show that the agent is better off, holding fixed the action at $\hat{a}$, to increase his announcement, sliding horizontally to the right until $\hat{\theta}$ is reached, and we are back on locus. In particular, consider any $\theta < \hat{\theta}$, and, consider the effect of a small increase in the announced type. Under *SCC* this increases the agent's income at high signals and lowers it at low signals. On the locus, the agent is indifferent about this trade-off by (11). But then, above the locus, where he is working harder, and thus more likely to attain high signals, the trade-off is profitable.

10.1 Decoupling with Exponential Families

Condition *SCC* gives another tractable class where we can provide primitives for decoupling to be valid–the exponential families. These includes a vast variety of common parameterized density functions including the Bernoulli, normal, exponential, Poisson, Gamma, Beta distributions, and

[28] Note that neither *SCC* nor *IMC* implies the other. To see the first direction, note that it may be that *IMC* holds, but that the contracts at different θ cross more than once, so that *SCC* fails. To see the other direction–that *SCC* can succeed while *IMC* fails–consider a setting in which contracts cross only once. In such a setting, the contracts will cross in the right direction (i.e., *SCC* holds), if and only if the second-order condition $\int v_\theta(x,\theta) f_a(x|\alpha(\theta))dx \geq 0$ is satisfied. This follows from Beesack's inequality, since f_a/f is increasing, and by (11).

truncations thereof. This thus provides another set of "plug and play" environments in which the modeler can study models with both moral hazard and adverse selection.

Recall that f is an exponential family if it can be written as $f(x|a) = m(a)g(x)e^{K(a)j(x)}$. Letting $k = K' > 0$, we have $l(x|a) = k(a)j(x) + m'(a)/m(a)$, so that *MLRP* holds if and only if j is increasing in x, and $l_{ax}(x|a) = k'(a)j'(x)$ is positive if k is increasing in a. In the next result we will assume that B (the gross benefit to the principal of the agent's effort) is linear in a, while the expected value of the signal given action can have an arbitrary shape.

Recall that ψ solves $1/u'(\psi(\tau)) = \tau$, and that $\rho(\tau) = u(\psi(\tau))$.

Assumption 5 *ψ is convex, ρ is concave, and $1/\rho'$ is strictly concave.*

It can be checked that ψ convex is equivalent to $P \geq 2A$, that ρ concave is equivalent to $P \leq 3A$, and that $1/\rho'$ strictly concave is equivalent to $6A - 4P - P'/A < 0$. Thus, given $P \in [2A, 3A]$, we have that $1/\rho'$ is strictly concave if P' is not excessively negative. In Example 3, Assumption 5 holds when $\gamma \in (0, 1/2)$.[29]

We now present our central result of this section.

Theorem 5 *Let f be an exponential family with $l_{ax} \geq 0$, $F_{aa} \geq 0$, and $F_{aaa} \leq 0$. Assume that $c_{aa}/c_{a\theta}$ decreases in a for each θ, $B_{aa} = 0$, and that Assumptions 1 and 5 hold. Then, any solution to the decoupling problem with $\alpha' \geq 0$ satisfies* SCC *and hence is optimal.*

The proof of this relies on Theorem 4. The key is that since f is an exponential family, any two distinct *HM* contracts can cross at most once, since for any a and a', $l_x(x|a')/l_x(x|a) = k(a')/k(a)$ and so is independent of x. Assume that the contracts cross the "wrong" way as θ increases. Then we show that the marginal cost C_a of inducing effort must have decreased, which we show is inconsistent with a solution to the screening problem.

11 Extensions

11.1 Optimal Exclusion

In this section, we explore optimal exclusion in our setting with both moral hazard and adverse selection. We are unaware of any related results at this level of generality, especially as we will show that our conditions are not only necessary, but also sufficient. For any given θ^*, let

$$\hat{\alpha}(\cdot, \theta^*) = \arg\max_{\alpha} \int_{\underline{\theta}}^{\bar{\theta}} \left(B(\alpha(\theta)) - C(\alpha(\theta), S(\theta), \theta)\right) h(\theta) d\theta \tag{16}$$
$$s.t. \;\; S(\theta^*) = \bar{u}, \text{ and } \; S'(\theta) = -c_\theta(\alpha(\theta), \theta),$$

[29]Note that $u' = a\kappa^{\gamma-1}$ and $u'' = -a^2\kappa^{\gamma-2}$. Hence, $\psi' = (u')^2/(-u'') = \kappa^\gamma$, and so since κ is linear in w, and increasing when $\gamma \in [0, 1)$, ψ is convex for all $\gamma \in [0, 1)$. Also, $\rho' = (u')^3/(-u'') = a\kappa^{2\gamma-1}$ and so ρ is concave for all $\gamma \in [0, 1/2)$. Finally, $1/\rho' = (1/a)\kappa^{1-2\gamma}$ which is strictly concave for all $\gamma \in (0, 1/2)$.

noting that we have replaced $S(\underline{\theta}) = \bar{u}$ by $S(\theta^*) = \bar{u}$. Appendix B shows that, for each θ, $\hat{\alpha}(\theta, \cdot)$ is differentiable in θ^*, and that the restriction of $\hat{\alpha}(\cdot, \theta^*)$ to $[\theta^*, \bar{\theta}]$, characterizes the unique optimal solution to the principal's relaxed problem subject to excluding types below θ^*.

For any given θ_c and θ^*, and for any $\theta \geq \theta_c$, let $\hat{S}(\theta, \theta_c, \theta^*) = \bar{u} - \int_{\theta_c}^{\theta} c_\theta(\hat{\alpha}(\tau, \theta^*), \tau) d\tau$ be the surplus to the agent of type θ if the principal excludes types below θ_c and chooses action profile $\hat{\alpha}(\cdot, \theta^*)$ for types above θ_c. We then have the following proposition.

Proposition 7 *Assume that one of the sufficient conditions for feasibility of the decoupled solution holds, and that $C(\cdot, \cdot, \theta)$ is convex for each θ.*[30] *Cutoff level θ_c is optimal only if $(B - C)h = -c_\theta \int_{\theta_c}^{\bar{\theta}} C_{u_0} h d\theta$, evaluated at θ_c, $\hat{\alpha}(\cdot, \theta_c)$ and $\hat{S}(\cdot, \theta_c, \theta_c)$. If $c_{\theta\theta} \geq 0$ and $C_{u_0 a} > 0$ then this condition is sufficient as well.*

So, at the optimal cutoff, $B - C$ is positive. The necessity direction is both simple and intuitive. As one expands the range of included types, the direct benefit of adding types near θ is given by the left-hand side (*lhs*) of the equation. The *rhs* reflects that expanding the set of included types increases the information rent that must be paid to types above the cutoff. At an optimum, these effects must balance. That this condition is also sufficient is more involved. They key is to show that the convexity of $C(\cdot, \cdot, \theta)$ implies that profits are concave in the cutoff.

Mirroring previous results in the literature, if h is sufficiently small at $\underline{\theta}$, then there is a strictly positive region of exclusion. This is intuitive, as a small amount of exclusion then destroys surplus on very few agents, but reduces rents to the remaining agents in a first order way. Similarly, if B is large enough, and $h(\underline{\theta}) > 0$, then there is no exclusion.

11.2 Common Values

As the insurance application and Remark 1 show, many important examples with a common value aspect can be fruitfully analyzed with the tools we developed for the private values case. But, especially once one moves away from the two-outcome case, most problems with common values cannot be appropriately reparameterized. For example, if a more capable manager can both accomplish tasks more easily (so that θ enters the disutility of the action) and better (so that θ enters the distribution over outcomes), then only in very special cases would it be the case that one needed only to know a single summary statistic of ability and effort to know both the distribution over outcomes and the cost of the action to the agent.

Given this, we now turn to the general common-values problem. We consider the variation on the maximization problem (P) in which each of c, f, and B may depend on θ, were we recall that depending on the context, B may be expected output, but need not be so. Our main goal in this section is to establish an analog to Theorem 4, showing that a set of conditions centered on

[30] See Appendix B for a discussion and primitives that ensure the convexity of $C(\cdot, \cdot, \theta)$ for each θ.

SCC imply feasibility. Intriguingly, an analog to Theorem 1, which centers on *IMC*, seems more difficult, a topic to which we will return. We leave exploration of primitives that guarantee *SCC* in the common value case for future work.

Our central assumption generalizes the conditions needed on f in the private values case.

Assumption 6 *Each of f_a/f and f_θ/f is increasing in x, with $F_{a\theta} \leq 0$.*

The next example, whose proof is in the Appendix, provides a far from pathological class of densities that satisfies Assumption 6.

Example 5 *Let g be a parameterized family of densities satisfying* MLRP*, where for each a, $g(\cdot|a)$ is strictly increasing, and where $g(\underline{x}|\cdot)$ is bounded away from zero. Let r be an arbitrary measurable strictly positive function on $[\underline{x}, \overline{x}]$, and define*

$$f(x|a,\theta) = \frac{r(x)g^\theta(x|a)}{\int r(s)g^\theta(s|a)ds}.$$

Assume that $(f_\theta/f)\theta + 1 \geq 0$ for all (x, a, θ).[31] *Then, Assumption 6 holds.*[32,33]

To begin our analysis, note that the three first order conditions (FOC) become

$$\int v_\theta(x,\theta)f(x|\alpha(\theta),\theta)dx = 0,$$

$$\int v(x,\theta)f_a(x|\alpha(\theta),\theta)dx - c_a(\alpha(\theta),\theta) = 0,$$

and

$$S'(\theta) = \int v(x,\theta)f_\theta(x|\alpha(\theta),\theta)dx - c_\theta(\alpha(\theta),\theta),$$

where S' now reflects that as the agent's type changes, there is a direct effect through f_θ, and where $S(\underline{\theta}) = \bar{u}$ as before. This in hand, let us turn to global incentive compatibility.

Proposition 8 *Assume that f satisfies Assumption 6. If (α, v) is a regular menu that satisfies the FOCs, with α increasing, and with v satisfying SCC, where for each θ, $v(\cdot,\theta)$ is increasing and bounded, then (α, v) is feasible.*

[31] That is, the elasticity of f with respect to θ is at least -1. Equivalent is that for all a,

$$\log g(\underline{x}|a) - \int \log g(s|a)\frac{r(s)g^{\bar{\theta}}(s|a)}{\int r(t)g^{\bar{\theta}}(t|a)dt}ds + \frac{1}{\bar{\theta}} \geq 0,$$

which is automatic as long as either $\bar{\theta}$ is not too large or $g(\underline{x}|a)$ is not too much below its (weighted) average.

[32] See the Appendix for the proof.

[33] The density f can be reparameterized as a function of transformed effort (as in Remark 1) if and only if f is an exponential family, which holds if and only if g is also an exponential family. Except in this case, f depends in a non-trivial way on both a and θ.

To see why a generalization of *IMC* is hard in the common values case, return to the argument from Section 5. Critical to that argument was that movements from left to right as one worked along the locus L were irrelevant. But, in the common-values setting, this argument falls apart, since different types–who have different distributions over output for any given action–have different preferences about changes in the contract.

11.3 Random Mechanisms

So far, we have restricted the principal to deterministic menus. In this section, we leverage an idea of Strausz (2006) to show that when decoupling works, then the (deterministic) menu it generates is in fact optimal even when randomization is allowed. In particular, consider a setting in which first the agent announces a type, and then, based on the announcement, the principal randomizes over pairs $(\tilde{v}, a)$ consisting of a compensation scheme and recommended action. The agent needs to be willing to report his type honestly given the lottery he faces, and to follow the recommended action for each realized pair $(\tilde{v}, a)$.

Proposition 9 *Let $C(\cdot, \cdot, \theta)$ be strictly convex for each θ, and let the (deterministic) decoupled solution (α, v) be feasible.*[34] *Then, (α, v) remains optimal even if randomization is allowed.*

The key to the proof is to consider any randomized solution to the relaxed screening problem. Since $B - C$ is strictly concave, replacing actions and surplus by their expections raises the value of the objective function. But, further, because $-c_\theta$ is convex in effort, this menu requires less surplus to be given to the agent than the expected surplus in the randomized mechanism.

12 Concluding Remarks

We study a canonical problem with both moral hazard and screening. We derive necessary conditions for a menu (α, v) to be feasible. In contrast to the standard screening problem it is not enough that the recommended action rises as the agent becomes more capable. Rather, our necessary conditions require that the recommended action rises fast enough that, in a specific sense, more capable agents face stronger incentives. We also provide two different sufficient conditions for a menu to be feasible. The first requires that as the agent becomes more capable, the recommended action rises fast enough so that his marginal cost of effort rises. The second requires that as the agent becomes more capable, his contract becomes steeper in the sense of single crossing. Each of our two sufficient conditions can thus be interpreted as a stronger version of the requirement that more capable agents face stronger incentives.

We explore the question of when one can decouple the problem by first solving the moral hazard problem for each action-surplus-type triple and then solving a (non-quasilinear) screening

[34]For example, assume that the conditions of one of Propositions 4, 5, or 6, or of Theorems 2 or 5 hold.

problem for the resultant cost function. We provide several broad classes of primitives where decoupling is guaranteed to work, providing conditions under which decoupling is valid for output linear in effort, for the two outcome case, for a very general case with large enough reservation utility, for square root utility, and when the conditional distribution of output is an exponential family. Altogether, these classes subsume a large number of economic applications.[35] We also provide a case where decoupling fails, highlighting the need for such results. When decoupling does work, we have the economically important implication that the optimal mechanism has the form that the agent, by choice of his announcement of type, is choosing from a menu over *HM* contracts. Thus, everything that we know from the pure moral hazard problem carries over into the setting which also includes adverse selection, and everything that we know from the pure adverse selection problem carries through when there is also moral hazard. We apply our model to derive new predictions about insurance markets when both adverse selection and moral hazard are at play, examining in particular the question of whether moral hazard and adverse selection together create larger or smaller distortions than either alone. Finally, we examine optimal exclusion, we begin the generalization of our set-up to one of common values, and we show that decoupled menus remain optimal even if the principal can randomize.

There are several open problems for future research. We name two: a full analysis of the common values case, and the extension to a dynamic setting.

Appendix A Proofs

The following lemma (Beesack (1957)) is central to our analysis.

Lemma 5 *(Beesack's inequality). Let $g : X \to \mathbb{R}$ be an integrable function with domain an interval $X \subseteq \mathbb{R}$. Assume that g is never first strictly positive and then strictly negative, and that $\int_X g(x)dx \geq 0$. Then, for any positive increasing function $h : X \to \mathbb{R}$ such that gh is integrable,*

$$\int_X g(x)h(x)dx \geq 0.$$

If h is strictly increasing, and g is non-zero on some interval of positive length, then the inequality is strict. If $\int_X g(x)dx = 0$, then h need not be positive.

[35]To name a few, they can be applied to the insurance problem that we thoroughly analyze in Section 9 and variations thereof; to the ubiquitous contracting problem between shareholders and CEO, with the added friction that the CEO talent is private information; or to extensions to a risk averse agent of problems traditionally analyzed under risk neutrality, such as Laffont and Tirole (1986) procurement problem (or a variation but with standard moral hazard), or the project-owner and project-operator contracting problem in Lewis and Sappington (2001) (see also Lewis and Sappington (2000)).

A.1 Proofs for Section 3

Proposition 10 *Assume that $c_{aa\theta\theta}$ and $c_{a\theta\theta\theta}$ exist. If h is log-concave and $-c_{a\theta}$ is log-convex in θ, then $\alpha' > 0$ everywhere.*

Proof Note that the numerator of (7) rearranges to

$$-\frac{c_{a\theta\theta}}{c_{a\theta}} + \frac{h'}{h} + \left(\frac{c_{a\theta}\hat{C}_{u_0} + \hat{C}_{a\theta} - c_\theta \hat{C}_{au_0}}{c_{a\theta}\hat{C}_{u_0}}\right) \frac{\hat{C}_{u_0} h}{\int_\theta^{\bar{\theta}} \hat{C}_{u_0} h} > 0.$$

Since in the pure adverse selection case $\hat{C}(a, u_0, \theta) = \varphi(u_0 + c(a, \theta))$, we have that $\hat{C}_a = \varphi' c_a$ and $\hat{C}_{u_0} = \varphi'$, and hence $\hat{C}_{a\theta} = \varphi'' c_\theta c_a + \varphi' c_{a\theta}$, and $\hat{C}_{au_0} = \varphi'' c_a$. From this, the term in parenthesis equals 2, and thus $\alpha' > 0$ for any given θ if and only if for all θ,

$$z(\theta) \equiv -\frac{c_{a\theta\theta}}{c_{a\theta}} + \frac{h'}{h} + \frac{2\varphi' h}{\int_\theta^{\bar{\theta}} \varphi' h} > 0. \tag{17}$$

Note that $z(\bar{\theta}) > 0$, for the first two terms are bounded while the last term diverges as θ goes to $\bar{\theta}$. Hence, by continuity, there is a smallest type $\theta_0 \in [\underline{\theta}, \bar{\theta})$ such that $z(\theta) > 0$ for all $\theta > \theta_0$. We wish to show that $\theta_0 = \underline{\theta}$. Towards a contradiction, assume that $\theta_0 > \underline{\theta}$. Then $z(\theta_0) = 0$, and $z'(\theta_0) \geq 0$ (since $z(\theta) > 0$ for all $\theta > \theta_0$). We will show that these two properties cannot hold simultaneously under the stated assumptions on h and $c_{a\theta}$, yielding the desired contradiction.

Assume that $z(\theta_0) = 0$ and consider $z'(\theta_0)$. The second term in (17) is decreasing in θ since h is log-concave. Note next that

$$\left(\frac{c_{a\theta\theta}}{-c_{a\theta}}\right)_\theta = \left(\frac{\partial}{\partial a}\frac{c_{a\theta\theta}}{-c_{a\theta}}\right)\alpha' + \frac{\partial}{\partial \theta}\frac{c_{a\theta\theta}}{-c_{a\theta}},$$

where we recall that we use $(\cdot)_\theta$ as shorthand for the total derivative with respect to θ. When we evaluate this expression at $\theta = \theta_0$, the first term vanishes since $\alpha'(\theta_0) = 0$, and the second term is negative since $-c_{a\theta}$ is log-convex in θ. Hence, a necessary condition for $z'(\theta_0) \geq 0$ is that $\left(\varphi' h / \int_\theta^{\bar{\theta}} \varphi' h\right)_\theta$ is positive at $\theta = \theta_0$, which holds if and only if

$$\varphi'' c_a \alpha' h \int_{\theta_0}^{\bar{\theta}} \varphi' h + \varphi' h' \int_{\theta_0}^{\bar{\theta}} \varphi' h + \varphi'^2 h^2 \geq 0$$

when evaluated at $\theta = \theta_0$. Since the first term vanishes at θ_0, we obtain $\varphi' h' \int_{\theta_0}^{\bar{\theta}} \varphi' h + \varphi'^2 h^2 \geq 0$, which holds if and only if

$$\frac{h'}{h} + \frac{\varphi' h}{\int_{\theta_0}^{\bar{\theta}} \varphi' h} \geq 0.$$

But this implies that

$$z(\theta_0) = -\frac{c_{a\theta\theta}}{c_{a\theta}} + \frac{h'}{h} + \frac{2\varphi' h}{\int_{\theta_0}^{\bar{\theta}} \varphi' h} > 0,$$

contradicting that $z(\theta_0) = 0$. Hence, $z(\theta_0) = 0$ and $z'(\theta_0) \geq 0$ cannot hold simultaneously. □

We now provide sufficient conditons for $\mu_a \geq 0$ and $\lambda_a \geq 0$.

Lemma 6 *Let l be submodular in x and a, i.e., $l_{xa} \leq 0$. Then, $\mu_a \geq 0$. If in addition f is log-concave in a and ρ is concave, then $\lambda_a \geq 0$ as well.*

Proof From the first-order condition of the cost-minimization problem plus the binding participation and incentive constraints, we obtain the following system of equations in λ and μ:

$$\int \rho(\lambda + \mu l(x|a)) f(x|a) dx = c(a, \theta) + u_0 \tag{18}$$

$$\int \rho(\lambda + \mu l(x|a)) f_a(x|a) dx = c_a(a, \theta). \tag{19}$$

By differentiating this system and manipulating (see *CS* for details), one arrives at

$$\lambda_a = -\mu_a \int l\xi - \mu \int l_a \xi \quad \text{and} \quad \mu_a = \frac{1}{var_\xi(l)} \left(\frac{1}{\int \rho' f} \left(c_{aa} - \int \rho f_{aa} \right) - \mu \, cov_\xi(l_a, l) \right), \tag{20}$$

where ξ is the density with kernel $\rho'(\lambda + \mu l(\cdot|\alpha(\theta))) f(\cdot|\alpha(\theta))$ To see that $\mu_a > 0$, note that $c_{aa} - \int \rho f_{aa} \geq 0$ by *FOP*, while $cov_\xi(l_a, l) < 0$ under the assumption $l_{ax} < 0$. Turning to λ_a, notice that $\int l\xi =_s \int l\rho' f = \int \rho' f_a$, where we recall that $=_s$ indicates that the objects on either side have strictly the same sign. Now, $\int \rho' f_a$ is negative by Beesack's inequality (see the beginning of this section), since f_a single-crosses zero from below, $\int f_a = 0$, and ρ' is positive and decreasing in x. Since $\mu_a \geq 0$, it follows that $\lambda_a \geq 0$ if $\int l_a \xi =_s \int l_a \rho' f \leq 0$. But this holds since f is log-concave in a, which is equivalent to $l_a \leq 0$. □

A.2 Proofs for Section 4

Proof of Proposition 1 Let us first show that α is weakly increasing. For any two types $\theta' > \theta$ incentive compatibility implies

$$\int v(x, \theta') f(x|\alpha(\theta')) dx - c(\alpha(\theta'), \theta') \geq \int v(x, \theta) f(x|\alpha(\theta)) dx - c(\alpha(\theta), \theta'), \text{ and}$$

$$\int v(x, \theta) f(x|\alpha(\theta)) dx - c(\alpha(\theta), \theta) \geq \int v(x, \theta') f(x|\alpha(\theta')) dx - c(\alpha(\theta'), \theta).$$

Adding these inequalities yields $c(\alpha(\theta'), \theta') + c(\alpha(\theta), \theta) \leq c(\alpha(\theta'), \theta) + c(\alpha(\theta), \theta')$, which, since c is submodular in (a, θ), implies that $\alpha(\theta') \geq \alpha(\theta)$.

The condition $S(\underline{\theta}) \geq \bar{u}$ is immediate from our simplifying assumption of full participation. Equation (9) is the first-order condition for the agent's choice of effort, while (11) is the one for his choice of what type to report (which holds given the assumed continuity of α and the validity of passing the derivative throught the integral). Condition (10) captures that for the agent not to want to locally misrepresent his type and change his action along the locus $(\theta, \alpha(\theta))$, it must be that $S'(\theta) = -c_\theta(\alpha(\theta), \theta)$. Formally, since c is continuously differentiable in (a, θ), the conditions of Theorem 2 of Milgrom and Segal (2002) hold, and so S is absolutely continuous and hence differentiable almost everywhere and thus satisfies the integral condition (10).

To see that α is strictly increasing, let us consider the second order necessary conditions. Fix the true type of the agent θ, and differentiate the agent's objective function $\int v(x, \theta')f(x|a)dx - c(a, \theta)$ as a function of the reported type, θ', and the chosen action, a, twice. Evaluating the derivatives at the candidate menu, we obtain the following Hessian matrix:

$$M = \begin{bmatrix} \int v_{\theta\theta}(x,\theta)f(x|\alpha(\theta))dx & \int v_\theta(x,\theta)f_a(x|\alpha(\theta))dx \\ \int v_\theta(x,\theta)f_a(x|\alpha(\theta))dx & \int v(x,\theta)f_{aa}(x|\alpha(\theta))dx - c_{aa}(\alpha(\theta),\theta) \end{bmatrix}.$$

The second-order conditions for optimality by the agent require that the diagonal elements of M are negative, and the determinant of M is positive.

Given that feasibility implies that α is increasing and hence almost everywhere differentiable, to show that α is strictly increasing, we need only to rule out that at some point, $\alpha' = 0$. To see this, begin by noting that (9) and (11) are identities in θ. Differentiating (11) yields

$$\int v_{\theta\theta}(x,\theta)f(x|\alpha(\theta))dx = -\alpha'(\theta)\int v_\theta(x,\theta)f_a(x|\alpha(\theta))dx. \tag{21}$$

Hence, if $\alpha' = 0$, then $\int v_{\theta\theta}(x,\theta)f(x|\alpha(\theta))dx = 0$. Similarly, differentiating (9) yields

$$\alpha'\left(\int v(x,\theta)f_{aa}(x|\alpha(\theta))dx - c_{aa}(\alpha(\theta),\theta)\right) = c_{a\theta}(\alpha(\theta),\theta) - \int v_\theta(x,\theta)f_a(x|\alpha(\theta))dx, \tag{22}$$

and so if $\alpha' = 0$, then, $\int v_\theta(x,\theta)f_a(x|\alpha(\theta))dx = c_{a\theta}(\alpha(\theta),\theta) < 0$. But then, $\det M = -(c_{a\theta}(\alpha(\theta),\theta))^2 < 0$, violating the second-order necessary conditions.

Finally, let us establish that the second order necessary conditions hold if and only if (12) holds. Since $\alpha' > 0$, it follows from (21) that $\int v_{\theta\theta}(x,\theta)f(x|\alpha(\theta))dx \leq 0$ if and only if (12) holds. Similarly, from (22), $\int v f_{aa} - c_{aa} \leq 0$ when evaluated at (α, v) if and only if $c_{a\theta} - \int v_\theta f_a \leq 0$, and since c is submodular in (a, θ), it suffices for this that (12) holds.

Finally, note that using (21) and (22) we can rewrite M as

$$M = \begin{bmatrix} -\alpha'(\theta)\int v_\theta(x,\theta)f_a(x|\alpha(\theta))dx & \int v_\theta(x,\theta)f_a(x|\alpha(\theta))dx \\ \int v_\theta(x,\theta)f_a(x|\alpha(\theta))dx & \frac{1}{\alpha'(\theta)}\left(c_{a\theta}(\alpha(\theta),\theta) - \int v_\theta(x,\theta)f_a(x|\alpha(\theta))dx\right) \end{bmatrix},$$

and so $\det M = -c_{a\theta}(\alpha(\theta),\theta)\int v_\theta(x,\theta)f_a(x|\alpha(\theta))dx$. Since $c_{a\theta}<0$, the determinant is positive if and only if $\int v_\theta(x,\theta)f_a(x|\alpha(\theta))dx \geq 0$, thereby completing the proof. $\square$

A.3 Proofs for Section 5

Proof of Lemma 1 Assume *wlog* that $\theta_A > \theta$. Then using (10) we obtain

$$S(\theta) - S(\theta_A) = \int_\theta^{\theta_A} c_\theta(\alpha(s),s)ds > \int_\theta^{\theta_A} c_\theta(\alpha(\theta_A),s)ds = c(\alpha(\theta_A),\theta_A) - c(\alpha(\theta_A),\theta),$$

where the inequality follows since α is strictly increasing and c is strictly submodular in (a,θ). Hence, $S(\theta) > S(\theta_A) + c(\alpha(\theta_A),\theta_A) - c(\alpha(\theta_A),\theta) = \int v(x,\theta_A)f(x|\alpha(\theta_A))dx - c(\alpha(\theta_A),\theta)$, as required. $\square$

Proof of Theorem 1 We proceed in several steps. Denote by γ the generalized inverse of α (α need not be continuous everywhere; it can jump up a countable number of times).

STEP 1. Since $S(\underline{\theta}) \geq \overline{u}$, (10) implies that $S(\theta) \geq \overline{u}$ for all θ, and thus participation holds.

STEP 2. From Lemma 1, it suffices to show that every deviation $(\theta_A,\hat{a}) \notin L$ is dominated by some on locus deviation. We focus on deviations with $\hat{a} > \alpha(\theta_A)$ (the other case is similar).

Let the agent's true type be θ_T. If $\theta_T \leq \theta_A$, then

$$\int v(x,\theta_A)f(x|\hat{a})dx - \int v(x,\theta_A)f(x|\alpha(\theta_A))dx \leq c(\hat{a},\theta_A) - c(\alpha(\theta_A),\theta_A) \leq c(\hat{a},\theta_T) - c(\alpha(\theta_A),\theta_T),$$

where the first inequality follows from the first-order condition (9), from *FOP*, and from $\hat{a} > \alpha(\theta_A)$, and the second since c is submodular. But then, the agent is better off with $(\theta_A,\alpha(\theta_A)) \in L$.

STEP 3. If for any given $\tilde{\theta}$, $\hat{a} > \alpha(\tilde{\theta})$ and $\theta_A \leq \tilde{\theta}$, then deviation $(\theta_A,\hat{a})$ is dominated for type $\tilde{\theta}$ by $(\theta_A,\alpha(\tilde{\theta}))$. To see this, consider any action $a \in [\alpha(\tilde{\theta}),\hat{a}]$. Then

$$\int v(x,\theta_A)f_a(x|a)dx \leq \int v(x,\theta_A)f_a(x|\alpha(\theta_A))dx = c_a(\alpha(\theta_A),\theta_A) \leq c_a(\alpha(\tilde{\theta}),\tilde{\theta}) \leq c_a(a,\tilde{\theta}),$$

where the first inequality follows from the *FOP*, the equality follows by (9), the second inequality follows by *IMC,* and the third by convexity of c in a. Hence, $\int v(x,\theta_A)f_a(x|a)dx \leq c_a(a,\tilde{\theta})$ for any $a \in [\alpha(\tilde{\theta}),\hat{a}]$, which implies that $\tilde{\theta}$'s expected utility is decreasing in a in that range, and so $\tilde{\theta}$ is at least as well off with deviation $(\theta_A,\alpha(\tilde{\theta}))$ as with $(\theta_A,\hat{a})$.

From Step 2, and from Step 3 applied to $\tilde{\theta} = \theta_T$, we can restrict attention to deviations $(\theta_A,\hat{a})$ with $\theta_A \leq \theta_T$ and $\hat{a} \leq \alpha(\theta_T)$.

STEP 4. Let $(\theta_A,\hat{a})$ be such that $\hat{a} > \alpha(\theta_A)$ and $(\gamma(\hat{a}),\hat{a}) \in L$, i.e., $\hat{a} = \alpha(\gamma(\hat{a}))$. We will show that $\int v(x,\gamma(\hat{a}))f(x|\hat{a})dx \geq \int v(x,\theta_A)f(x|\hat{a})dx$ and hence,

$$\int v(x,\gamma(\hat{a}))f(x|\hat{a})dx - c(\hat{a},\theta_T) \geq \int v(x,\theta_A)f(x|\hat{a})dx - c(\hat{a},\theta_T),$$

showing that $(\theta_A, \hat{a})$ is dominated for θ_T by $(\gamma(\hat{a}), \hat{a}) \in L$.

Define Q by

$$Q(a) \equiv S(\gamma(a)) + c(a,\gamma(a)) - \int v(x,\theta_A)f(x|a)dx. \tag{23}$$

It is immediate from the definition that $Q(\alpha(\theta_A)) = 0$, since $\gamma(\alpha(\theta_A)) = \theta_A$, and thus $S(\gamma(\alpha(\theta_A))) = S(\theta_A)$ while the second and third terms on the *rhs* of (23) sum to $-S(\theta_A)$. Also, Q is differentiable a.e. since S and γ are, and has derivative given by

$$\begin{aligned} Q'(a) &= S_\theta(\gamma(a))\gamma'(a) + c_a(a,\gamma(a)) + c_\theta(a,\gamma(a))\gamma'(a) - \int v(x,\theta_A)f_a(x|a)dx \\ &= c_a(a,\gamma(a)) - \int v(x,\theta_A)f_a(x|a)dx, \end{aligned}$$

where the equality follows since $S_\theta = -c_\theta$ on the locus and since $\gamma' = 0$ when the generalized inverse is not on the locus (i.e., where α jumps). Now, for any $a \in (\alpha(\theta_A), \hat{a})$, note that since $a > \alpha(\theta_A)$, then $\gamma(a) \geq \theta_A$. We claim that $c_a(a,\gamma(a)) \geq c_a(\alpha(\theta_A),\theta_A)$. This is immediate if $(\gamma(a), a) \in L$ by *IMC*. Otherwise, note that for all $\theta \in [\theta_A, \gamma(a))$, $c_a(a,\theta) \geq c_a(\alpha(\theta),\theta) \geq c_a(\alpha(\theta_A),\theta_A)$, where the first inequality is by convexity of c in a, noting that $\theta < \gamma(a)$ implies $\alpha(\theta) < a$, and the second inequality is by *IMC.* But then, taking $\theta \to \gamma(a)$, $c_a(a,\gamma(a)) \geq c_a(\alpha(\theta_A),\theta_A)$ as claimed. Note also that $\int v(x,\theta_A)f_a(x|a)dx \leq \int v(x,\theta_A)f_a(x|\alpha(\theta_A))dx$ by *FOP*. Hence, for each θ where Q is differentiable, $Q'(a) \geq c_a(\alpha(\theta_A),\theta_A) - \int v(x,\theta_A)f_a(x|\alpha(\theta_A))dx = 0$, where the equality is simply (9). Hence, in particular, $Q(\hat{a}) \geq 0$. But, by definition of S, and since $(\gamma(\hat{a}), \hat{a}) \in L$, we have $\int v(x,\gamma(\hat{a}))f(x|\hat{a})dx = S(\gamma(\hat{a})) + c(\hat{a},\gamma(\hat{a}))$, and hence

$$\int v(x,\gamma(\hat{a}))f(x|\hat{a})dx - \int v(x,\theta_A)f(x|\hat{a})dx = Q(\hat{a}) \geq 0,$$

completing the proof of this step.

Step 5. Let $(\theta_A, \hat{a})$ be such that $\hat{a} > \alpha(\theta_A)$ and $\hat{a} \neq \alpha(\gamma(\hat{a}))$. Then α jumps at $\theta_J = \gamma(\hat{a})$ with endpoints $\underline{a} = \lim_{\varepsilon \downarrow 0} \alpha(\theta_J - \varepsilon)$ and $\overline{a} = \lim_{\varepsilon \downarrow 0} \alpha(\theta_J + \varepsilon)$ and with $\hat{a} \in [\underline{a}, \overline{a}]$. Let $\underline{v}$ and $\bar{v}$ be the associated limit contracts. We claim

$$\int \underline{v}(x)f(x|\underline{a})dx - c(\underline{a},\theta_J) = S(\theta_J) \text{ and } \int \underline{v}(x)f_a(x|\underline{a})dx - c_a(\underline{a},\theta_J) = 0 \tag{24}$$

and similarly at $\bar{a}$. To see the first equality, note that by definition, $\int v(x,\theta)f(x|\alpha(\theta))dx - c(\alpha(\theta),\theta) = S(\theta)$ for all $\theta < \theta_J$, and then use the definitions of $\underline{a}$ and $\underline{v}$, uniform convergence of

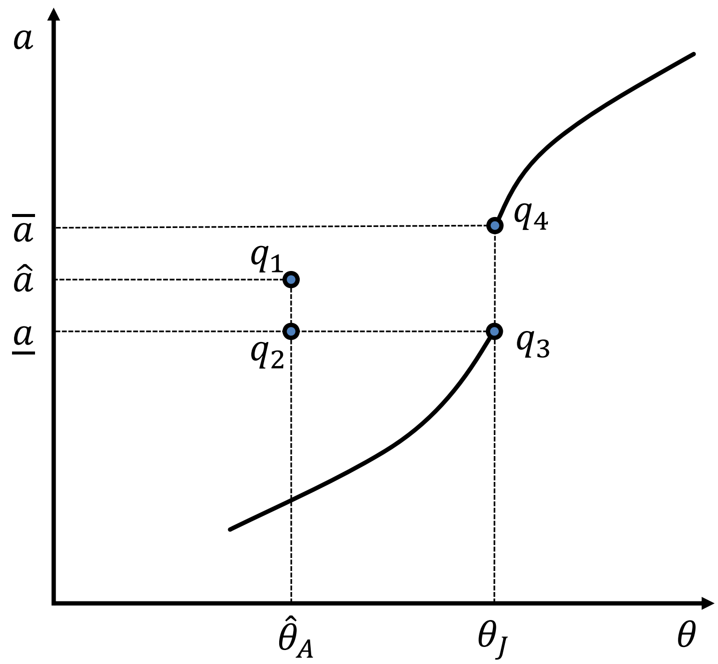

Figure 2: ***IMC***. Under *IMC*, a deviation by θ_J to q_1 is dominated by one to q_2, which in turn is dominated by q_3, which, from the point of view of θ_J is equivalent to q_4. But then, from the point of view of θ_T, who has a lower incremental cost of effort, the (on-locus) point q_4 also dominates q_1, and telling the truth and taking the recommended action is better yet.

$v(\cdot,\theta)$ to $\underline{v}$ as $\theta \uparrow \theta_J$, and continuity of S. The second equality similarly follows from (9).

Note that since $\hat{a} \leq \alpha(\theta_T)$, it follows that $\theta_T \geq \theta_J$. If $\theta_A = \theta_J$, then $(\theta_A, \hat{a})$ is no better than $(\theta_J, \alpha(\theta_J))$, by (9) and *FOP*. We are thus back on locus, and done. So, assume $\theta_A < \theta_J$ in what follows. We will show that $(\theta_T, \alpha(\theta_T))$ dominates $(\theta_A, \hat{a})$ for θ_T. To this end, define $v_1 = \int v(x,\theta_A) f(x|\hat{a})dx$, $v_2 = \int v(x,\theta_A) f(x|\underline{a})dx$, $v_3 = \int \underline{v}(x) f(x|\underline{a})dx$, and $v_4 = \int \bar{v}(x) f(x|\bar{a})dx$. These give the payoff from income at points $q_1, ..., q_4$, in Figure 2, where v_3 reflects a limit from the left and v_4 a limit from the right.

We claim

$$v_4 - c(\bar{a},\theta_J) \geq v_1 - c(\hat{a},\theta_J). \tag{25}$$

To see this, note that, $v_1 - c(\hat{a},\theta_J) \leq v_2 - c(\underline{a},\theta_J) \leq v_3 - c(\underline{a},\theta_J) = S(\theta_J) = v_4 - c(\bar{a},\theta_J)$, where the first inequality applies Step 3 at $\tilde{\theta} = \theta_J$, the second inequality applies Step 4, and the equalities are by (24). But then, since c is submodular, $v_1 - c(\hat{a},\theta_T) \leq v_4 - c(\bar{a},\theta_T) \leq S(\theta_T)$, where the second inequality uses Lemma 1 and the fact that $v_4 - c(\bar{a},\theta_T)$ is θ_T's limit payoff to imitating $\theta_J + \varepsilon$ on locus. This completes the proof of Step 5 and the theorem. □

A.4 Proofs for Section 6

Proof of Proposition 3 Sufficiency follows from Theorem 1, since *FOP* is automatic when $f_{aa} = 0$. For necessity, note that since (α, v) is feasible, α is increasing and hence differentiable almost everywhere. Where α jumps, c_a jumps as well. Thus, it is enough to show that where α is differentiable $(c_a(\alpha(\theta), \theta))_\theta \geq 0$. But since (9) holds as an identity in θ, we can differentiate both sides to arrive at

$$(c_a(\alpha(\theta), \theta))_\theta = \left(\int v(x, \theta) f_a(x|\alpha(\theta)) dx\right)_\theta = \int v_\theta(x, \theta) f_a(x|\alpha(\theta)) dx \geq 0,$$

where the second equality follows from $f_{aa} = 0$ and the inequality follows from the second-order condition using Proposition 1. □

Prior to proving Proposition 4, we will need three lemmas. The first two will play an ongoing role in our analysis.

Lemma 7 *If Assumption 1 holds, then a sufficient condition for the second-stage screening solution to satisfy IMC is that*

$$\left(2\lambda + \mu_a + \mu \frac{c_{aa\theta}}{c_{a\theta}}\right) c_{aa} \geq C_{aa} - B_{aa}. \tag{26}$$

for each $(\alpha(\theta), S(\theta), \theta)$.

Proof Recall that α' is given by (7), and that *IMC* is the condition that $\alpha' \geq -c_{a\theta}/c_{aa}$ for all θ. Hence, rearranging, *IMC* holds if and only if for all θ,

$$\left(-c_{a\theta\theta}c_{aa} + c_{a\theta}c_{aa}\frac{h'}{h} + c_{a\theta}c_{aa\theta}\right) \int_\theta^{\bar{\theta}} C_{u_0} h + (c_{a\theta}C_{u_0} - C_{au_0}c_\theta + C_{a\theta})\, hc_{aa} \leq -c_{a\theta}(B_{aa} - C_{aa})h, \tag{27}$$

noting that the denominator on the *rhs* of (7) is strictly negative. The bracketed term on the *lhs* has the sign of $(\log(-c_{a\theta}/hc_{aa}))_\theta$ which, for each a, is negative by assumption. Hence it suffices that

$$(c_{a\theta}C_{u_0} - C_{au_0}c_\theta + C_{a\theta})\, c_{aa} \leq -c_{a\theta}(B_{aa} - C_{aa}). \tag{28}$$

But, from Section 3.3, $C_{u_0} = \lambda$, $C_{au_0} = \lambda_a$, $C_\theta = \lambda c_\theta + \mu c_{a\theta}$, and $C_{\theta a} = \lambda_a c_\theta + \lambda c_{a\theta} + \mu_a c_{a\theta} + \mu c_{aa\theta}$. Thus

$$c_{a\theta}C_{u_0} - C_{au_0}c_\theta + C_{a\theta} = 2\lambda c_{a\theta} + \mu_a c_{a\theta} + \mu c_{aa\theta}, \tag{29}$$

and the result follows. □

The objects λ, μ, and C_{aa} are each complicated functions of the primitives, and so for further progress, we need to break them into more manageable objects. The following lemma goes part of the way in achieving this task. It relies on Lemma 7 of Chade and Swinkels (2019) (*CS*) characterizing C_{aa}. Recall that ρ is the function that transforms $1/u'$ into u.

Lemma 8 *If Assumption 1 holds, then a sufficient condition for the second-stage screening solution to satisfy IMC is that for each θ,*

$$\left(\lambda + \mu_a + \mu \frac{c_{aa\theta}}{c_{a\theta}}\right) c_{aa} \geq \mu \left(c_{aaa} - \int v f_{aaa} - \int v_x l F_{aa} \right) + \left(\mu_a^2 var_\xi(l) - \mu^2\, var_\xi(l_a)\right) \int \rho' f - B_{aa}, \tag{30}$$

where for each θ, v is the HM *contract implementing action $\alpha(\theta)$ at surplus $S(\theta)$ for θ and ξ is the density with kernel $\rho'(\lambda + \mu l(\cdot|\alpha(\theta)))f(\cdot|\alpha(\theta))$.*

Proof From *CS*, Lemma 7,

$$C_{aa} = \lambda c_{aa} + \mu \left(c_{aaa} - \int v f_{aaa} - \int v_x l F_{aa} \right) + \left(\int \rho' f \right) \left(\mu_a^2 var_\xi(l) - \mu^2\, var_\xi(l_a)\right)$$

and so, substituting into (26), and cancelling λc_{aa} from each side yields the result. □

Lemma 9 *Let G and $\hat{G}$ be two cdf's with finite support, with G MLRP-dominated by $\hat{G}$. Then,*

$$\frac{cov_G(s^2, s)}{var_G(s)} \leq \frac{cov_{\hat{G}}(s^2, s)}{var_{\hat{G}}(s)}.$$

For intuition, note that $cov_G(s^2, s)/var_G(s)$ is the slope of the best linear fit to s^2 under G, and that $\hat{G}$ moves probability mass rightward from G and hence to where s^2 has a higher slope.

Proof Let T be any continuous distribution. By Theorem 1 in Cuadras (2002) specialized to our setting, for any C^2 function ζ of q,

$$\begin{aligned} cov_T(\zeta(q), q) &= \int \left(\int (T(\min(q, y)) - T(q)T(y))dy \right) \zeta'(q)dq \\ &= \int \left(\int_{\underline{l}}^{q} (T(y) - T(q)T(y))dy + \int_{q}^{\bar{l}} (T(q) - T(q)T(y))dy \right) \zeta'(q)dq \\ &= \int M_T(q)\zeta'(q)dq, \end{aligned}$$

where $M_T(q) = (1 - T(q)) \int_{\underline{l}}^{q} T(y)dy + T(q) \int_{q}^{\bar{l}} (1 - T(y))dy$. Thus, since $var_T(q) = cov_T(q, q)$,

$$\frac{cov_T(q^2, q)}{var_T(q)} = \frac{2 \int M_T(q)qdq}{\int M_T(q)dq} = 2 \int m_T(q)qdq$$

where $m_T(\cdot)$ is the density given by $M_T(\cdot)/\int M_T(q)dq$. Since q is increasing, it is thus sufficient for the result that $m_{\hat{G}}/m_G$, or equivalently, that $M_{\hat{G}}/M_G$ is increasing.

Now, $M_T(q) = T\left(\bar{l} - q - \int T\right) + \int_{\underline{l}}^{q} T = T(\mu_T - q) + \int_{\underline{l}}^{q} T$, where μ_T is the expectation of q under T. Thus, $M_T' = q(\mu_T - q)$, and so,

$$\frac{1}{g}\left(\frac{M_{\hat{G}}(q)}{M_G(q)}\right)_q =_s \frac{\hat{g}}{g}(\mu_{\hat{G}} - q)\left(G(\mu_G - q) + \int_{\underline{l}}^{q} G\right) - (\mu_G - q)\left(\hat{G}(\mu_{\hat{G}} - q) + \int_{\underline{l}}^{q} \hat{G}\right) \equiv Z(q).$$

We thus have

$$\begin{aligned} Z' = & \left(\frac{\hat{g}}{g}\right)_q (\mu_{\hat{G}} - q)\left(G(\mu_G - q) + \int_{\underline{l}}^{q} G\right) - \frac{\hat{g}}{g}\left(G(\mu_G - q) + \int_{\underline{l}}^{q} G\right) + \frac{\hat{g}}{g}(\mu_{\hat{G}} - q)\, g(\mu_G - q) \\ & + \hat{G}(\mu_{\hat{G}} - q) + \int_{\underline{l}}^{q} \hat{G} - (\mu_G - q)\,\hat{g}(\mu_{\hat{G}} - q) \\ = & \left(\left(\frac{\hat{g}}{g}\right)_q (\mu_{\hat{G}} - q) - \frac{\hat{g}}{g}\right)\left(G(\mu_G - q) + \int_{\underline{l}}^{q} G\right) + \hat{G}(\mu_{\hat{G}} - q) + \int_{\underline{l}}^{q} \hat{G}. \end{aligned}$$

Now, if $\mu_G - q \neq 0$, then, solving $Z = 0$ for $\hat{G}(\mu_{\hat{G}} - q) + \int_{\underline{l}}^{q} \hat{G}$, replacing and cancelling $G(\mu_G - q) + \int_{\underline{l}}^{q} G$, we have

$$Z' =_s \left(\frac{\hat{g}}{g}\right)_q (\mu_{\hat{G}} - q) + \left(\frac{\hat{g}}{g}\right)\frac{\mu_{\hat{G}} - \mu_G}{\mu_G - q}.$$

This expression is strictly positive on $[\underline{l}, \mu_G)$, since $q < \mu_G < \mu_{\hat{G}}$, and since by the premise, $\hat{g}/g$ is increasing. Thus, any crossing of 0 by Z on $[\underline{l}, \mu_G)$ is strictly upward, and so, since $Z(\underline{l}) = 0$, Z is positive on $[\underline{l}, \mu_G)$. Similarly, any crossing of 0 by Z on $(\mu_{\hat{G}}, \bar{l}]$ is strictly downward, and so, since $Z(\underline{l}) = 0$, Z is positive on $(\mu_{\hat{G}}, \bar{l}]$. Note finally that $Z(q) > 0$ on $[\mu_G, \mu_{\hat{G}}]$ since $\mu_{\hat{G}} - q$ and $-(\mu_G - q)$ are positive, with one of them strictly so. Thus, Z is everywhere positive. But then, $M_{\hat{G}}/M_G$ is increasing, and we are done. □

Proof of Proposition 4 For the linear case (30) reduces to

$$\left(\lambda + \mu_a + \mu\frac{c_{aa\theta}}{c_{a\theta}}\right) c_{aa} \geq \mu c_{aaa} + \left(\mu_a^2 var_\xi(l) - \mu^2\, var_\xi(l_a)\right)\int \rho' f,$$

where, since $c_{aa}/c_{a\theta}$ is increasing in a, and since in the linear case $l_a = -l^2$, it suffices that

$$\lambda c_{aa} \geq -\mu_a c_{aa} + \left(\mu_a^2 var_\xi(l) - \mu^2\, var_\xi(l^2)\right)\int \rho' f.$$

But, from (20),

$$\mu_a = \frac{1}{var_\xi(l)}\left(\frac{1}{\int \rho' f}\left(c_{aa} - \int \rho f_{aa}\right) - \mu\, cov_\xi(l_a, l)\right) = \frac{1}{var_\xi(l)}\left(\frac{1}{\int \rho' f} c_{aa} + \mu\, cov_\xi(l^2, l)\right).$$

Substituting, it suffices that

$$\lambda c_{aa} \geq -\frac{1}{var_\xi(l)}\left(\frac{1}{\int \rho' f}c_{aa} + \mu\, cov_\xi(l^2,l)\right)c_{aa} + \left(\frac{1}{var_\xi(l)}\left(\frac{1}{\int \rho' f}c_{aa} + \mu\, cov_\xi(l^2,l)\right)^2 - \mu^2\, var_\xi(l^2)\right)\int \rho' f,$$

and so, expanding the squared term, cancelling, and rearranging, it suffices that

$$\lambda c_{aa} \geq c_{aa}\mu\,\frac{cov_\xi(l^2,l)}{var_\xi(l)} + \left(\int \rho' f\right)\frac{\mu^2}{var_\xi(l)}\left(\left(cov_\xi(l^2,l)\right)^2 - var_\xi(l^2)var_\xi(l)\right).$$

We are thus done if (i) $cov_\xi(l^2,l) \leq 0$ and (ii) $\left(cov_\xi(l^2,l)\right)^2 - var_\xi(l^2)var_\xi(l) \leq 0$. Now, $cov_F(l^2,l) = skew_F(l)$, since $\mathbb{E}_F(l) = 0$. But, since ρ is concave, ξ is *MLRP*-dominated by f. It follows that the distribution $\hat{\xi}$ on l generated by ξ is *MLRP*-dominated by the distribution $\hat{f}$ on l generated by f.[36] Result (i) then follows from Lemma 9. To see (ii), note that

$$\left(cov_\xi(l^2,l)\right)^2 - var_\xi(l^2)var_\xi(l) = \left(\mathbb{E}_\xi\left[\left(l^2 - \mathbb{E}_\xi\left(l^2\right)\right)\left(l - \mathbb{E}_\xi\left(l\right)\right)\right]\right)^2 - \mathbb{E}_\xi\left[\left(l^2 - \mathbb{E}_\xi\left(l^2\right)\right)^2\right]\mathbb{E}_\xi\left[\left(l - \mathbb{E}_\xi\left(l\right)\right)^2\right],$$

which is negative by the Cauchy-Schwartz inequality. □

Proof of Lemma 2 For each a, and since $\mathbb{E}_F(l) = 0$, we need

$$skew_F(l) = \int l^3 f = \int\left(\frac{f_H - f_L}{af_H + (1-a)f_L}\right)^3 (af_H + (1-a)f_L)dx \leq 0.$$

But,

$$\frac{\partial}{\partial a}\int \frac{(f_H - f_L)^3}{(af_H + (1-a)f_L)^2}dx = -2\int \frac{(f_H - f_L)^4}{(af_H + (1-a)f_L)^3}dx < 0.$$

So, it is enough that the relevant inequality holds at $a = 0$, i.e., that, letting $r = f_H/f_L$,

$$\int\left(\frac{f_H - f_L}{f_L}\right)^3 f_L dx = \int (r(x) - 1)^3 f_L(x)dx \leq 0.$$

Let $r(x^*) = 1$, and for $x \in [x^*, 1]$, define $\chi(x) \in [0, x^*]$ by

$$\int_{\chi(x)}^{x^*}(1 - r(s))f_L(s)ds = \int_{x^*}^{x}(r(s) - 1)f_L(s)ds,$$

where $\chi(x^*) = x^*$, $\chi(1) = 0$ (using that f_H and f_L are densities), and for all $x > x^*$,

$$\chi'(x) = \frac{r(x) - 1}{r(\chi(x)) - 1}\frac{f_L(x)}{f_L(\chi(x))} < 0.$$

[36] That is, $\hat{\xi}(l) = \hat{\xi}\left(l^{-1}(l|a)\right)/l_x\left(l^{-1}(l|a)\right)$, and similarly for $\hat{f}$.

Now, changing variables using $s = \chi(x)$, we obtain

$$\begin{aligned}\int_0^{x^*}(r(s)-1)^3 f_L(s)ds &= \int_{\chi(1)}^{\chi(x^*)}(r(s)-1)^3 f_L(s)ds\\ &= \int_1^{x^*}(r(\chi(x))-1)^3 f_L(\chi(x))\chi'(x)dx\\ &= \int_1^{x^*}(r(\chi(x))-1)^3 f_L(\chi(x))\frac{r(x)-1}{r(\chi(x))-1}\frac{f_L(x)}{f_L(\chi(x))}dx\\ &= -\int_{x^*}^{1}(r(\chi(x))-1)^2(r(x)-1)f_L(x)dx,\end{aligned}$$

where the first equality is by construction of χ, the second by the Substitution Theorem, and the third by the expression for χ'. Thus,

$$\int_0^1 (r(x)-1)^3 f_L(x)dx = \int_{x^*}^1 [(r(x)-1)^2 - (1-r(\chi(x)))^2](r(x)-1)f_L(x)dx.$$

and so, since $r(x)-1>0$ on $(x^*,1]$, it is enough that the term in square brackets is everywhere negative, i.e., that $1-r(\chi(x)) \geq r(x)-1$, or, equivalently, that

$$j(x) \equiv 2 - r(\chi(x)) - r(x)$$

is positive for all $x \in [x^*,1]$. Now, since $r(x^*)=1$, and since $\chi(x^*)=x^*$, we have $j(x^*)=0$, and

$$\begin{aligned}j' &= -r'(\chi(x))\chi'(x) - r'(x)\\ &= r'(\chi(x))\frac{r(x)-1}{1-r(\chi(x))}\frac{f_L(x)}{f_L(\chi(x))} - r'(x),\end{aligned}$$

and so, where $j \leq 0$, and so $r(x)-1 \geq 1-r(\chi(x)) \geq 0$,

$$j' \geq r'(\chi(x))\frac{f_L(x)}{f_L(\chi(x))} - r'(x)$$

which is positive as long as r is concave and f_L is increasing. Hence, j is everywhere positive on $[x^*,1]$, and we are done. □

Proof of Proposition 5 Since $C(a,u_0,\theta) = a\varphi_h + (1-a)\varphi_l$,

$$C_a(a,u_0,\theta) = \varphi_h - \varphi_l + a\,(1-a)\,c_{aa}\left(\varphi_h' - \varphi_l'\right). \tag{31}$$

Thus,

$$\begin{aligned} C_{aa}(a) &= \left(\varphi_h'(2-3a) - \varphi_l'(1-3a)\right) c_{aa} + \left(a-a^2\right)\left(c_{aaa}\left(\varphi_h' - \varphi_l'\right) + c_{aa}^2\left(\varphi_h''(1-a) + \varphi_l''a\right)\right) \\ &\geq \left(\varphi_l' + \varphi_h'(2-3a) - \varphi_l'(2-3a)\right) c_{aa} + \left(a-a^2\right) c_{aaa}\left(\varphi_h' - \varphi_l'\right) \\ &> \left(\varphi_h' - \varphi_l'\right)(2-3a) c_{aa} + \left(a-a^2\right) c_{aaa}\left(\varphi_h' - \varphi_l'\right), \end{aligned}$$

and so (14) is sufficient for $C_{aa} > 0$.

From Lemma (7) and (28), and using that $B_{aa} = 0$, strict *IMC* is guaranteed if

$$\left(c_{a\theta}C_{u_0} - C_{au_0}c_\theta + C_{a\theta}\right) c_{aa} < c_{a\theta}C_{aa}. \tag{32}$$

But, $C_{u_0} = a\varphi_h' + (1-a)\varphi_l'$, and so $C_{au_0} = \varphi_h' - \varphi_l' + a(1-a)c_{aa}\left(\varphi_h'' - \varphi_l''\right)$, and

$$\begin{aligned} C_{a\theta} &= \varphi_h'\left(c_\theta + (1-a)c_{a\theta}\right) - \varphi_l'\left(c_\theta - ac_{a\theta}\right) + a(1-a)c_{aa\theta}\left(\varphi_h' - \varphi_l'\right) \\ &\quad + a(1-a)c_{aa}\left(\varphi_h''\left(c_\theta + (1-a)c_{a\theta}\right) - \varphi_l''\left(c_\theta - ac_{a\theta}\right)\right). \end{aligned}$$

Substituting these expressions into (32) and manipulating, we want

$$\begin{aligned} &\left(c_{a\theta}\left(\varphi_l' + \varphi_h'\right) + a(1-a)\left[c_{aa}c_{a\theta}\left(\varphi_h''(1-a) + \varphi_l''a\right) + c_{aa\theta}\left(\varphi_h' - \varphi_l'\right)\right]\right) c_{aa} \\ < \;& c_{a\theta}c_{aa}\left(\varphi_h'(1-a) + \varphi_l'a\right) + c_{a\theta}\left((1-2a)c_{aa} + \left(a-a^2\right)c_{aaa}\right)\left(\varphi_h' - \varphi_l'\right) \\ &+ c_{a\theta}a(1-a)c_{aa}^2\left(\varphi_h''(1-a) + \varphi_l''a\right), \end{aligned}$$

or, $c_{aa}c_{a\theta}\varphi_l' + \left[a(1-a)\left(c_{aa}c_{aa\theta} + c_{a\theta}c_{aaa}\right) + (3a-1)c_{a\theta}c_{aa}\right]\left(\varphi_h' - \varphi_l'\right) < 0$, and so, since $\varphi_l' > 0$, it is sufficient that the term in square brackets is weakly negative, or, equivalently, (15). □

A.5 Proofs for Section 7

Proof of Lemma 3 Lemma 12 in Appendix C provides conditions under which a solution to the system (18)–(19) is a pair $\lambda(a,\theta,u_0)$ and $\mu(a,\theta,u_0)$ that is twice continuously differentiable. And since $\hat{v}(x,a,u_0,\theta) = \rho(\lambda + \mu l(x|a))$ for all x, it follows that $\hat{v}$ is twice continuously differentiable as well, and thus so is $C(a,u_0,\theta) = \int \varphi(\hat{v}(x,a,u_0,\theta))f(x|a)dx$.

Recall that for each θ, α is defined by $\kappa(\alpha(\theta),\theta) + \zeta(\theta) = 0$,where

$$\kappa(a,\theta) = \frac{(B_a - C_a)h}{c_{a\theta}} \leq 0, \text{ and } \zeta(\theta) = \int_\theta^{\overline{\theta}} C_{u_0}h \geq 0.$$

Consider any point (a,θ) with $\theta < \overline{\theta}$, where $\kappa(a,\theta) + \zeta(\theta) = 0$. Then, $B_a - C_a > 0$, and since $B_a - C_a$ is strictly decreasing in a given the assumption that $C_{aa} > 0$, and since $c_{aa\theta} \leq 0$, it follows that $\kappa_a > 0$. And since κ and ζ are continuous in θ, it follows that α is continuous in θ.

Now, the fact that α is continuous implies that $S(\theta) = \bar{u} - \int_{\underline{\theta}}^{\theta} c_\theta(\alpha(s), s)ds$ is continuously differentiable. Hence, ζ is continuously differentiable, since the integrand $C_{u_0}h$ is continuous by Lemma 3. But, κ is continuously differentiable as well, and so, as $\kappa_a > 0$, α is continuously differentiable by the Implicit Function Theorem. Note finally that $\bar{v}(\cdot, \theta)$ and $\underline{v}(\cdot, \theta)$ are trivially well defined for all θ, since $\lambda(\alpha(\cdot), S(\cdot), \cdot)$ and $\mu(\alpha(\cdot), S(\cdot), \cdot)$ are continuous. The last claim follows since for all (x, θ), $v(x, \theta) = \hat{v}(x, \alpha(\theta), S(\theta, \alpha), \theta)$. □

A.6 Proofs for Section 8

Proof of Theorem 2 Using Lemma 8, it is enough to show that (30) holds for sufficiently large $\bar{u}$. But, B_{aa} and all terms involving c are finite by assumption, while $\int v f_{aaa}$ (which is equal to $\int v_x F_{aa}$ by integration by parts) and $\int v_x l F_{aa}$ each have a finite limit from Chade and Swinkels (2019), Lemma 3, and $\mu_a^2 var_\xi(l) \int \rho' f$ is finite by Lemma 4. Since each of λ, λ/μ, and λ/μ_a diverges in $\bar{u}$ from Lemma 4, we are done. □

Proof of Proposition 6 Note that $\int \rho(\lambda+\mu l) f = \int (\lambda+\mu l) f = \lambda$, and hence, from the participation constraint, $\lambda = u_0 + c(a, \theta)$. Similarly, letting $\hat{I}(a) = 1/\int l^2 f$ be the reciprocal of the Fisher Information that the output carries about a, then $\int(\lambda + \mu l) f_a = \mu/\hat{I}(a)$, and hence $\mu = c_a(a, \theta)\hat{I}(a)$. From this,

$$C(a, u_0, \theta) = \frac{1}{2}\left((u_0 + c(a, \theta))^2 + c_a^2(a, \theta)\hat{I}\right) = \frac{1}{2}(\lambda^2 + \mu c_a). \tag{33}$$

This will allow us to apply Lemma 7 directly. In particular, $C_a = \lambda c_a + \frac{1}{2}\mu_a c_a + \frac{1}{2}\mu c_{aa}$, and hence

$$C_{aa} = c_a^2 + \lambda c_{aa} + \frac{1}{2}\mu_{aa} c_a + \mu_a c_{aa} + \frac{1}{2}\mu c_{aaa}.$$

Substituting this into (26), rearranging and using $\mu = c_a\hat{I}$, sufficient for *IMC* is

$$\bar{u} \geq -c - c_a\left(\hat{I}\frac{c_{aa\theta}}{c_{a\theta}} - \frac{c_a}{c_{aa}} - \frac{1}{2}\frac{\mu_{aa}}{c_{aa}} - \frac{\hat{I}}{2}\frac{c_{aaa}}{c_{aa}}\right).$$

Now, since $\mu = c_a\hat{I}$, we have $\mu_a = c_{aa}\hat{I} + c_a\hat{I}_a$, and $\mu_{aa} = c_{aaa}\hat{I} + 2c_{aa}\hat{I}_a + c_a\hat{I}_{aa}$ and so, substituting and rearranging gives

$$\bar{u} \geq -c + c_a\left(\hat{I}\left(\frac{c_{aaa}}{c_{aa}} - \frac{c_{aa\theta}}{c_{a\theta}}\right) + \hat{I}_a + \left(1 + \frac{\hat{I}_{aa}}{2}\right)\frac{c_a}{c_{aa}}\right) \geq -c,$$

using that $c_{aa}/c_{a\theta}$ is decreasing in a for each θ, $\hat{I}_a \geq 0$ and $\hat{I}_{aa} \geq -2$.[37] □

Proof of Example 4 Simplifying (27) and dividing by α, *IMC* holds in this case at any given θ if

[37]Substituting μ_{aa} into C_{aa} and gathering terms shows that these conditions also imply $C_{aa} > 0$.

and only if

$$\zeta(\theta) \equiv \int_{\theta}^{1} C_{u_0} + \frac{1}{\alpha}(c_{a\theta}C_{u_0} - C_{au_0}c_\theta + C_{a\theta})(1-\theta) + C_{aa} \leq 0,$$

where, with some algebra, $C_{aa} = (1-\theta)(S(1+\alpha)+(1-\theta)(3\alpha-5\alpha^2))$, and thus $\zeta(1)=0$. But,

$$\zeta' = -C_{u_0} + \left(\frac{1}{\alpha}(c_{a\theta}C_{u_0} - C_{au_0}c_\theta + C_{a\theta})\right)_\theta (1-\theta) - \frac{1}{\alpha}(c_{a\theta}C_{u_0} - C_{au_0}c_\theta + C_{a\theta}) + (C_{aa})_\theta$$

which, evaluated at $\theta = 1$ gives $\zeta' = -C_{u_0} - \frac{1}{\alpha}(c_{a\theta}C_{u_0} - C_{au_0}c_\theta + C_{a\theta}) + (C_{aa})_\theta$. In this case $\lambda = u_0 + c = u_0 + (1-\theta)a^2/2$, and $\mu = \hat{I}c_a = (1-\theta)a^2(1-a)$, and hence, using (29),

$$\begin{aligned} c_{a\theta}C_{u_0} - C_{au_0}c_\theta + C_{a\theta} &= 2\lambda c_{a\theta} + \mu_a c_{a\theta} + \mu c_{aa\theta} = -2a\lambda - a\mu_a - \mu \\ &= -a\left(2u_0 + (1-\theta)\,a^2\right) - (1-\theta)\left(2a^2 - 3a^3\right) - (1-\theta)a^2(1-a) \\ &= -2au_0 - 3(1-\theta)a^2(1-a), \end{aligned}$$

which, evaluated at $\theta = 1$ is equal to $-2\alpha S$. And,

$$(C_{aa})_\theta = -\left(S\,(1+\alpha) + (1-\theta)\left(3\alpha - 5\alpha^2\right)\right) + (1-\theta)\left(S\,(1+\alpha) + (1-\theta)\left(3\alpha - 5\alpha^2\right)\right)_\theta,$$

which, evaluated at $\theta = 1$ is $-S(1+\alpha)$.[38] Finally, at $\theta = 1$, $C_{u_0} = \lambda = S$. We thus have that at $\theta = 1$, $\zeta' = -S - \frac{1}{\alpha}(-2\alpha S) - S(1+\alpha) = -S\alpha < 0$. Thus, since $\zeta(1) = 0$, ζ is strictly positive near 1, violating *IMC*. □

A.7 Proofs for Section 9

Proof of Theorem 3 By *IMC*, $v_h - v_l$ is increasing in θ. Let θ^* be such that $v_h - v_l \geq u(\omega) - u(\omega - \ell)$ if and only if $\theta \geq \theta^*$. Fix an optimal menu, and assume some type θ is receiving strictly negative insurance, i.e., $v_h(\theta) > u(\omega) > u(\omega - \ell) > v_l(\theta)$. By *IMC*, insurance is strictly negative on all higher types as well. But then, since the agent is strictly risk averse, the principal is strictly losing money on all such types, since

$$(1-\alpha(\theta))u(\omega-\ell) + \alpha(\theta)u(\omega) - c(\alpha(\theta),\theta) \leq \bar{U}(\theta) \leq (1-\alpha(\theta))v_l(\theta) + \alpha(\theta)v_h(\theta) - c(\alpha(\theta),\theta),$$

and so since $v_h(\theta) > u(\omega) > u(\omega - \ell) > v_l(\theta)$,

$$(1-\alpha(\theta))\varphi(u(\omega-\ell)) + \alpha(\theta)\varphi(u(\omega)) < (1-\alpha(\theta))\varphi(v_l(\theta)) + \alpha(\theta)\varphi(v_h(\theta)),$$

[38] Since C_{aa} is 0 at $\theta = 1$ from above, and since $(C_{aa})_\theta < 0$ at $\theta = 1$, it follows that $C_{aa} > 0$ for θ near 1.

and hence

$$B(\alpha(\theta)) = \omega - (1-\alpha(\theta))\ell < (1-\alpha(\theta))\varphi(v_l(\theta)) + \alpha(\theta)\varphi(v_h(\theta)) = C(\alpha(\theta), S(\theta), \theta).$$

Consider the mechanism in which all types above θ^* are offered zero insurance at a zero premium. That is, $\alpha = \alpha_{NI}$, and $S = \bar{U}$. For types below θ^*, keep α at its original level, but raise t so as to reduce S by the constant amount $S(\theta^*) - \bar{U}(\theta^*)$. This menu is strictly more profitable, since a set of unprofitable types are now zero profit, while remaining types take the same action but at a higher premium. Since α is unchanged, *IMC* continues to hold on $[\underline{\theta}, \theta^*]$, while *IMC* holds on $[\theta^*, \bar{\theta}]$ since $c_a(\alpha_{NI}(\cdot), \cdot)$ is constant and equal to $u(\omega) - u(\omega - \ell)$ on that range. Hence incentive compatibility is satisfied. To see that the participation constraints hold, note that each $\theta < \theta^*$ gets strictly positive insurance and so takes an action $\alpha(\theta) < \alpha_{NI}(\theta)$. Thus,

$$S'(\theta) = -c_\theta(\alpha(\theta), \theta) < -c_\theta(\alpha_{NI}(\theta), \theta) = \bar{U}'(\theta),$$

since $c_{\theta a} < 0$. But then, since now $S(\theta^*) = \bar{U}(\theta^*)$, we have that $S(\theta) > \bar{U}(\theta)$ for all $\theta < \theta^*$.

To see that $\theta^* > \underline{\theta}$, let $\Delta_0 = u(\omega) - u(\omega - \ell)$, and define $\alpha(\Delta)$ by $c_a(\alpha(\Delta), \underline{\theta}) = \Delta$, so that $\alpha' = 1/c_{aa}$. Now, define $v_l(\Delta)$ by

$$v_l(\Delta) + \alpha(\Delta)\Delta - c(\alpha(\Delta), \underline{\theta}) = \bar{U}(\underline{\theta}), \tag{34}$$

so that $\underline{\theta}$ is indifferent between the policy $(v_l(\Delta), v_l(\Delta) + \Delta)$ and the outside option. The profit to the principal from $\underline{\theta}$ accepting is

$$\hat{\Pi} = \omega - (1-\alpha(\Delta))\ell - (1-\alpha(\Delta))\varphi(v_l(\Delta)) - \alpha(\Delta)\varphi(v_l(\Delta) + \Delta).$$

From (34), $v_l' = -\alpha'\Delta - \alpha + (c_a/c_{aa}) = -\alpha$, where the second equality comes from $\alpha' = 1/c_{aa}$ and $c_a = \Delta$. Thus, with a little manipulation, $\hat{\Pi}' = \alpha'(\ell - (\varphi_h - \varphi_l)) - (1-\alpha)\alpha(\varphi_h' - \varphi_l')$, where the first term reflects that as Δ increases, coverage is paid out less often, and the second that as Δ increases, the agent is bearing more risk. But then, since at Δ_0, coverage is zero, $\hat{\Pi}'(\Delta_0) < 0$, and hence, since $\hat{\Pi}(\Delta_0) = 0$, the policy $(v_l(\Delta), v_l(\Delta) + \Delta)$ is strictly profitable for Δ near Δ_0. Thus, $(v_l(\Delta) + \varepsilon, v_l(\Delta) + \Delta + \varepsilon)$ is accepted by an interval of types near $\underline{\theta}$ and is strictly profitable on any type that accepts it (since types above $\underline{\theta}$ pay the same premium but are less likely to incur losses). It follows that the principal gives a positive measure of agents strictly positive insurance at any optimal menu, and so $\theta^* > \underline{\theta}$.

To see upward distortion, note that the *rhs* of $(\ell - C_a)h = c_{a\theta} \int_{\underline{\theta}}^{\theta} C_{u_0} h$ is negative, and strictly so for $\theta > \underline{\theta}$. Hence, since at the constrained optimum, $\ell - C_a = 0$, any solution to the screening problem must involve a strictly higher effort. Insurance must thus be less than full, since by the incentive constraint, $v_h - v_l = c_a > 0$. □

A.8 Proofs for Section 10

Proof of Theorem 4 We proceed in several steps.

Step 1. Since $S(\overline{\theta}) \geq \overline{u}$, (10) implies that $S(\theta) \geq \overline{u}$ for all θ, and thus participation holds.

Step 2. Recall from Lemma 1 that conditional on being on L, the agent strictly prefers to announce his true type. It thus suffices to show that any deviation to $(\theta_A, \hat{a})$ for θ_T, with $\hat{a} \neq \alpha(\theta_A)$ is dominated by a deviation such that the action and the report are on L. We will show that this holds for any deviation that is above L, i.e., with $\hat{a} > \alpha(\theta_A)$. A symmetric argument handles deviations below L.

Step 3. We will show that fixing $\hat{a}$, and for any $\tilde{\theta}$ that the agent is contemplating announcing with $\hat{a} > \alpha(\tilde{\theta})$, the agent is better off by modifying his deviation so as to slightly raise θ from $\tilde{\theta}$. To do so, consider first the case where v is differentiable in θ at $\tilde{\theta}$. Notice that $\int v_\theta(x, \tilde{\theta}) f(x|\alpha(\tilde{\theta}))dx = 0$ by the first-order condition (11), and v_θ has sign pattern $-/+$ by hypothesis. Hence

$$\int v_\theta(x, \tilde{\theta}) f(x|\hat{a})dx = \int v_\theta(x, \tilde{\theta}) f(x|\alpha(\tilde{\theta})) \frac{f(x|\hat{a})}{f(x|\alpha(\tilde{\theta}))} dx \geq 0,$$

where we have used *MLRP*, $\hat{a} > \alpha(\tilde{\theta})$, and Beesack's inequality. Thus, the agent's expected utility is increasing in θ at $(\tilde{\theta}, \hat{a})$.

Consider now a jump point at $\tilde{\theta}$ with endpoints $\underline{a}$ and $\bar{a}$, and where $\hat{a} \geq \bar{a}$. It is enough to show that

$$\int \left(\overline{v}(x, \tilde{\theta}) - \underline{v}(x, \tilde{\theta}) \right) f(x|\overline{a})dx \geq 0, \tag{35}$$

for then, since $\bar{v}(\cdot, \tilde{\theta}) - \underline{v}(\cdot, \tilde{\theta})$ has sign pattern $-/+$ and since as before $f(\cdot|\hat{a})/f(\cdot|\overline{a})$ is increasing in x, we have $\int \left(\overline{v}(x, \tilde{\theta}) - \underline{v}(x, \tilde{\theta}) \right) f(x|\overline{a}) \left(f(x|\hat{a})/f(x|\overline{a}) \right) dx \geq 0$.Thus, the agent is again better off to raise the report of his type. To show (35), note that

$$\int \overline{v}(x, \tilde{\theta}) f(x|\overline{a})dx - c(\overline{a}, \tilde{\theta}) = S(\tilde{\theta}) = \int \underline{v}(x, \tilde{\theta}) f(x|\underline{a})dx - c(\underline{a}, \tilde{\theta}) \geq \int \underline{v}(x, \tilde{\theta}) f(x|\overline{a})dx - c(\overline{a}, \tilde{\theta}),$$

where the first two equalities use (24), and the inequality uses (24) and *FOP*. Comparing the outer terms and cancelling $c(\overline{a}, \tilde{\theta})$ gives (35).

Step 4. Suppose that $\hat{a} > \alpha(\bar{\theta})$. We will show that the agent is better off with a deviation to $(\bar{\theta}, \alpha(\bar{\theta}))$. To see this, notice that the previous step shows that

$$\int v(x, \bar{\theta}) f(x|\hat{a})dx \geq \int v(x, \theta_A) f(x|\hat{a})dx,$$

so the agent prefers deviation $(\bar{\theta}, \hat{a})$ to $(\theta_A, \hat{a})$. By *FOP* and c submodular we obtain

$$\int v(x,\bar{\theta})f(x|\hat{a})dx - \int v(x,\bar{\theta})f(x|\alpha(\bar{\theta}))dx \le c(\hat{a},\bar{\theta}) - c(\alpha(\bar{\theta}),\bar{\theta})$$
$$\le c(\hat{a},\theta_T) - c(\alpha(\bar{\theta}),\theta_T),$$

which rearranges to

$$\int v(x,\bar{\theta})f(x|\alpha(\bar{\theta}))dx - c(\alpha(\bar{\theta}),\theta_T) \ge \int v(x,\bar{\theta})f(x|\hat{a})dx - c(\hat{a},\theta_T),$$

and thus $(\bar{\theta}, \alpha(\bar{\theta}))$ is an even better deviation for the agent. Deviations with $\hat{a} < \alpha(\underline{\theta})$ are similarly ruled out and we will hence restrict attention to deviations with $\hat{a} \in [\alpha(\underline{\theta}), \alpha(\bar{\theta})]$.

Step 5. If there is a $\tilde{\theta}$ such that $\alpha(\tilde{\theta}) = \hat{a}$, then by Step 3, the agent is better off with deviation $(\tilde{\theta}, \alpha(\tilde{\theta})) \in L$. Suppose instead that for some θ_J there is a jump at θ_J containing $\hat{a}$, i.e., $\underline{a} \le \hat{a} \le \overline{a}$. Assume first that $\theta_J > \theta_T$. Then, by Step 3, $\int v(x,\theta_T)f(x|\hat{a})dx \ge \int v(x,\theta_A)f(x|\hat{a})dx$, and so type θ_T prefers the deviation $(\theta_T, \hat{a})$ to $(\theta_A, \hat{a})$. But, by *FOP,* $(\theta_T, \alpha(\theta_T))$ is better still. So, assume $\theta_J \le \theta_T$. Define $\hat{S}_1 = \int v(x,\theta_A)f(x|\hat{a})dx - c(\hat{a},\theta_T)$, $\hat{S}_2 = \int \underline{v}(x)f(x|\hat{a})dx - c(\hat{a},\theta_T)$, $\hat{S}_3 = \int \bar{v}(x)f(x|\bar{a})dx - c(\bar{a},\theta_T)$, and $\hat{S}_4 = S(\theta_T)$. These are the expected utilities for type θ_T at the points q_i, i = 1,2,3,4, in Figure 3, where q_2 reflects a limit from the left, and q_3 from the right.

By Lemma 1 and (24), we have $\hat{S}_4 \ge \hat{S}_3$, while by Step 3, $\hat{S}_2 \ge \hat{S}_1$. It remains only to show that $\hat{S}_3 \ge \hat{S}_2$. Note that

$$\int \bar{v}(x)f(x|\bar{a})dx - c(\bar{a},\theta_J) = S(\theta_J) = \int \underline{v}(x)f(x|\underline{a})dx - c(\underline{a},\theta_J) \ge \int \underline{v}(x)f(x|\hat{a})dx - c(\hat{a},\theta_J)$$

where the two equalities follow from (24) and the inequality by (24) and by *FOP.* But then, since $\theta_T \ge \theta_J$ and since c is submodular,

$$\hat{S}_3 = \int \bar{v}(x)f(x|\bar{a})dx - c(\bar{a},\theta_T) \ge \int \underline{v}(x)f(x|\hat{a})dx - c(\hat{a},\theta_T) = \hat{S}_2,$$

and we are done. Thus, the menu (α, v) is incentive compatible. □

Proof of Theorem 5 We will show that v_θ has sign pattern $-/+$. It follows by Theorem 4, and by the fact that a solution to the decoupling program satisfies *BFC* and *FOP* by construction, that the decoupling menu is feasible and hence, as the solution to a relaxed program, optimal.

We can assume $\alpha' \le -c_{a\theta}/c_{aa}$. Note that if at any given point $\alpha' \ge -c_{a\theta}/c_{aa}$, then it is automatic that v_θ has sign pattern $-/+$. This follows as in the discussion at the end of Section 5, since by construction, $\int v_\theta f = 0$ and hence, if v_θ is $+/-$, then by Beesack's inequality, $\int v_\theta f_a \le 0$, a contradiction. So, in what follows, we can assume that $\alpha' \le -c_{a\theta}/c_{aa}$.

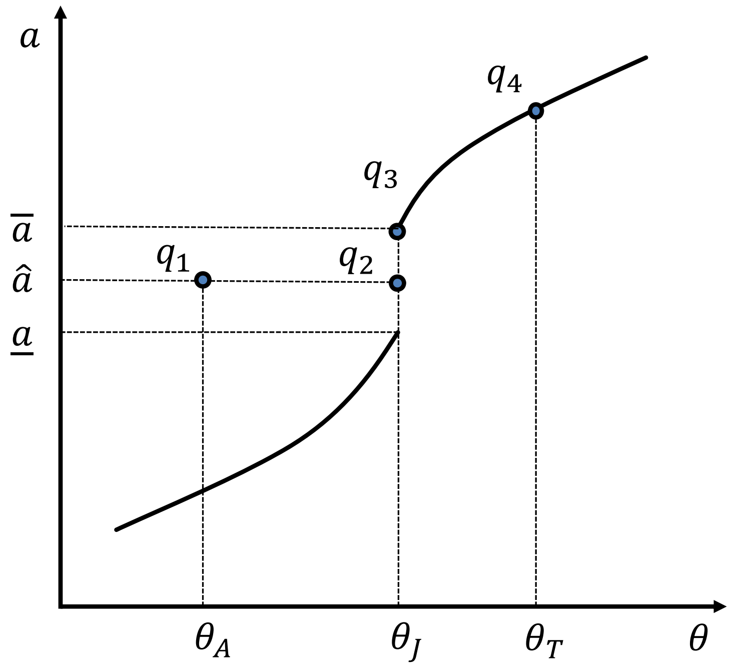

Figure 3: ***SCC***. Under *SCC*, a deviation by θ_T to q_1 is dominated by one to q_2. But that deviation in turn is dominated by a deviation to q_3 and, since q_3 is on locus, it is dominated by telling the truth and taking the recommended action at point q_4.

$(C_a)_\theta \geq 0$ IN ANY SOLUTION TO THE SCREENING PROBLEM. Recall that $(B_a - C_a) = (-c_{a\theta}/h)\int_\theta^{\bar{\theta}} C_{u_0} h$ is an identity, and so, since $B_{aa} = 0$, we have

$$(-C_a)_\theta = \left(\frac{-c_{a\theta}}{h}\right)_\theta \int_\theta^{\bar{\theta}} C_{u_0} h - \frac{-c_{a\theta}}{h} C_{u_0} h \leq 0, \tag{36}$$

Note that

$$\begin{aligned} \left(\frac{-c_{a\theta}}{h}\right)_\theta \underset{s}{=} \left(\log \frac{-c_{a\theta}}{h}\right)_\theta &= \frac{-c_{a\theta\theta}}{-c_{a\theta}} - \frac{h'}{h} + \alpha' \frac{-c_{aa\theta}}{-c_{a\theta}} \\ &\leq \frac{-c_{a\theta\theta}}{-c_{a\theta}} - \frac{h'}{h} - \frac{c_{aa\theta}}{c_{aa}} = \left(\log \frac{-c_{a\theta}}{hc_{aa}}\right)_\theta \leq 0, \end{aligned}$$

where the first inequality follows since $c_{aa\theta} \leq 0$, and the second since $c_{a\theta}/(hc_{aa})$ is increasing. From (36), it follows that $(C_a)_\theta \geq 0$. We will show that if v_θ has sign pattern $+/-$ (strictly) then in fact $(C_a)_\theta < 0$, a contradiction.

RE-EXPRESSING π. Since $l_x(x|a')/l_x(x|a)$ is independent of x, we can fix some reference action $\hat{a}$, let $\hat{l} = l(\cdot|\hat{a})$, and express all relevant contracts as $1/u'(\pi(\theta, x)) = m(\theta) + s(\theta)\hat{l}(x) > 0$, where

$s(\theta) > 0$, and where the condition that v_θ is $+/-$ is equivalent to $s_\theta < 0$.[39]

Setting up the contradiction. From the Envelope Theorem, $C_a = \int \pi f_a + \mu \left(c_{aa} - \int v f_{aa}\right)$. We will show that under the assumption that v_θ is $+/-$, each of $\int \pi f_a$, μ, and $c_{aa} - \int v f_{aa}$ decreases with θ, with $\int \pi f_a$ decreasing strictly. Since $c_{aa} - \int v f_{aa} \geq 0$ (as $c_{aa} \geq 0$ and since $F_{aa} \geq 0$, and hence $\int v f_{aa} \leq 0$), it would then follow that $(C_a)_\theta < 0$, contradicting that v_θ is $+/-$ and completing the proof.

μ is decreasing. Since $s(\theta)\, \hat{l}_x(x) = \mu(\theta)\, l_x(x|\alpha(\theta))$ for all θ, we have

$$s_\theta(\theta)\, \hat{l}_x(x) = \mu_\theta(\theta)\, l_x(x|\alpha(\theta)) + \mu(\theta)\, \alpha'(\theta)\, l_{xa}(x|\alpha(\theta)).$$

By the premise, $s_\theta < 0$, and thus $s_\theta(\theta)\, \hat{l}_x(x) < 0$. The second term on the *rhs* is positive by the assumption that $l_{ax} \geq 0$ and $\alpha' > 0$. Hence, $\mu_\theta < 0$.

$\int \pi f_a$ is strictly decreasing. Note that $\left(\int \pi f_a\right)_\theta = \int \pi_\theta f_a + \alpha' \int \pi f_{aa}$, where the second term is negative since $\alpha' > 0$ and since $\int \pi f_{aa} = \int \pi_x (-F_{aa}) \geq 0$ using that $\pi_x > 0$, and $F_{aa} \geq 0$.

It is thus enough to show that $\int \pi_\theta f_a < 0$. To see this, recall that $\int v_\theta f = 0$. Thus, by the premise that v_θ is $+/-$ (strictly), and since $1/u'$ is strictly increasing, we have by Beesack's inequality that $\int \pi_\theta f = \int (1/u')\, v_\theta f < 0$.But, since $\pi_x = \psi'(m + s\hat{l})s\hat{l}_x$, we have $\pi_{x\theta} = \psi''(m + s\hat{l})s\hat{l}_x(m_\theta + s_\theta \hat{l}) + \psi'(m + s\hat{l})s_\theta \hat{l}_x$. By assumption, $\psi'' \geq 0$. It follows that $\pi_{x\theta} < 0$ whenever $\pi_\theta =_s m_\theta + s_\theta \hat{l} \leq 0$. Let $\hat{m}$ be such that$\int (\pi_\theta + \hat{m})\, f = 0$, where since $\int \pi_\theta f < 0$, we have $\hat{m} > 0$. Since $\pi_{x\theta} < 0$ whenever $\pi_\theta \leq 0$, it follows that $\pi_\theta + \hat{m}$ single-crosses 0 from above (and does so strictly). Hence, since l is strictly increasing in x, Beesack's inequality implies that $\int \pi_\theta f_a = \int (\pi_\theta + \hat{m})\, f_a = \int (\pi_\theta + \hat{m})\, l f < 0$, and it follows that $\int \pi f_a$ strictly decreases in θ.

Analyzing $c_{aa} - \int v f_{aa}$. Since $(c_{aa})_\theta = c_{aaa}\alpha' + c_{aa\theta}$, and using $c_{aa\theta} \leq 0$, if $c_{aaa} \leq 0$, then $(c_{aa})_\theta \leq 0$ since $\alpha' \geq 0$, while if $c_{aaa} > 0$, then, since $\alpha' \leq -c_{a\theta}/c_{aa}$,

$$c_{aaa}\alpha' + c_{aa\theta} \leq c_{aaa}\frac{-c_{a\theta}}{c_{aa}} + c_{aa\theta} =_s \left(\frac{c_{aa}}{-c_{a\theta}}\right)_a \leq 0.$$

Thus, it suffices that $\left(\int v f_{aa}\right)_\theta = \int v_\theta f_{aa} + \alpha' \int v f_{aaa} \geq 0$. But, $\alpha' \int v f_{aaa} = -\alpha' \int v_x F_{aaa} \geq 0$, since $\alpha' \geq 0$, $v_x > 0$, and $F_{aaa} \leq 0$. It thus suffices that $\int v_\theta f_{aa} \geq 0$, or equivalently, that $\int v_{\theta x} F_{aa} \leq 0$. We would be done if we could show that $v_{\theta x}$ was everywhere negative, since F_{aa} is everywhere positive.

$v_{\theta x}$ is negative. Since $v_\theta = (\rho')\,(m_\theta + s_\theta \hat{l})$, we have $v_{\theta x} = z\hat{l}_x$, where

$$z = \rho'' s(m_\theta + s_\theta \hat{l}) + \rho' s_\theta, \tag{37}$$

[39] To see this, note that $v_\theta = (\rho')\,(m_\theta + s_\theta \hat{l})$ and $v_{\theta x} = (\rho'')\,(m_\theta + s_\theta \hat{l})s\hat{l}_x + \rho' s_\theta \hat{l}_x$. Hence, at any point where $v_\theta = 0$, we have $v_{\theta x} = \rho' s_\theta \hat{l}_x$, which is strictly negative if and only if $s_\theta > 0$.

and it is enough to show that z is everywhere negative. Let $v_\theta(\hat{x}) = 0$. To the left of $\hat{x}$, $v_\theta =_s m_\theta + s_\theta \hat{l} \geq 0$, and so, since $s_\theta < 0$ and $\rho'' \leq 0$, we have $z < 0$. It is thus enough to show that z does not cross zero from below anywhere to the right of $\hat{x}$.

Now, $z_x = \left(\rho''' s^2 (m_\theta + s_\theta \hat{l}) + 2\rho'' s s_\theta\right) \hat{l}_x$, and, since $\rho' s_\theta < 0$, where $z = 0$, we have by (37) that $\rho'' < 0$, and so $m_\theta + s_\theta \hat{l} = -(\rho'/\rho'')(s_\theta/s)$. Inserting this expression into z_x yields

$$z_x = \left(\rho''' s^2 \left(-\frac{\rho'}{\rho''}\frac{s_\theta}{s}\right) + 2\rho'' s s_\theta\right) \hat{l}_x =_s \frac{\rho'''}{\rho''} - 2\frac{\rho''}{\rho'} =_s \frac{-\rho''}{(\rho')^2}\left(\frac{\rho'''}{\rho''} - 2\frac{\rho''}{\rho'}\right) = \left(\frac{1}{\rho'}\right)'' < 0,$$

where the first equality of sign uses that $s_\theta < 0$ and the second that $\rho'' < 0$. Thus, any crossing point of z is strictly from above, and we are done. □

A.9 Proofs For Section 11

Proof of Proposition 7 If we replace $\underline{\theta}$ in (16) by θ^*, we have the relaxed problem subject to exclusion. By way of contradiction, assume this problem admits a strictly better solution $\tilde{\alpha}$ than $\hat{\alpha}(\cdot, \theta^*)|_{[\theta^*, \bar{\theta}]}$. Then, paste $\tilde{\alpha}$ and $\hat{\alpha}(\cdot, \theta^*)|_{[\underline{\theta}, \theta^*]}$ together, holding fixed $S(\theta^*) = \bar{u}$, to obtain a strictly better feasible solution to (16). But, the optimal solution to the relaxed problem is unique by Appendix B, and we have a contradiction.

Now, let $Z(\theta_c, \theta^*) = \int_{\theta_c}^{\bar{\theta}} (B(\hat{\alpha}(\theta, \theta^*)), \theta) - C(\hat{\alpha}(\theta, \theta^*), \hat{S}(\theta, \theta_c, \theta^*), \theta) h(\theta)\, d\theta$ be the profit to the principal of excluding types below θ_c and choosing action profile $\hat{\alpha}(\cdot, \theta^*)$ for types above θ_c. But, by the first paragraph, $Z(\theta_c, \cdot)$ is optimized at $\theta^* = \theta_c$, and thus, since $\hat{\alpha}(\theta, \theta^*)$ is differentiable in θ^*, $Z_{\theta^*}(\theta_c, \theta_c) = 0$. The first order condition on the choice of the cutoff θ_c for the principal is thus $0 = \frac{d}{d\theta_c}(Z(\theta_c, \theta_c)) = Z_{\theta_c}(\theta_c, \theta_c) + Z_{\theta^*}(\theta_c, \theta_c) = Z_{\theta_c}(\theta_c, \theta_c)$.

But, note that

$$\begin{aligned} Z_{\theta_c}(\theta_c, \theta_c) = &- (B(\hat{\alpha}(\theta_c, \theta_c)), \theta_c) - C\left(\hat{\alpha}(\theta_c, \theta_c), \hat{S}(\theta_c, \theta_c, \theta_c), \theta_c\right) h(\theta_c) \\ &- c_\theta(\hat{\alpha}(\theta_c, \theta_c), \theta_c) \int_{\theta_c}^{\bar{\theta}} C_{u_0}(\hat{\alpha}(\theta, \theta_c), \hat{S}(\theta, \theta_c, \theta_c), \theta) h(\theta)\, d\theta, \end{aligned} \tag{38}$$

using that for all $\theta > \theta_c$, $\hat{S}_{\theta_c}(\theta, \theta_c, \theta_c) = -c_\theta(\hat{\alpha}(\theta_c, \theta_c), \theta_c)$. Rearranging gives the necessity result.

It remains to show that if $c_{\theta\theta} \geq 0$, then, the function $Z(\theta_c, \theta_c)$ is concave in θ_c, and hence, a solution to the first order condition characterizes the optimal cutoff. Recall that for all θ_c, $\frac{d}{d\theta_c}(Z(\theta_c, \theta_c)) = Z_{\theta_c}(\theta_c, \theta_c)$, and hence $\frac{d^2}{d\theta_c^2}(Z(\theta_c, \theta_c)) = \frac{d}{d\theta_c}(Z_{\theta_c}(\theta_c, \theta_c))$, which by (38), is equal to

$$-\left((B_a - C_a) h - c_{\theta a}\int_{\theta_c}^{\bar{\theta}} C_{u_0}\right)\frac{d}{d\theta_c}(\hat{\alpha}(\theta_c, \theta_c)) + C_\theta h - c_{\theta\theta}\int_{\theta_c}^{\bar{\theta}} C_{u_0} - c_\theta \frac{d}{d\theta_c}\int_{\theta_c}^{\bar{\theta}} C_{u_0}.$$

The first term is zero using the *FOC* with respect to the action at θ_c. The terms $C_\theta h$ and

$-c_{\theta\theta}\int_{\theta_c}^{\bar{\theta}} C_{u_0}$ are each negative by assumption. Since $-c_\theta > 0$, it remains to show that

$$\frac{d}{d\theta_c}\int_{\theta_c}^{\bar{\theta}} C_{u_0}(\hat{\alpha}\left(\tau,\theta_c\right), \hat{S}(\tau,\theta_c,\theta_c),\tau)h\left(\tau\right)d\tau \leq 0,$$

where the Leibniz term is strictly negative since $C_{u_0} > 0$. It would thus suffice to show that for each $\theta \geq \theta_c$,

$$k\left(\theta\right) = \int_{\theta}^{\bar{\theta}} \frac{d}{d\theta_c} C_{u_0}(\hat{\alpha}\left(\tau,\theta_c\right), \hat{S}(\tau,\theta_c,\theta_c),\tau)h\left(\tau\right)d\tau \leq 0.$$

Note that $k\left(\bar{\theta}\right) = 0$. Hence, it is enough to show that for any $\hat{\theta} \geq \theta_c$, $k'(\hat{\theta}) \geq 0$. But,

$$k'(\hat{\theta}) =_s -\frac{d}{d\theta_c} C_{u_0}(\hat{\alpha}(\hat{\theta},\theta_c), \hat{S}(\hat{\theta},\theta_c,\theta_c),\hat{\theta}),$$

and so, evaluating the *rhs*, we desire to show that

$$C_{u_0a}(\hat{\alpha}(\hat{\theta},\theta_c), \hat{S}(\hat{\theta},\theta_c,\theta_c),\hat{\theta})\hat{\alpha}_{\theta_c}(\hat{\theta},\theta_c) + C_{u_0u_0}(\hat{\alpha}(\hat{\theta},\theta_c), \hat{S}(\hat{\theta},\theta_c,\theta_c),\hat{\theta})\frac{d}{d\theta_c}\hat{S}(\hat{\theta},\theta_c,\theta_c) \leq 0. \tag{39}$$

Now, $C_{u_0a} > 0$ by assumption. Consider $\frac{d}{d\theta_c}\hat{S}(\hat{\theta},\theta_c,\theta_c)$. Fix any $\theta_H > \theta_L$. Then, we claim that for all θ, $\hat{S}(\theta,\theta_H,\theta_H) \leq \hat{S}(\theta,\theta_L,\theta_L)$. To see this, note first that $\bar{u} = \hat{S}(\theta_H,\theta_H,\theta_H) < \hat{S}(\theta_H,\theta_L,\theta_L)$. Assume that at some point $\tilde{\theta}$, $\hat{S}(\tilde{\theta},\theta_H,\theta_H) = \hat{S}(\tilde{\theta},\theta_L,\theta_L)$. Then, as in the first paragraph of the proof, the optimal solutions $\hat{\alpha}\left(\cdot,\theta_H\right)$ and $\hat{\alpha}\left(\cdot,\theta_L\right)$ coincide for all $\theta > \tilde{\theta}$, and hence so do $\hat{S}(\cdot,\theta_H,\theta_H)$ and $\hat{S}(\cdot,\theta_L,\theta_L)$. It follows that $\frac{d}{d\theta_c}\hat{S}(\hat{\theta},\theta_c,\theta_c) \leq 0$,

Now, from the optimality of $\hat{\alpha}(\cdot,\theta_c)$, we have that for all θ_c,

$$B_a(\hat{\alpha}(\hat{\theta},\theta_c)) - C_a(\hat{\alpha}(\hat{\theta},\theta_c), \hat{S}(\hat{\theta},\theta_c,\theta_c),\hat{\theta}) = -\frac{1}{h(\hat{\theta})}c_{a\theta}(\hat{\alpha}(\hat{\theta},\theta_c),\hat{\theta})\int_{\hat{\theta}}^{\bar{\theta}} C_{u_0}(\hat{\alpha}\left(\tau,\theta_c\right), \hat{S}(\tau,\theta_c,\theta_c),\tau)h\left(\tau\right)d\tau,$$

and hence, differentiating by θ_c,

$$\left(B_{aa} - C_{aa} + \frac{1}{h}c_{aa\theta}\int_{\hat{\theta}}^{\bar{\theta}} C_{u_0}\right)\hat{\alpha}_{\theta_c}(\hat{\theta},\theta_c) = C_{au_0}\frac{d}{d\theta_c}\hat{S}(\hat{\theta},\theta_c,\theta_c)$$

where the omitted terms can be ignored since $k(\hat{\theta}) = 0$.

We thus have that

$$\hat{\alpha}_{\theta_c}(\hat{\theta},\theta_c) = \frac{C_{au_0}}{B_{aa} - C_{aa} + \frac{1}{h}c_{aa\theta}\int_{\hat{\theta}}^{\bar{\theta}} C_{u_0}}\frac{d}{d\theta_c}\hat{S}(\hat{\theta},\theta_c,\theta_c) \leq \frac{C_{au_0}}{-C_{aa}}\frac{d}{d\theta_c}\hat{S}(\hat{\theta},\theta_c,\theta_c)$$

where we use that $B_{aa} \leq 0$ and $c_{aa\theta} \leq 0$, and that $(d/d\theta_c)\hat{S}(\hat{\theta},\theta_c,\theta_c) \leq 0$. The *lhs* of (39) is thus at most $\left(C_{u_0u_0}C_{aa} - C_{u_0a}^2\right)(1/C_{aa})(d/d\theta_c)\hat{S}(\hat{\theta},\theta_c,\theta_c) \leq 0$,where the inequality follows since the bracketed term is positive by the convexity of C in a and u_0. □

Proof of Example 5 Since $\log f = \log r + \theta \log g - \log \int rg^\theta$, we have $f_\theta/f = \log g - \int rg^\theta(\log g) \int rg^\theta$, and so $(f_\theta/f)_x = g_x/g > 0$. Similarly, $f_a/f = \theta g_a/g - \int \theta rg^{\theta-1} g_a / \int rg^\theta$, and so $(f_a/f)_x = \theta(g_a/g)_x > 0$. It remains to show that $F_{a\theta} \leq 0$. But,

$$f_a = f\theta \left(\frac{g_a}{g} - \frac{\int rg^\theta \frac{g_a}{g}}{\int rg^\theta} \right) = f\theta \left(\frac{g_a}{g} - \int \frac{g_a}{g} f \right) = f\theta \left(\frac{g_a}{g} - \gamma \right),$$

where $\gamma = \int (g_a/g)f$. Note that $\gamma_\theta = \int (g_a/g) f_\theta = \int_x (g_a/g)_x(-F_\theta) > 0$, using that g is *lsm* in a and x, and that since f_θ/f is increasing, $-F_\theta > 0$ on $[\underline{x}, \bar{x}]$. Thus,

$$f_{a\theta} = \left(\left(\frac{f_\theta}{f}\theta + 1 \right) \left(\frac{g_a}{g} - \gamma \right) - \theta\gamma_\theta \right) f.$$

To show that $F_{a\theta} \leq 0$, it is enough to show that $f_{a\theta}(\cdot|a,\theta)$ single-crosses zero from below, using that $F_{a\theta}(x|a,\theta) = \int_{\underline{x}}^x f_{a\theta} ds$, and that $F_{a\theta}(\underline{x}|a,\theta) = F_{a\theta}(\bar{x}|a,\theta) = 0$. But, since $\gamma_\theta > 0$, and since by assumption $(f_\theta/f)\theta + 1 \geq 0$, it follows that at any point where $f_{a\theta}(s|a,\theta) = 0$, both $(f_\theta/f)\theta + 1$ and $(g_a/g) - \gamma$ are positive and increasing in x, and the result follows. □

For contract $\hat{v}$, action a, and type θ, define $U(\hat{v}, a, \theta) = \int \hat{v}(x) f(x|a,\theta) dx - c(a,\theta)$. Note that $U_a = \int \hat{v} f_a - c_a = \int \hat{v}_x(-F_a) - c_a$, and hence $U_{a\theta} = \int \hat{v}_x(-F_{a\theta}) - c_{a\theta} \geq 0$ since $F_{a\theta} \leq 0$, and since c is submodular.

Lemma 10 *Let (α, v) satisfy the FOCs, let α be increasing, and let v satisfy SCC, where for each θ, $v(\cdot,\theta)$ is increasing and bounded. Let $\hat{S}(\theta_T, \hat{\theta}) = U(v(\cdot,\hat{\theta}), \alpha(\hat{\theta}), \theta_T)$ be the value to type θ_T of imitating $\hat{\theta}$'s announcement and action. Then, $\hat{S}(\theta_T, \cdot)$ is single-peaked at θ_T for all θ_T.*

Proof Since α is increasing, it is differentiable almost everywhere, and jumps at a countable set of points. To show single-peakedness, it is enough to show that for $\hat{\theta} < \theta_T$, where α is differentiable, $\hat{S}$ is also differentiable, with $\hat{S}_{\hat{\theta}}(\theta_T, \hat{\theta}) \geq 0$, and that at jump points of α, $\hat{S}(\theta_T, \cdot)$ also jumps up. The zero measure set of points where α is continuous but not differentiable is irrelevant, and the case $\hat{\theta} > \theta_T$ is symmetric.

First, consider any point $\hat{\theta}$ at which α is differentiable. By regularity, $v_\theta(x, \hat{\theta})$ is defined for almost all x. Thus,

$$\hat{S}_{\hat{\theta}}(\theta_T, \hat{\theta}) = \int v_\theta(x,\hat{\theta}) f(x|\alpha(\hat{\theta}), \theta_T) dx + \alpha'(\hat{\theta}) \left(\int v(x,\hat{\theta}) f_a(x|\alpha(\hat{\theta}), \theta_T) dx - c_a(\alpha(\hat{\theta}), \theta_T) \right).$$

It suffices to show that each of the two terms on the *rhs* is positive evaluated at $\hat{\theta} = \theta_T$.

By *FOC*, $\int v_\theta(x,\hat{\theta}) f(x|\alpha(\hat{\theta}), \hat{\theta}) dx = 0$, and so, since $f(\cdot|\alpha(\hat{\theta}), \theta_T)/f(\cdot|\alpha(\hat{\theta}), \hat{\theta})$ is increasing, the first term is positive using Beesack's Inequality. The second term is positive at $\hat{\theta} = \theta_T$, using that $\alpha'(\theta) \geq 0$, that $U_a(\hat{\theta}, \alpha(\hat{\theta}), \hat{\theta}) = 0$, and that $U_{a\theta} \geq 0$.

Consider a point $\hat{\theta}$ at which α jumps, with $\underline{a}$, $\bar{a}$, $\underline{v}$, and $\bar{v}$ defined as usual. Then, for any $\tilde{\theta}$,

$$\lim_{\theta \uparrow \hat{\theta}} U(v(\cdot, \theta), \alpha(\theta), \tilde{\theta}) = U(\underline{v}, \underline{a}, \tilde{\theta}) \text{ and } \lim_{\theta \uparrow \hat{\theta}} U_a(v(\cdot, \theta), \alpha(\theta), \tilde{\theta}) = U_a(\underline{v}, \underline{a}, \tilde{\theta}),$$

and similarly for $\bar{a}$, $\bar{v}$ and $\theta \downarrow \hat{\theta}$. We wish to establish that

$$0 \leq \lim_{\theta \downarrow \hat{\theta}} \hat{S}(\theta_T, \theta) - \lim_{\theta \uparrow \hat{\theta}} \hat{S}(\theta_T, \theta) = U(\bar{v}, \bar{a}, \theta_T) - U(\underline{v}, \underline{a}, \theta_T). \tag{40}$$

But, note that $S(\hat{\theta}) = \lim_{\theta \uparrow \hat{\theta}} \hat{S}(\hat{\theta}, \theta) = \lim_{\theta \uparrow \hat{\theta}} U(v(\cdot, \theta), \alpha(\theta), \hat{\theta}) = U(\underline{v}, \underline{a}, \hat{\theta})$, and similarly, $S(\hat{\theta}) = U(\bar{v}, \bar{a}, \hat{\theta})$, and thus $U(\bar{v}, \bar{a}, \hat{\theta}) = U(\underline{v}, \underline{a}, \hat{\theta})$. Further, since $U_a(v(\cdot, \theta), \alpha(\theta), \theta) = 0$ for all θ by *FOC*, $U_a(\underline{v}, \underline{a}, \hat{\theta}) = 0$, and so by *FOP*, $U(\underline{v}, \underline{a}, \hat{\theta}) \geq U(\underline{v}, \bar{a}, \hat{\theta})$, and hence, combining these two equations, $U(\bar{v}, \bar{a}, \hat{\theta}) \geq U(\underline{v}, \bar{a}, \hat{\theta})$. Similarly, $U_a(\bar{v}, \bar{a}, \hat{\theta}) = 0$, and so $U(\bar{v}, \underline{a}, \hat{\theta}) \leq U(\underline{v}, \underline{a}, \hat{\theta})$. There is thus $\tilde{a} \in [\underline{a}, \bar{a}]$ such that $U(\bar{v}, \tilde{a}, \hat{\theta}) = U(\underline{v}, \tilde{a}, \hat{\theta})$.

Now, for any θ, $U(\bar{v}, \bar{a}, \theta) - U(\underline{v}, \underline{a}, \theta)$ is equal to

$$(U(\bar{v}, \bar{a}, \theta) - U(\bar{v}, \tilde{a}, \theta)) + (U(\bar{v}, \tilde{a}, \theta) - U(\underline{v}, \tilde{a}, \theta)) + (U(\underline{v}, \tilde{a}, \theta) - U(\underline{v}, \underline{a}, \theta)).$$

Evaluated at $\theta = \hat{\theta}$, this difference is 0. We will show that each of the three bracketed terms is larger evaluated at $\theta = \theta_T$ than at $\theta = \hat{\theta}$, establishing (40). But, $U_{a\theta}(\hat{v}, a, \theta) \geq 0$, and so the first bracketed term, which is equal to $\int_{\tilde{a}}^{\bar{a}} U_a(\bar{v}, a, \theta) da$, is increasing in θ, and similarly for the third term. The middle term is $U(\bar{v}, \tilde{a}, \theta) - U(\underline{v}, \tilde{a}, \theta) = \int (\bar{v}(x) - \underline{v}(x)) f(x|\tilde{a}, \theta) dx$. By definition of $\tilde{a}$, this is zero evaluated at $\theta = \hat{\theta}$. But then, since $f(\cdot|\tilde{a}, \theta_T)/f(\cdot|\tilde{a}, \hat{\theta})$ is increasing, $U(\bar{v}, \tilde{a}, \theta_T) - U(\underline{v}, \tilde{a}, \theta_T) \geq 0$ by Beesack's Inequality and we are done. □

Proof of Proposition 8 Consider a type θ_T, and deviation $(\hat{a}, \hat{\theta})$. We focus on the case where $\hat{\theta} \leq \theta_T$, and then appeal to symmetry. Given Lemma 10, the key, as before, is to show that there is some "on locus" deviation $(\alpha(\theta), \theta)$ that θ_T prefers to $(\hat{a}, \hat{\theta})$.

Assume first that $\hat{a} \leq \alpha(\hat{\theta})$. Then, since $U_a(v(\cdot, \hat{\theta}), \alpha(\hat{\theta}), \hat{\theta}) = 0$, it follows from *FOP* that for any $a \in [\hat{a}, \alpha(\hat{\theta})]$, $U_a(v(\cdot, \hat{\theta}), a, \hat{\theta}) \geq 0$, and so, since $U_{a\theta} \geq 0$, the deviation $(\hat{a}, \hat{\theta})$ is dominated for θ_T by the on-locus deviation $(\alpha(\hat{\theta}), \hat{\theta})$.

Assume next that $\hat{a} > \alpha(\hat{\theta})$. Informally, we will show that, holding fixed $\hat{a}$, type θ_T is better off to increase his announced type until either he reaches the locus or θ_T. In the latter case, using *FOC* and *FOP*, $(\alpha(\theta_T), \theta_T)$ is better still.

Consider in particular, any $\tilde{\theta} < \theta_T$ at which $\hat{a} \geq \alpha(\tilde{\theta})$. Assume first that α is differentiable at $\tilde{\theta}$. Then, by regularity, $v_\theta(\cdot, \theta)$ is well-defined and from *FOC*, $\int v_\theta(x, \tilde{\theta}) f(x|\alpha(\tilde{\theta}), \tilde{\theta}) dx = 0$. But then, since v_θ is $-/+$ by assumption, and using Beesack's Inequality, it is enough that

$$\frac{f(x|\hat{a},\theta)}{f(x|\alpha(\tilde{\theta}),\tilde{\theta})} = \frac{f(x|\hat{a},\theta)}{f(x|\alpha(\tilde{\theta}),\theta)} \frac{f(x|\alpha(\tilde{\theta}),\theta)}{f(x|\alpha(\tilde{\theta}),\tilde{\theta})} \tag{41}$$

increases in x. Since each of f_a/f and f_θ/f are increasing in x, it follows that $f(x|\hat{a},\theta)/f(x|\alpha(\tilde{\theta}),\tilde{\theta})$ is the product of positive increasing functions, and so is increasing.

Consider a jump point $\tilde{\theta}$, and, as in the proof of Lemma 10, define the corresponding effort levels $\underline{a}$ and $\bar{a}$ and contracts $\underline{v}$ and $\bar{v}$. Assume first that $\bar{a} \leq \hat{a}$. Then, as in the proof of Lemma 10, $\int(\bar{v}(x) - \underline{v}(x))f(x|\tilde{a},\tilde{\theta})dx = 0$ for some $\tilde{a} \in [\underline{a}, \bar{a}]$, and so, as (41), $\int(\bar{v}(x) - \underline{v}(x))f(x|\hat{a},\theta_T)dx \geq 0$.

Finally, consider the case that $\hat{a} \in [\underline{a}, \bar{a}]$. Then, for all $\theta < \tilde{\theta}$, the reasoning above shows that the agent is better to locally increase his announced type, and thus, $U(\underline{v}, \hat{a}, \theta_T) \geq U(v(\cdot, \hat{\theta}), \hat{a}, \theta_T)$, and so, to get "on-locus" it would be enough to establish that $U(\bar{v}, \bar{a}, \theta_T) \geq U(\underline{v}, \hat{a}, \theta_T)$.

Note in particular

$$U(\bar{v}, \bar{a}, \tilde{\theta}) = S(\tilde{\theta}) = U(\underline{v}, \underline{a}, \tilde{\theta}) \geq U(\underline{v}, \hat{a}, \tilde{\theta}), \tag{42}$$

where the inequality follows *FOC*, *FOP*, and convexity of c in a. Consider first the case that $\tilde{a} \geq \hat{a}$. Then, for each θ, $U(\bar{v}, \bar{a}, \theta) - U(\underline{v}, \hat{a}, \theta)$ equals

$$\int_{\hat{a}}^{\tilde{a}} U_a(\underline{v}, a, \theta)da + \int(\bar{v}(x) - \underline{v}(x))f(x|\tilde{a}, \theta)dx + \int_{\tilde{a}}^{\bar{a}} U_a(\bar{v}, a, \theta)da,$$

which is increasing in θ as in the proof of Lemma 10. Thus,

$$U(\bar{v}, \bar{a}, \theta_T) - U(\underline{v}, \hat{a}, \theta_T) \geq U(\bar{v}, \bar{a}, \tilde{\theta}) - U(\underline{v}, \hat{a}, \tilde{\theta}) \geq 0,$$

where the last inequality is from (42).

Consider next the case that $\tilde{a} < \hat{a}$. Then, for all $\theta \geq \tilde{\theta}$,

$$U(\bar{v}, \bar{a}, \theta) - U(\underline{v}, \hat{a}, \theta) = \int(\bar{v}(x) - \underline{v}(x))f(x|\hat{a}, \theta)dx + \int_{\hat{a}}^{\bar{a}} U_a(\bar{v}, a, \theta)da.$$

The first term on the *rhs* is positive as before since $\hat{a} > \tilde{a}$ and $\theta \geq \tilde{\theta}$. And, by the *FOP* and convexity of c in a, $U_a(\bar{v}, a, \tilde{\theta}) \geq 0$, so, since $U_{a\theta} \geq 0$, $U_a(\bar{v}, a, \theta_T) \geq 0$. Hence, in either case, type θ prefers $(\bar{v}, \bar{a})$ to $(\underline{v}, \hat{a})$, and we are done. □

Proof of Proposition 9 A randomized mechanism is a map σ that for each θ generates a distribution $\sigma(\cdot|\theta)$ over pairs consisting of a compensation scheme $\tilde{v}$ and recommended action a. There is an obvious mapping between deterministic mechanisms and degenerate random mechanisms.

For given $\tilde{v}$ and a, let $\tilde{s}(\tilde{v}, a, \theta) = \int \tilde{v}(x) f(x|a)dx - c(a, \theta)$. The principal's problem is

$$\max_{\sigma} \int \left(\int \left(B(a) - \int \varphi(\tilde{v}) f(x|a)dx \right) d\sigma(\tilde{v}, a|\theta) \right) h(\theta)d\theta,$$

subject to incentive compatibility and participation, where incentive compatibility requires first that

$$\theta \in \arg\max_{\theta_A} \left(\int \max_{a'} \tilde{s}(\tilde{v}, a', \theta) d\sigma(\tilde{v}, a|\theta_A) \right), \tag{43}$$

so that the agent truthfully reports his type, and second that, with σ-probability one,

$$\tilde{s}(\tilde{v}, a, \theta) = \max_{a'} \tilde{s}(\tilde{v}, a', \theta), \tag{44}$$

so that the agent follows the recommended action. In turn, participation holds if

$$\int \tilde{s} \left(\tilde{v}, a, \theta \right) d\sigma \left(\tilde{v}, a|\theta \right) \geq \bar{u}.$$

Let V_{FR} (FR for "full-random") be the value of this program.

Fix any feasible mechanism σ. Then, with σ-probability one,

$$\int \varphi(\tilde{v}(x)) f(x|a)dx \geq C(a, \tilde{s}(\tilde{v}, a, \theta), \theta),$$

since the local incentive constraint is necessary for (44), and C is the minimized cost subject only to the local incentive constraint. Further, let

$$\varrho(\theta, \theta_A) = \int \tilde{s}(\tilde{v}, a, \theta) d\sigma(\tilde{v}, a|\theta_A) = \int \left(\int \tilde{v}(x) f(x|a)dx - c(a, \theta) \right) d\sigma(\tilde{v}, a|\theta_A)$$

be the surplus to type θ of announcing θ_A and then taking the recommended action. Then, letting $S(\theta) = \int \tilde{s}(\tilde{v}, a, \theta) d\sigma(\tilde{v}, a|\theta)$ be the surplus to type θ, $S(\theta) \geq \varrho(\theta, \theta_A)$, with equality at $\theta_A = \theta$. Hence, a necessary condition for incentive compatibility is that

$$S'(\theta) = \varrho_{\theta}(\theta, \theta) = \int (-c_{\theta}(a, \theta)) d\sigma(\tilde{v}, a|\theta).$$

Thus, noting that each choice of σ generates a distribution μ on actions cross surplus, V_{FR} is at most V_{RR} (RR for "relaxed-random"), the value of the program

$$\max_{\mu} \int \left(\int \left(B(a) - C(a, s, \theta) \right) d\mu(a, s|\theta) \right) h(\theta)d\theta$$
$$s.t. \int s d\mu(a, s|\theta) = \bar{u} + \int_{\underline{\theta}}^{\theta} \left(\int (-c_{\theta}(a, \tau)) d\mu(a, s|\tau) \right) d\tau.$$

Let V_{RD} (RD for "relaxed-deterministic") be the value of our original relaxed screening problem, in which menus are restricted to be deterministic. We claim $V_{RD} = V_{RR}$. To see this, let μ^* be optimal in the relaxed random program. Let $a^*(\theta) = \int a d\mu^*(a, s|\theta)$, $S^*(\theta) = \int s d\mu^*(a, s|\theta)$, and $S^{**}(\theta) = \bar{u} + \int_{\underline{\theta}}^{\theta} (-c_\theta(a^*(\tau), \tau)) d\tau$. Since $-c_\theta$ is convex in a (recall $c_{aa\theta} \leq 0$), we have

$$S_\theta^{**} = -c_\theta(a^*(\theta), \theta) \leq \int (-c_\theta(a, \theta)) d\mu^*(a, s|\theta) = S_\theta^*,$$

and so, since $S^{**}(\underline{\theta}) = S^*(\underline{\theta}) = \bar{u}$, we have $S^{**} \leq S^*$. But then, since $B - C$ is strictly concave in (a, u_0), and decreasing in u_0,

$$\begin{aligned} V_{RR} &= \int \left(\int (B(a) - C(a, s, \theta)) d\mu^*(a, s|\theta) \right) h(\theta) d\theta \\ &\leq \int (B(a^*(\theta)) - C(a^*(\theta), S^*(\theta), \theta)) h(\theta) d\theta \\ &\leq \int (B(a^*(\theta)) - C(a^*(\theta), S^{**}(\theta), \theta)) h(\theta) d\theta \\ &\leq V_{RD}, \end{aligned}$$

where the last inequality follows since by construction (a^*, S^{**}) is feasible in the relaxed deterministic problem. So, $V_{FR} \leq V_{RD}$, and thus if the solution to the relaxed deterministic program is feasible, then it is in fact optimal even if randomization is allowed. □

Appendix B Existence in the Relaxed Pure Adverse Selection Problem

To show existence, we will need the assumption, maintained for this section, that $\hat{C}(\cdot, \cdot, \theta)$ is strictly convex for each θ. For the canonical setting without moral hazard, $\hat{C}(a, u_0, \theta) = \varphi(u_0 + c(a, \theta))$, where $\varphi = u^{-1}$, and so this is immediate. The situation is more complicated in the decoupling program where $\hat{C} = C$ comes from the cost minimization step of the pure moral hazard problem. Although primitives for C convex in a are known (see Jewitt, Kadan, and Swinkels (2008) and Chade and Swinkels (2019) (CS)), ensuring joint convexity in (a, u_0) is harder. For the square root utility case analyzed in Section 15, all the assumptions are easily satisfied. Moreover, checking the convexity of a numerically generated C for any given set of primitives is straightforward. Finally, we have the following result, showing convexity on the relevant range as long as $\bar{u}$ is large enough.

Lemma 11 *Let $F \in \mathcal{C}^4$, let Assumptions 2-4 hold, and let $\overline{a} < \infty$. Then for all $\bar{u}$ sufficiently large, $C(\cdot, \cdot, \theta)$ is strictly convex for each θ and for all (a, u_0) with $u_0 \geq \bar{u}$.*

Proof From *CS*, Lemma 7,

$$C_{aa}=\lambda c_{aa}+\mu\left(c_{aaa}-\int vf_{aaa}-\int v_x lF_{aa}\right)+\left(\int\rho' f\right)\left(\mu_a^2 var_\xi(l)-\mu^2\, var_\xi(l_a)\right),$$

which is strictly positive for all $u_0>\bar{u}$ when $\bar{u}$ is sufficiently large (*CS*, Corollary 1), while from *CS*, Lemma 5, $C_{au_0}=\lambda_a=-\mu_a\int l\xi-\mu\int l_a\xi$ and $C_{u_0u_0}=\lambda_{u_0}=\int l^2\xi/\left(var_\xi(l)\int\rho' f\right)>0$. It remains only to show that for u_0 sufficiently large, the determinant $C_{aa}C_{u_0u_0}-\left(C_{au_0}\right)^2$ is strictly positive. But, this has the same sign as

$$\begin{aligned}
&\frac{C_{aa}}{\lambda}-\frac{\left(C_{au_0}\right)^2}{\lambda C_{u_0u_0}}\\
=\;& c_{aa}+\frac{\mu}{\lambda}\left(c_{aaa}-\int vf_{aaa}-\int v_x lF_{aa}\right)+\frac{\mu_a}{\lambda}\left(\mu_a\int\rho' f\right)var_\xi(l)-\frac{\mu}{\lambda}\left(\mu\int\rho' f\right)\,var_\xi(l_a)\\
&-\left(\frac{\mu_a}{\lambda}\left(\mu_a\int\rho' f\right)\left(\int l\xi\right)^2+2\frac{\mu_a}{\lambda}\left(\mu\int\rho' f\right)\int l\xi\int l_a\xi+\frac{\mu}{\lambda}\left(\mu\int\rho' f\right)\left(\int l_a\xi\right)^2\right)\frac{var_\xi(l)}{\int l^2\xi},
\end{aligned}$$

where each term other than the first consists of a ratio that by Lemma 4 goes to 0, along with terms that remain bounded using Lemma 4 and the fact that $\xi\to f$ by *CS*, Lemma 4. □

We are now ready to prove our existence and uniqueness result.

Proposition 11 *Let $\hat{C}$ be continuous, strictly convex in (a,u_0) for each θ, and satisfy $\hat{C}_a(0,u_0,\theta)=0$ and $\lim_{a\to\bar{a}}\hat{C}_a(a,u_0,\theta)=\infty$ for all (u_0,θ). Let $\overline{u}$ be in the interior of the range of u. Then there is a solution to the relaxed pure adverse selection problem*

$$\begin{aligned}
\max_{\alpha,S}\quad & \int_{\underline{\theta}}^{\overline{\theta}}\left(B(\alpha(\theta))-\hat{C}(\alpha(\theta),S(\theta),\theta)\right)h(\theta)d\theta\\
s.t.\quad & S(\theta)=\bar{u}-\int_{\underline{\theta}}^{\theta}c_\theta(\alpha(\tau),\tau)d\tau\ \ \textit{for almost all}\,\theta,
\end{aligned}\tag{45}$$

This solution is unique.

Proof Recall from Footnote 12 that the Hamiltonian of the problem is $\mathcal{H}=(B-\hat{C})h-\eta c_\theta$, where $\eta\le 0$ is the co-state variable. To see that $\mathcal{H}$ is concave in (a,u_0), note that joint concavity requires (i) $\mathcal{H}_{aa}\le 0$, which follows from B concave in a, $c_{aa\theta}\le 0$, and $\hat{C}_{aa}\ge 0$; (ii) $\mathcal{H}_{u_0u_0}\le 0$, which follows since $\hat{C}_{u_0u_0}\ge 0$; and (iii) $\mathcal{H}_{aa}\mathcal{H}_{u_0u_0}-\mathcal{H}_{au_0}^2\ge 0$, which follows since $\hat{C}_{aa}\hat{C}_{u_0u_0}-\hat{C}_{au_0}^2\ge 0$.

Given the boundary conditions on $\hat{C}_a$, the optimality conditions are $\partial\mathcal{H}/\partial a=0$, $\eta'(\theta)=-\partial\mathcal{H}/\partial S$, and $\eta(\overline{\theta})=0$, from which we obtain

$$B_a-\hat{C}_a=-\frac{c_{a\theta}}{h}\int_\theta^{\bar{\theta}}\hat{C}_{u_0}h,\tag{46}$$

plus (45). The concavity of $\mathcal{H}$ ensures that (45)–(46) are also sufficient. As a result, we will focus on them in our search for a solution (α, S) to the problem.

Define $a^*(s, z, \theta)$ as the solution to

$$B_a(a) - \hat{C}_a(a, s, \theta) = -\frac{c_{a\theta}(a, \theta)}{h(\theta)} z \tag{47}$$

where a^* exists from the boundary conditions on $\hat{C}_a$, and is unique from the convexity of $\hat{C}$ and $-c_\theta$ in a and the concavity of B. We will then be done if we find a solution to the following system of ordinary differential equations:

$$\begin{bmatrix} S'(\theta) \\ Z'(\theta) \end{bmatrix} = \begin{bmatrix} g^S(S(\theta), Z(\theta), \theta) \\ g^Z(S(\theta), Z(\theta), \theta) \end{bmatrix}.$$

with boundary conditions $S(\underline{\theta}) = \overline{u}$ and $Z(\bar{\theta}) = 0$, where

$$\begin{bmatrix} g^S(S(\theta), Z(\theta), \theta) \\ g^Z(S(\theta), Z(\theta), \theta) \end{bmatrix} = \begin{bmatrix} -c_\theta(a^*(S(\theta), Z(\theta), \theta), \theta) \\ -C_{u_0}(a^*(S(\theta), Z(\theta), \theta), S(\theta), \theta) h(\theta) \end{bmatrix}.$$

Indeed if we take $\alpha(\theta) = a^*(S(\theta), Z(\theta), \theta)$ then $Z(\theta) = \int_\theta^{\bar{\theta}} C_{u_0}(\alpha(t), S(t), t) h(t) dt$. Hence, by definition of a^* and comparing (46) and (47), (α, S) satisfies the relevant conditions.

Define $u_{\max} = \bar{u} + (\overline{\theta} - \underline{\theta}) \max_{(a,\theta) \in [0,\overline{a}] \times [\underline{\theta}, \overline{\theta}]} (-c_\theta(a, \theta))$, and

$$z_{\max} = (\overline{\theta} - \underline{\theta}) \max_{(a,\theta) \in [0,\overline{a}] \times [\overline{u}, u_{\max}] \times [\underline{\theta}, \overline{\theta}]} C_{u_0}(a, s, \theta) h(\theta).$$

Choose $\delta > 0$ such that $\overline{u} - \delta$ remains in the domain of u, and let $R = [\overline{u}, u_{\max}] \times [0, z_{\max}]$ and $R_\delta = [\overline{u} - \delta, u_{\max} + \delta] \times [-\delta, z_{\max} + \delta]$. Then a^* is Lipschitz on $R_\delta \times [\underline{\theta}, \overline{\theta}]$, and hence so are g^S and g^Z.

Let $\zeta : \mathbb{R}^2 \to [0, 1]$ be a Lipschitz function such that $\zeta(s, z) = 1$ if $(s, z) \in R$ and $\zeta(s, z) = 0$ if $(s, z) \notin R_{\delta/2}$. In an abuse of notation, write ζg^S for the function that is $\zeta(s, z) g^S(s, z, \theta)$ on R_δ, and zero otherwise, and similarly for ζg^Z. Then $(\zeta g^S, \zeta g^Z)$ is Lipschitz on $\mathbb{R}^2 \times [\underline{\theta}, \overline{\theta}]$, and so by standard results in the theory of differential equations (see, e.g., Theorems 2.3 and 2.6 in Khalil (1992)) there exist continuous functions $\hat{S}$ and $\hat{Z}$ mapping $\mathbb{R} \times [\underline{\theta}, \overline{\theta}]$ into $\mathbb{R}$ such that $\hat{S}(u_0, \bar{\theta}) = u_0$, $\hat{Z}(u_0, \bar{\theta}) = 0$, and

$$\begin{bmatrix} \hat{S}_\theta(u_0, \theta) \\ \hat{Z}_\theta(u_0, \theta) \end{bmatrix} = \begin{bmatrix} (\zeta g^S)(\hat{S}(u_0, \theta), \hat{Z}(u_0, \theta), \theta) \\ (\zeta g^Z)(\hat{S}(u_0, \theta), \hat{Z}(u_0, \theta), \theta) \end{bmatrix}.$$

Note that $\hat{S}(u_{\max}, \underline{\theta}) \geq \overline{u}$ by the definition of g^S and $u_{\max}$. Similarly, since $\hat{S}_\theta \geq 0$, $\hat{S}(\overline{u}, \underline{\theta}) \leq \overline{u}$.

Hence, by continuity, there exists $u^* \in [\bar{u}, u_{\max}]$ such that $\hat{S}(u^*, \underline{\theta}) = \overline{u}$. But then, since $\hat{S}_\theta \geq 0$, $\hat{S}(u^*, \theta) \in [\bar{u}, u_{\max}]$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Similarly, since $\hat{Z}_\theta \leq 0$, and using the definition of $z_{\max}$, we have $\hat{Z}(u^*, \theta) \in [0, z_{\max}]$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Thus, $(\hat{S}(u^*, \theta), \hat{Z}(u^*, \theta)) \in R$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$, and so since $\zeta = 1$ on R, the pair $(S(\cdot), Z(\cdot)) = (\hat{S}(u^*, \cdot), \hat{Z}(u^*, \cdot))$ satisfies the required conditions.

To see uniqueness, let (α^1, S^1) and (α^2, S^2) be optimal and differ on a positive measure set. Consider $\hat{\alpha} = (\alpha^1 + \alpha^2)/2$, and note that since $c_{aa\theta} \leq 0$, $-c_\theta(\hat{\alpha}, \theta) \leq (-c_\theta(\alpha^1, \theta) - c_\theta(\alpha^2, \theta))/2$. Hence, $\hat{S} = \bar{u} - \int_{\underline{\theta}}^{\theta} c_\theta(\hat{\alpha}(\tau), \tau)\, d\tau \leq (1/2)(S^1 + S^2)$. But then, because $B - C$ is strictly concave in a and u_0, and decreasing in u_0, $(\hat{\alpha}, \hat{S})$ is strictly superior, contradicting that either (α^1, S^1) or (α^2, S^2) was optimal. □

Appendix C Existence and Differentiability in the Pure Moral Hazard Problem

Let W be the domain of the utility function, an interval with infimum $\underline{w}$ and supremum $\bar{w}$. Let $\underline{v} = \lim_{w \to \underline{w}} u(w)$, and let $\bar{v} = \lim_{w \to \bar{w}} u(w)$. Let $\mathcal{E}$ be the set of (a, u_0, θ) such that the relaxed moral hazard problem in Section 3.3 admits a solution $\hat{v}$ where $\hat{v}(\underline{x}) > \underline{v}$ and $\hat{v}(\bar{x}) < \bar{v}$. If we let $\underline{\tau} = \lim_{w \to \underline{w}} \frac{1}{u'(w)}$, and $\bar{\tau} = \lim_{w \to \bar{w}} \frac{1}{u'(w)}$, then it is easy to show that $\hat{v}(\underline{x}) > \underline{v}$ if and only if $\lambda + \mu l(\underline{x}|a) > \underline{\tau}$ for the associated Lagrange multipliers, and similarly, that $\hat{v}(\bar{x}) < \bar{v}$ if and only if $\lambda + \mu l(\bar{x}|a) < \bar{\tau}$.

Lemma 12 *The set $\mathcal{E}$ is open. The multipliers λ and μ are twice continuously differentiable functions of (a, u_0, θ) on $\mathcal{E}$.*

Proof Let

$$G(\lambda, \mu, a, u_0, \theta) = \begin{pmatrix} g_1(\lambda, \mu, a, u_0, \theta) \\ g_2(\lambda, \mu, a, u_0, \theta) \end{pmatrix},$$

where

$$g_1(\lambda, \mu, a, u_0, \theta) = \int \rho(\lambda + \mu l(x|a)) f(x|a) dx - c(a, \theta) - u_0,$$
$$g_2(\lambda, \mu, a, u_0, \theta) = \frac{\partial}{\partial a} \int \rho(\lambda + \mu l(x|a)) f(x|a) dx - c(a, \theta) - u_0.$$

Let $(a^0, u_0^0, \theta^0) \in \mathcal{E}$, let λ^0 and μ^0 be the associated Lagrange multipliers, and let $\kappa^0 = (\lambda^0, \mu^0, a^0, u_0^0, \theta^0)$. Then, $G(\kappa^0) = 0$, and by definition of $\mathcal{E}$, $\lambda^0 + \mu^0 l(\underline{x}|a^0) > \underline{\tau}$, and $\lambda^0 + \mu^0 l(\bar{x}|a^0) < \bar{\tau}$. We need to show that λ and μ are implicitly defined as C^1 functions of (a, u_0, θ) on a neighborhood of (a^0, u_0^0, θ^0). Since $\lambda + \mu l(\underline{x}|a)$ and $\lambda + \mu l(\bar{x}|a)$ are continuous in (λ, μ, a), it would follow from this that $\mathcal{E}$ is open. We proceed in several steps.

STEP 1. We first show that $g_{1\lambda}$ exists at κ^0, and is equal to $\int \rho'(\lambda^0 + \mu^0 l(x|a^0))f(x|a^0)dx$. To show this, we must first show that it is valid to differentiate under the integral. This requires that $\rho(\lambda + \mu l(x|a))f(x|a)$ be integrable. Since f is continuous on the compact interval $[\underline{x}, \bar{x}]$, it is bounded, and so it is enough to show that $|\rho(\lambda + \mu l(x|a))|$ is bounded. But,

$$\rho(\lambda + \mu l(x|a)) \leq \rho(\lambda^0 + \mu^0 l(\bar{x}|a^0)) < \infty,$$

where we use that $\lambda^0 + \mu^0 l(\bar{x}|a^0) < \bar{\tau}$ by hypothesis, and similarly, $\rho(\lambda + \mu l(x|a)) \geq \rho(\lambda^0 + \mu^0 l(\underline{x}|a^0)) > \infty$, and we are done. Another requirement for passing the derivative through the integral is that $\rho'(\lambda^0 + \mu^0 l(x|a^0))f(x|a^0)$ is bounded above by an integrable function on some neighborhood of (λ^0, μ^0, a^0). To see this, choose $\underline{\delta}$ and $\bar{\delta}$ such that $\underline{\tau} < \underline{\delta} < \lambda^0 + \mu^0 l(\underline{x}|a^0)$ and $\lambda^0 + \mu^0 l(\bar{x}|a^0) < \bar{\delta} < \bar{\tau}$. Then, since $\lambda + \mu l(\underline{x}|a)$ and $\lambda + \mu l(\bar{x}|a)$ are continuous in (λ, μ, a), there is a neighborhood N of (λ^0, μ^0, a^0) such that $\underline{\delta} < \rho(\lambda + \mu l(\underline{x}|a)) < \rho(\lambda + \mu l(\bar{x}|a)) < \bar{\delta}$ on N. But then, for all x, and everywhere on N, $\rho'(\lambda + \mu l(x|a)) \leq \max_{\sigma \in [\underline{\delta}, \bar{\delta}]} \rho'(\sigma) < \infty$, where the second inequality follows since ρ is continuously differentiable (with $\rho'(\sigma) = ((u')^3 / - u'')(\psi(\sigma)))$ and $[\underline{\delta}, \bar{\delta}]$ is compact. By Corollary 5.9 in Bartle (1966) (and Billingsley (1995), problem 16.5), we can pass the derivative through the integral and this provides an expression for $g_{1\lambda}$.

STEP 2. $g_{1\lambda} = \int \rho'(\lambda + \mu l(x|a))f(x|a)dx$ is itself continuous in (λ, μ, a) at (λ^0, μ^0, a^0). This follows since $\lambda + \mu l(x|a)$ is, under our conditions, uniformly continuous in (λ, μ, a), and ρ' is uniformly continuous in its argument on $[\underline{\delta}, \bar{\delta}]$.

STEP 3. By similar arguments, $g_{1\mu}$, g_{1a}, $g_{2\lambda}$, $g_{2\mu}$, and g_{2a} are defined as the integral of the relevant derivative, and are continuous. Finally, $g_{i\theta}$ and g_{iu_0} are trivially continuous. Hence, G is continuously differentiable on a neighborhood of κ^0. Indeed, by similar arguments, G is twice continuously differentiable, noting in specific that

$$\rho''(\sigma) = \frac{(u')^3}{-u''} \left[3\frac{u''}{u'} - \frac{u'''}{u''} \right] (\psi(\sigma)),$$

and so since u is C^3, ρ'' is continuous on the compact interval $[\underline{\delta}, \bar{\delta}]$, and hence it is bounded.

STEP 4. By the argument in Jewitt *et al.* (2008), $\nabla G(\kappa^0) \neq 0$. Hence, by the Implicit Function Theorem for C^k functions (Fiacco (1983), Theorem 2.4.1), λ and μ are twice continuously differentiable functions of (a, u_0, θ) in a neighborhood of (a^0, u_0^0, θ^0). □

The reader may wonder at the level of detail displayed in this proof. To see that there is something to prove, consider $u = \log w$. Then (see Moroni and Swinkels (2014) for details), it is easy to exhibit first, combinations of c_a, c, and u_0 for which no optimal contract exists, and second, combinations of c_a, c, and u_0 for which the optimal contract has $v(\underline{x}) = -\infty$, and at which the relevant integrals cease to be continuous (let alone differentiable) in the relevant parameters.

Lemma 12 implies that the cost function C is twice differentiable on $\mathcal{E}$, and also that α is continuously differentiable.

Another differentiability argument we have used in the text is about the integrals $\int v_\theta f$ and $\int v f_a$. It can be justified as follows:

Lemma 13 *Let $(\alpha(\theta^0), S(\theta^0), \theta^0) \in \mathcal{E}$. Then, for all a, $\int v(x,\theta) f(x|a) dx$ is differentiable in θ at θ^0, with*

$$\frac{\partial}{\partial \theta} \int v(x,\theta^0) f(x|a) dx = \int v_\theta(x,\theta^0) f(x|a) dx,$$

and similarly, $\int v(x,\theta^0) f(x|a) dx$ is differentiable in a at a, with

$$\frac{\partial}{\partial a} \int v(x,\theta^0) f(x|a) dx = \int v(x,\theta^0) f_a(x|a) dx.$$

Proof We will show the result for the case of differentiation by θ since the other case is similar. We must show first that $v(x,\theta^0) f(x|a)$ is integrable. This follows as before since

$$|v(x,\theta^0)| \leq \max\left(|v(\underline{x},\theta^0)|, |v(\bar{x},\theta^0)|\right) < \infty.$$

Next we show that, under decoupling, v_θ exists and it is uniformly bounded. To see this, note first that $v(x,\theta) = \rho(\lambda(\theta) + \mu(\theta) l(x|\alpha(\theta)))$ and so

$$v_\theta(x,\theta) = \rho'(\lambda(\theta) + \mu(\theta) l(x|\alpha(\theta)))(\lambda'(\theta) + \mu'(\theta) l(x|\alpha(\theta)) + \mu(\theta) l_a(x|\alpha(\theta))\alpha'(\theta)).$$

As before, let $\underline{\tau} < \underline{\delta} < \lambda^0 + \mu^0 l(\underline{x}|a^0)$, and let $\lambda^0 + \mu^0 l(\bar{x}|a^0) < \bar{\delta} < \bar{\tau}$. Since α is continuous, for all θ sufficiently close to θ^0, $\lambda(\theta) + \mu(\theta) l(x|\alpha(\theta)) \in [\underline{\delta}, \bar{\delta}]$, and so, as before, $\rho'(\lambda(\theta) + \mu(\theta) l(x|\alpha(\theta)))$ is uniformly bounded on a neighborhood of θ^0. Also, since α and S are C^1, $\lambda(\theta)$ and $\mu(\theta)$ are continuously differentiable on some neighborhood of θ^0. But then, since l and l_a are uniformly bounded, we can also uniformly bound $(\lambda'(\theta) + \mu'(\theta) l(x|\alpha(\theta)) + \mu(\theta) l_a(x|\alpha(\theta))\alpha'(\theta))$ on the relevant neighborhood. It follows that v_θ is uniformly bounded on the neighborhood, and the lemma follows from Bartle (1966), Corollary 5.9. □

Of course, for decoupling to work, it has to be that the resultant moral-hazard subproblem has a solution for each θ. That is, we need to know that $(\alpha(\theta), S(\theta), \theta) \in \mathcal{E}$ for all θ. By Moroni and Swinkels (2014), one set of conditions is given by the following lemma.

Lemma 14 *Assume that $\bar{w} = \bar{v} = \infty$, $\underline{w} = \underline{v} = -\infty$, $\underline{\tau} = 0$, and $\bar{\tau} = \infty$. Then, for all (a, u_0, θ), $(a, u_0, \theta) \in \mathcal{E}$.*

Proof Direct from Moroni and Swinkels (2014). □

This Lemma, however, does not cover important cases such as $u = \ln(w)$ or $u = \sqrt{w}$, because in each case, $\underline{w} = 0 > -\infty$. Our next lemma covers $u = \sqrt{w}$, but does not cover $u = \ln w$.

Lemma 15 *Let $\bar{w} = \bar{v} = \infty$, $\underline{w} = 0$, and $\bar{\tau} = \infty$. Assume further that $\rho'(\tau)\tau$ is increasing and diverges in τ. Then, there is a threshold $\hat{u}$ such that for all $\bar{u} \geq \hat{u}$, $(\alpha(\theta), S(\theta), \theta) \in \mathcal{E}$ for all θ.*

Proof For any given a, and $\mu > 0$, let $i(\mu, a) = \int \rho(\mu(l(x|a) - l(\underline{x}|a)))f_a(x|a)dx$. Note that

$$\begin{aligned} i(\mu, a) &= \int \rho'(\mu(l(x|a) - l(\underline{x}|a)))\mu l_x(x|a)(-F_a(x|a))dx \\ &= \int \frac{1}{l(x|a) - l(\underline{x}|a)}[\rho'(\mu(l(x|a) - l(\underline{x}|a)))\mu(l(x|a) - l(\underline{x}|a))]l_x(x|a)(-F_a(x|a))dx, \end{aligned}$$

and so, since $\rho'(\tau)\tau$ is increasing in τ, it follows that the bracketed term, and hence $i(\cdot, a)$, is increasing in μ. Let $m = \min_a l(\bar{x}|a) - l(\underline{x}|a) > 0$, and let

$$\sigma = -\min_{\left\{(x,a)|\frac{m}{2} \leq l(x|a) - l(\underline{x}|a) \leq \frac{3m}{4}\right\}} l_x(x|a)F_a(x|a) > 0.$$

Then,

$$\begin{aligned} & i(\mu, a) \\ & \geq \int_{\left\{x|\frac{m}{2} \leq l(x|a) - l(\underline{x}|a) \leq \frac{3m}{4}\right\}} \frac{\rho'(\mu(l(x|a) - l(\underline{x}|a)))\mu(l(x|a) - l(\underline{x}|a))}{l(x|a) - l(\underline{x}|a)} l_x(x|a)(-F_a(x|a))dx \\ & \geq \frac{4\sigma}{3m} \int_{\left\{x|\frac{m}{2} \leq l(x|a) - l(\underline{x}|a) \leq \frac{3m}{4}\right\}} \rho'(\mu(l(x|a) - l(\underline{x}|a)))\mu(l(x|a) - l(\underline{x}|a))dx \\ & \geq \frac{4\sigma}{3m}\rho'\left(\mu\frac{m}{2}\right)\mu\frac{m}{2} \int_{\left\{x|\frac{m}{2} \leq l(x|a) - l(\underline{x}|a) \leq \frac{3m}{4}\right\}} dx \geq \frac{4\sigma}{3m}\frac{m}{4\max_{\{x,a\}} l_x(x|a)}\rho'\left(\mu\frac{m}{2}\right)\mu\frac{m}{2} \\ & = \frac{\sigma}{3\max_{\{x,a\}} l_x(x|a)}\rho'\left(\mu\frac{m}{2}\right)\mu\frac{m}{2}, \end{aligned}$$

where the first inequality follows from the fact that the integrand is positive, the second from $l(x|a) - l(\underline{x}|a) \leq 3m/4$, the third from the monotonicity of $\rho'(\tau)\tau$, and the fourth by integration. Notice that the lower bound on $i(\mu, a)$ thus obtained diverges in μ. Hence, there exists $\hat{\mu}$ such that $i(\mu, a) > c_a(a, \bar{\theta})$ for all a, and $\mu > \hat{\mu}$. Let

$$\hat{u} = \max_a \int \rho(\hat{\mu}(l(x|a) - l(\underline{x}|a)))f(x|a)dx \leq \rho\left(\hat{\mu}\max_a(l(\bar{x}|a) - l(\underline{x}|a))\right) < \infty.$$

It follows from Proposition 1 of Moroni and Swinkels (2014), along with $i(\cdot, a)$ increasing, that $(\alpha(\theta), S(\theta), \theta) \in \mathcal{E}$ for all θ for any $\bar{u} > \hat{u}$. In particular, at any θ, $S(\theta) + c(\alpha(\theta), \theta) > \bar{u} > \hat{u}$. □

Finally, let us consider the case $u = \log w$ (for which $\rho'(\tau)\tau$ is identically 1, so the previous result does not apply). Then, as in the proof of the previous lemma,

$$\begin{aligned} i\left(\mu,a\right) &\geq \frac{4\sigma}{3m}\int_{\left\{x\mid\frac{m}{2}\leq l(x|a)-l(\underline{x}|a)\leq\frac{3m}{4}\right\}}\rho'(\mu(l(x|a)-l(\underline{x}|a)))\mu(l(x|a)-l(\underline{x}|a))dx \\ &= \frac{4\sigma}{3m}\int_{\left\{x\mid\frac{m}{2}\leq l(x|a)-l(\underline{x}|a)\leq\frac{3m}{4}\right\}}dx \geq \frac{4\sigma}{3m}\frac{m}{4\max_{x,a}l_x\left(x|a\right)} \equiv s, \end{aligned}$$

and so, if we assume that $c_a(\bar{a},\bar{\theta})\leq s$, then Proposition 1 of Moroni and Swinkels (2014) applies.

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