RISKY MATCHING

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Abstract

We develop a model where risk-averse workers can costly invest in their skills before matching with heterogeneous firms. At the investment stage, workers face multiple sources of risk. They are uncertain about how skilled they will turn out and also about their income shock realizations at the time of employment. We analyze in a unified way the equilibria in two versions of the model that depend on when uncertainty resolves, which determines the available risk-sharing possibilities between workers and firms. We provide a thorough analysis of the equilibrium comparative statics regarding changes in risk, worker and firm heterogeneity, and technology. We derive conditions on the match output function and risk attitudes under which these shifts lead to more investment and show how this affects matching and wages. To illustrate the applied relevance of our theory, we provide a stylized quantitative assessment of the model and analyze the sources (risk, heterogeneity, or technology) of rising U.S. wage inequality. We find that changes in risk were the most important driver behind the surge in inequality, followed by technological change. We show that these conclusions are significantly altered if one neglects the key feature of our model that educational investment is endogenous.

Keywords. Matching, Investment, Changes in Risk, Comparative Statics, Sorting, Inequality.

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# Contents

1. **Introduction**  ·  1

2. **The Model**  ·  5

3. **Equilibrium Analysis**  ·  7
   - 3.1 Existence and Uniqueness  ·  7
   - 3.2 Efficiency Properties  ·  11

4. **Comparative Statics**  ·  12
   - 4.1 Changes in Background Risk  ·  13
   - 4.2 Changes in Controllable Risk, Initial Heterogeneity and Technology  ·  19
   - 4.3 Effects on Equilibrium Matching and Wages  ·  20
   - 4.4 Equilibrium Investment With and Without Risk-Sharing  ·  22

5. **Robustness**  ·  22

6. **Economic Relevance**  ·  23
   - 6.1 Estimation  ·  24
   - 6.2 Wage Inequality Changes Through the Lens of our Model  ·  26
   - 6.3 Wage Inequality Changes with Exogenous Investment  ·  28

7. **Concluding Remarks**  ·  30

A. **Appendix**  ·  31
   - A.1 Proof of Walrasian Equilibrium  ·  31
   - A.2 Proof of Proposition 1  ·  34
   - A.3 Proof of Proposition 2  ·  36
   - A.4 Proof of Proposition 3  ·  37
   - A.5 Proof of Proposition 4  ·  38
   - A.6 Decreasing Convex Order Shift in \( \mathcal{L} \)  ·  39
   - A.7 FOSD Shift in \( \mathcal{L} \) in RS  ·  40
   - A.8 Proof of Proposition 5  ·  41
   - A.9 Proof of Proposition 6  ·  41
   - A.10 Shift in Technology  ·  43
   - A.11 Proof of Proposition 7  ·  44
# B Online Appendix: Theory

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.1 Example with Multiplicity</td>
<td>1</td>
</tr>
<tr>
<td>B.2 Ex-Ante Inefficiency</td>
<td>1</td>
</tr>
<tr>
<td>B.3 Ex-Ante Homogeneous Workers</td>
<td>4</td>
</tr>
<tr>
<td>B.4 Reinterpretation of $\alpha$ as Idiosyncratic Income Shock</td>
<td>5</td>
</tr>
<tr>
<td>B.5 Remarks on Integration and Differentiation</td>
<td>6</td>
</tr>
<tr>
<td>B.6 IR Shift in $G, Q, H_1$</td>
<td>7</td>
</tr>
<tr>
<td>B.7 Two-Sided Risk Aversion with CRRA</td>
<td>10</td>
</tr>
<tr>
<td>B.8 Two-Sided Investments</td>
<td>17</td>
</tr>
<tr>
<td>B.9 Continuous Investments</td>
<td>19</td>
</tr>
<tr>
<td>B.10 The Partnership Model</td>
<td>21</td>
</tr>
</tbody>
</table>

# C Online Appendix: Quantitative Assessment

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>C.1 Data Sources and Variable Construction</td>
<td>23</td>
</tr>
<tr>
<td>C.1.1 CPS</td>
<td>23</td>
</tr>
<tr>
<td>C.1.2 DOT and ONET</td>
<td>24</td>
</tr>
<tr>
<td>C.1.3 NLSY 79/97</td>
<td>27</td>
</tr>
<tr>
<td>C.2 Summary Statistics</td>
<td>28</td>
</tr>
<tr>
<td>C.3 Estimation</td>
<td>29</td>
</tr>
<tr>
<td>C.3.1 Parameters</td>
<td>29</td>
</tr>
<tr>
<td>C.3.2 Identification</td>
<td>29</td>
</tr>
<tr>
<td>C.3.3 Moments</td>
<td>31</td>
</tr>
<tr>
<td>C.3.4 Simulated Method of Moments</td>
<td>31</td>
</tr>
<tr>
<td>C.4 Results with Endogenous Investment</td>
<td>33</td>
</tr>
<tr>
<td>C.5 Results with Exogenous Investment</td>
<td>34</td>
</tr>
<tr>
<td>C.5.1 Statistical Decomposition</td>
<td>34</td>
</tr>
<tr>
<td>C.5.2 Model Fit and Decomposition with Exogenous Investment</td>
<td>36</td>
</tr>
</tbody>
</table>
1 Introduction

In many economic situations agents make investments before entering a market in order to improve their prospects. Arguably the most prominent example is that of workers who invest in their skills before entering the labor market, where they match with firms that differ in productivity. These investments often take place under multiple sources of risks that can be idiosyncratic or aggregate. For example, workers do not know a priori what will be their ex-post skills after investment, and also they are uncertain about the state of the labor market that will prevail at the time of employment. A natural question in this context is how changes in these risks — in the sense of “better” or “more variable” risks — affect the incentives to invest, the allocation of workers and jobs, as well as wages and inequality. And how do these effects contrast with those of better or more spread out distributions of worker and firm heterogeneity (workers’ initial ability and firms’ productivity), or technological change?

These questions are of significant theoretical as well as applied interest. Theoretically, finding answers is challenging since they depend on a rich trade-off between workers’ risk attitudes and complementarities in an endogenous wage function. From an applied perspective, distinguishing between cross-sectional heterogeneity, labor market risk and technology is particularly important for understanding the determinants of income inequality and for why it has increased over time. Yet, little is known about these equilibrium comparative statics effects in the matching literature.

This paper helps fill this gap. We develop a model of the labor market in which risk-averse workers, before they match with firms, make pre-match investments in their skills under both aggregate and idiosyncratic risks. We study two versions of the model that depend on whether or not all uncertainty resolves before matching, determining the risk-sharing possibilities within each firm-worker pair. We derive results on existence, uniqueness, and efficiency of equilibrium. Our main contribution is to provide a novel and thorough analysis of the equilibrium comparative statics. For a large class of primitives we provide clear-cut answers to our questions above on the effects of changes in risk, heterogeneity and technology.

To illustrate the economic relevance of our framework, we use it to shed light on the determinants of rising U.S. wage inequality. We estimate our stylized model on data from the 1980s and today and decompose the rise in inequality into the shares driven by changes in risk, heterogeneity and technology. We then show that the conclusions from this decomposition significantly differ from those obtained under alternative approaches that assume educational investment and thus the skill distribution are exogenous. This comparison highlights the importance of our model’s key feature — endogenous educational investment — when analyzing sources of wage inequality.

The model consists of a large number of risk-averse workers and risk-neutral firms. Firms differ in productivity and, initially, workers differ in ability, with less able workers facing a higher...
disutility of investment. In the investment stage, workers first make a binary decision of whether or not to invest in their skills, and then all of them draw their skill from a distribution, which is better — in first-order stochastic dominance (FOSD) sense — for those who invest. That is, workers face an idiosyncratic ‘controllable’ risk regarding their skills and, importantly, the resulting aggregate distribution of skills is endogenous. In the matching stage, agents face a second source of risk, which affects match output and thus workers’ wages, and can be aggregate or idiosyncratic. It is a ‘background’ risk since workers cannot control it through their actions. Depending on whether this output shock is realized before or after matching, we are in the no-risk-sharing case with an uninsurable shock or in the risk-sharing case where risk-neutral firms optimally insure risk-averse workers. Both sides then match pairwise in a frictionless labor market, whose Walrasian equilibrium pins down the worker-firm allocation and the market-clearing wage function.

We provide a complete equilibrium analysis of both the no-risk-sharing and the risk-sharing cases. We first show equilibrium existence, and provide sufficient conditions for uniqueness. Moreover, when equilibrium is unique, equilibrium investment is efficient.

We then move to the centerpiece of our paper: a novel and thorough equilibrium comparative statics analysis. We consider the effects of changes in controllable and background risks, in heterogeneity of workers and firms, and in technology both under no-risk-sharing and risk-sharing; and we also analyze the effects of changes in risk-sharing opportunities. The overarching theme underlying all of our results is a fundamental trade-off between risk aversion and complementarities in the equilibrium wage function (inherited from similar properties of the match output function). If workers were risk-neutral or if the wage function was additively separable, this analysis would be much simpler but also considerably less rich.

We start with deriving a large class of match output functions and risk preferences under which a better (in the sense of FOSD) or a riskier (in the sense of an increase in risk, IR, à la Rothschild and Stiglitz (1970)) distribution of the background risk induces more workers to invest in equilibrium. These are the intuitively natural comparative statics to seek. First, workers will be more prone to invest if risk improves (FOSD) to take advantage of better market conditions. This result obtains if the two forces — risk attitude and complementarities — are balanced in the right way. Under a FOSD shift of background risk, complementarities between skill and shock provide agents with more incentives to invest (substitution effect). But, this shift leads to higher wages on average even if workers do not invest, reducing the incentives for risk-averse workers to invest (income effect). Sufficiently strong complementarities and a bound on absolute risk aversion cause the substitution effect to prevail. Second, it is also natural that, without risk-sharing, workers invest more if there is more uncertainty (IR), since investment can serve as a substitute for the lack of insurance (the intuition is different with risk-sharing and developed later). This result obtains if workers are sufficiently prudent relative to the complementarities in match output. It is well-known in consumption-saving problems that prudence triggers precautionary
savings to insure against bad realizations of a shock. But we believe that our precautionary investment motive is novel in the matching literature. We show that these conditions on risk attitudes and match output are weak, in the sense that they are satisfied by the most common parametric specifications of our model, including the ubiquitous multiplicatively separable class of technology and constant relative risk aversion utility.

We derive similar results when there is a FOSD or IR shift in the distributions of the controllable skill risk, or initial heterogeneity (worker ability and firm productivity). Once again we argue from economic principles which are the natural results to seek. For instance, it is intuitive that a better distribution of firm productivity leads to more investment, since this shift improves the matching of all workers, but more so for those with higher skills due to complementarities. Similarly, one would expect that an improvement in the skill distribution from which a worker draws when investing (akin to an improvement in the skill risk) induces more workers to invest. As before, these are not forgone conclusions due to the interaction between risk aversion and complementarities in match output. But we provide broad classes of primitives where income and substitution effects are tamed such that these natural comparative statics materialize.

We also derive sufficient conditions under which a shift in the match output function (i.e. technological change) leads to more investment. And we show for all of our comparative statics how these changes in investment translate into changes in worker-firm matching and wages.

Finally, we ask how the possibility to share risk within worker-firm pairs affects investment. Without risk-sharing, one would expect workers to have more incentives to invest since wages are more spread out, inducing workers to make precautionary investments. Indeed, this result obtains if workers are prudent enough: More workers invest under no-risk-sharing than under risk-sharing.

Our main insights are robust to relaxing several underlying assumptions. We show that they extend to allowing for two-sided risk aversion, partnership problems, continuous instead of binary investments, and two-sided investments.

To illustrate the applied relevance of our model, we use it for a stylized quantitative assessment of the rise in U.S. wage inequality. Understanding the sources of this surge, and to what extent it is due to changes in income risk, firm and worker heterogeneity (initial conditions), or technological change, is central for the design of effective redistributive policies (Heathcote, Storesletten, and Violante (2009)). Depending on the main driving force, policy interventions can be directed at providing insurance against initial conditions (worker heterogeneity), technological change (changes in technology or firm heterogeneity) or shocks (risk). But disentangling risk from heterogeneity is known to be challenging.

We first provide an identification argument for our model’s primitives, including risk, heterogeneity and technology, which then guides our estimation exercise. The estimates reveal that between the 1980s and today, the U.S. data is consistent with FOSD and IR shifts in risk and heterogeneity in line with those considered in our theory. We address the following two questions.
First, through the lens of our model featuring endogenous investment, how much of the increase
in wage inequality is due to changes in risk, heterogeneity of workers and jobs, and technological
change? Second, how would our conclusions change if we considered education as exogenous?

To address the first question, we perform a decomposition of rising inequality into its driving
forces, an exercise inspired by our theoretical comparative statics. We find that the main driver of
inequality is an increase in idiosyncratic background risk, followed by technological change. Dig-
ging deeper, we find that the increase in educational investment and in the skill premium over the
last few decades are both driven by the exact same primitives (IR shift in job heterogeneity, FOSD
shift in skill risk, and technological change) but they move wage inequality in opposite directions.
This is why the effect of those primitives on inequality is attenuated (relative to background risk).

To address the second question, we use two alternative approaches to analyze the drivers
of inequality, both of which treat education as exogenous. One is a statistical decomposition
of inequality into the returns to skill (which we also refer to as ‘skill premium’) and education
distribution (in the spirit of Eika, Mogstad, and Zafar (2018)); and the other is the estimation
of the ‘naive’ version of our model with exogenous skills. By not taking into account that both
skill returns and educational investment are driven by the same forces and push wage inequality
in opposite directions, these exercises show that holding education fixed while varying the wage
structure is misleading. Indeed, they overstate the impact of the wage structure (skill premium)
on inequality and understate the importance of risk changes relative to our model. Our model
with endogenous investment gives a richer and, most likely, more reliable decomposition of the
rise in wage inequality into its primitives, providing guidance for policies that aim to counter it.

Related Literature. Our theory is related to the large matching literature initiated by Becker
(1973), Shapley and Shubik (1972), Gale and Shapley (1962), and generalized by Legros and
Newman (2007) (see Chade, Eeckhout, and Smith (2017) for a survey), which has been extensively
applied to labor and marriage markets. We add a pre-match investment stage, a feature that is
also in Cole, Mailath, and Postlewaite (2001), Mailath, Samuelson, and Postlewaite (2013), and
Nöldeke and Samuelson (2015) (see also Peters and Siow (2002)). All these references focus
on pre-match investments with deterministic returns, and zero in on the important question of
efficiency of ex-post equilibria. A notable exception is Bhaskar and Hopkins (2016), who allow
for stochastic returns under nontransferable utility, and provide strong inefficiency results of ex-
post equilibria, in stark contrast with the transferable utility case with deterministic returns in
the papers cited above. Another paper is Zhang (2015), who shows that investments can be
extreme under linear surplus. Unlike all these references, we analyze pre-match investment with
stochastic returns by risk-averse agents that face multiple risks, and provide a complete analysis
of the equilibrium comparative statics of the model, plus a quantitative application. To the best

4
of our knowledge, ours is the first paper to analyze the equilibrium effects of changes in aggregate and idiosyncratic risks as well as in initial heterogeneity (FOSD and IR) and technology in a general matching framework, highlight investment as a precautionary action, and flesh out the economic relevance of these insights for studying inequality.

Our paper is also related to the large literature on the economics of uncertainty that studies the comparative statics of changes in risk in decision problems. For a survey, see Gollier (2004), where he also points out that intuitive comparative statics need not obtain without some restrictions on primitives, something we need to wrestle with as well. Our model deals with two complications: First, our risks affect agents’ expected utility through the equilibrium wage function, while this literature focuses on decision problems where the payoff/wage function is exogenous. Second, these risks enter our wage function in a non-additive way. Both features, equilibrium and non-additive risks, not only significantly complicate the analysis but they also enrich the economics of our comparative statics results — something that becomes particularly clear in our application.

2 The Model

There is a unit measure of heterogeneous workers and the same measure of heterogeneous firms.

Workers are risk averse with von Neumann-Morgestern utility function for income given by $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, which is continuous on $\mathbb{R}_+$ and three times continuously differentiable on $\mathbb{R}_{++}$, with $u' > 0$, $u'' \leq 0$, and $u''' \geq 0$. Initially, all workers have the same skill level normalized to zero, and they differ in a characteristic $\theta$, their ability, distributed according to a continuously differentiable cumulative distribution function (cdf) $Q$ on $[0, 1]$, with density $q > 0$.

Before entering the labor market workers make a binary investment choice $a \in \{0, 1\}$. This choice has stochastic returns: If $a = 1$, the worker draws a skill $x \in [0, 1]$ from a cdf $H_1$, and if $a = 0$ he draws $x$ from a cdf $H_0$. The drawn skill level is publicly observable. These distributions are ordered by strict FOSD: $H_1(x) \leq H_0(x)$ for all $x$, with strict inequality over some interval. Moreover, $H_i$ is continuously differentiable with density $h_i > 0$, $i = 0, 1$. Investment is costly, and how costly depends on the worker’s ability $\theta$. The utility cost is given by a function $c$, with $c(\theta) \geq 0$ for all $\theta$, and where $c$ is strictly decreasing and differentiable on $(0, 1]$, with $c(1) = 0$. To rule out uninteresting cases we assume that $\lim_{\theta \rightarrow 0} c(\theta) = +\infty$. If a worker with ability $\theta$ invests and obtains income $w$, then his payoff is $u(w) - c(\theta)$; in turn, it is $u(w)$ if he does not invest.

Firms are risk neutral, profit-maximizers, and differ in a publicly observable productivity attribute $y \in [0, 1]$, distributed with a continuously differentiable cdf $G$, with density $g > 0$.

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3We use the following conventions. Given $z : X \rightarrow \mathbb{R}$, $X \subset \mathbb{R}^n$, we write $z \geq 0$ ($z \leq 0$) or $z > 0$ ($z < 0$) if it holds for all $x \in X$. We use increasing, decreasing, concave, convex, positive, etc., in the weak sense, adding “strictly” if needed. When there is no confusion, we suppress the arguments of functions and limits of integrals.

4A random variable $X \in [0, 1]$ dominates $Y \in [0, 1]$ in FOSD if $F_X(s) \leq F_Y(s)$ for all $s$, where $F_X$ and $F_Y$ are the cdf’s of $X$ and $Y$. Equivalently, $X$ FOSD $Y$ if $E[z(X)] \geq E[z(Y)]$ for all increasing functions $z$. 

5
After workers’ skills are realized, firms and workers match pairwise in a competitive labor market. The amount of output a worker-firm pair produces depends on the realization of a shock \( \alpha \in [0, 1] \), drawn from a continuous cdf \( L \), and whose properties and interpretation we discuss below. Formally, match output is given by a positive function \( f \), so if a worker with skill \( x \) matches with a firm of productivity \( y \), and the shock realization is \( \alpha \), then they produce \( f(x, y, \alpha) \).

The function \( f \) is increasing and supermodular (spm), three times continuously differentiable, with positive derivatives \( f_x, f_y, f_\alpha, f_{xy}, f_{x\alpha}, \) and \( f_{y\alpha} \), strictly so on \((0, 1]^3\). Also, \( f(0, y, \alpha) = f(x, 0, \alpha) = 0 \) for all \( \alpha \), so in particular, a pair with the lowest attributes is unproductive. A simple example of such technology is \( f(x, y, \alpha) = \alpha xy \). If a worker is unmatched, then his income is zero.

Similarly, a firm that remains unmatched obtains a payoff of zero.

Apart from skill risk in \( x \), which is a controllable risk via investment, the shock \( \alpha \) is another source of risk in the model, a background risk, and it will play an important role in our analysis. We explore two different cases depending on when this shock is realized. In the first case, the shock is an aggregate shock to match output that realizes after workers draw their skills but before matching takes place, and thus firms cannot share this risk with workers. We call this the no-risk-sharing case (henceforth NRS). In the second case, the shock is realized after matching takes place, and it can be an aggregate shock to match output or an idiosyncratic shock to the pair’s match output. Hence, under this timing firms and workers can share this risk, and we call it the risk-sharing case (henceforth RS). We will analyze both cases in a unified fashion.

Let \( a : [0, 1] \to \{0, 1\} \) be a measurable investment function, where \( a(\theta) = 0 \) if a worker with ability \( \theta \) does not invest, and \( a(\theta) = 1 \) if he does. For a given \( a \), the distribution of skill \( x \) is \( H(\cdot, a) \), a mixture of \( H_1 \) and \( H_0 \) with weights given by the measure of workers who invest and do not invest, respectively. Letting \( I \) be the indicator function given by \( I(\theta) = 1 \) if \( a(\theta) = 1 \) and by zero otherwise, we have

\[
H(x, a) = \left( \int I(\theta) dQ(\theta) \right) H_1(x) + \left( 1 - \int I(\theta) dQ(\theta) \right) H_0(x). \tag{1}
\]

Let \( \mu(\cdot, a) : [0, 1] \to [0, 1] \) be a measurable and measure-preserving matching function that assigns each worker with skill \( x \) to a firm with productivity \( y = \mu(x, a) \), and let \( w(\cdot, \cdot, a) : [0, 1]^2 \to \mathbb{R}_+ \) be a measurable wage function so that \( w(x, \alpha, a) \) is the wage of a worker with \( x \) when the shock is \( \alpha \).

An equilibrium consists of an investment function \( a \), a matching function \( \mu \), and wage function \( w \), such that, given \( (w, \mu) \), workers invest optimally, that is, for all \( \theta \), \( a(\theta) = 1 \) if and only if the

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5 A twice continuously differentiable function \( z : \mathbb{R}^2 \to \mathbb{R} \) is spm in \((x, y)\) if \( z_{xy} \geq 0 \).

6 Assuming that outside income is zero will ensure that matching is individually rational for all agents, since match output \( f \) is a positive function. If their outside income was a strictly positive constant, the only change would be to pin down the mass of agents on each side who prefer to be unmatched.

7 Given our assumptions on \( f \) and the distributions (all atomless with connected supports), we can focus without loss of generality on assignments that are “pure,” given by a function \( \mu \) described above.
net utility from investing is higher than from not investing, \( U_1(w) - c(\theta) \geq U_0(w) \), where

\[
U_i(w) = \int \int u(w(x, \alpha, a))dH_i(x)dL(\alpha), \quad i = 0, 1;
\]

is the expected utility from investment choice \( i = 0, 1 \), taking both sources of risk into account; and for any \( a \) (which determines skill cdf \( H(\cdot, a) \)) the resulting pair \((w, \mu)\) given \( a \) (and given \( \alpha \) in NRS) is a Walrasian equilibrium of the labor market.

Remark 1. We assume that investment is binary and workers are ex-ante heterogeneous in ability. We could instead assume, in line with Bhaskar and Hopkins (2016), that workers are ex-ante identical and each chooses an investment level \( a \in [0, 1] \), with \( x \) distributed as \( H(\cdot | a) \), with common support for all \( a \), with \( H(x | \cdot) \) decreasing in \( a \) and strictly so on some interval of values of \( x \). The disutility of investment is now \( \kappa \), strictly increasing, strictly convex, and continuously differentiable, with \( \kappa(0) = \kappa'(0) = 0 \) and \( \lim_{a \to 1} \kappa'(a) = \infty \). If one focuses on symmetric equilibria where all workers choose the same level of investment \( a \), then the insights are the same as in our binary formulation. See Online Appendix B.3 for details.

Remark 2. With some minor modifications, in NRS one could assume instead that \( \alpha \) is an uninsurable idiosyncratic income shock (as opposed to aggregate shock), realized after matching. The only modification to the model is that now \( f \) is independent of \( \alpha \), and that the worker’s income is determined by his wage and the shock realization. All the insights go through with minor changes (see Online Appendix B.4). We use this interpretation in the quantitative analysis in Section 6.

3 Equilibrium Analysis

In this section we show that an equilibrium exists, provide conditions for uniqueness, and analyze the efficiency properties of equilibria.

3.1 Existence and Uniqueness

For each investment function \( a \) (and shock realization \( \alpha \) if it is drawn before matching), we obtain the Walrasian equilibrium of the labor market. Given the binary nature of the investment decision, matching and wages are parameterized by an ability threshold above which workers optimally decide to invest. Equilibrium existence obtains by solving an equation for that threshold.

No risk-sharing case (NRS). Consider the matching stage when \( \alpha \) realizes before matching takes place. Fix an investment function \( a \) and a shock realization \( \alpha \). Since matching takes place under certainty, this is an assignment game as in Becker (1973), but in a large market setting.

\[\text{That is, given } w, \text{ each agent of each attribute maximizes their payoff by choosing the partner described by } \mu \text{ instead of choosing another partner or none at all, and } \mu \text{ clears the market. In RS, it will also be the case that there is efficient risk-sharing for every firm-worker pair.}\]
Production exhibits complementarities since $f$ is strictly spm in $(x,y)$ for each $\alpha$. Thus, any equilibrium exhibits positive sorting, and so the matching $\mu(\cdot,a)$ solves $H(x,a) = G(\mu(x,a))$ for all $x$, and is strictly increasing and continuously differentiable in $x$.

We show in Appendix A.1 that the wage function $w(\cdot,\alpha,a)$ that supports the positive sorting allocation is $w(x,\alpha,a) = \int_0^x f(x,s,\mu(s,a),\alpha)ds$, and that the pair $(w(\cdot,\alpha,a),\mu(\cdot,a))$ is the unique Walrasian equilibrium of the labor market given $(\alpha,a)$. The payoffs of a matched worker-firm pair $(x,y)$ is $u(w(x,\alpha,a))$ for the worker with skill $x$ (minus the sunk investment cost if he invested) and $\pi(y,\alpha,a) = f(\mu^{-1}(y,a),y,\alpha) - w(\mu^{-1}(y,a),\alpha,a)$ for the firm with attribute $y$.

Consider now the investment stage. For any investment choices of other agents that a worker with ability $\theta$ anticipates, with the corresponding wage function in the matching stage, the worker invests if and only if $U_1(w) - c(\theta) \geq U_0(w)$. Since the wage function $w$ strictly increases in skill $x$, and $H_1$ strictly FOSD $H_0$, we have that $U_1(w) - U_0(w) > 0$. Thus, in any equilibrium we have

$$a(\theta) = \begin{cases} 1 & \text{if } \theta \geq \theta^* \\ 0 & \text{if } \theta < \theta^*, \end{cases}$$

where the ability threshold $\theta^* \in (0,1)$ characterizes the function $a$, and where without loss of generality we have set $a(\theta^*) = 1$.

Equilibrium existence reduces to finding an ability threshold $\theta^*$ such that a worker with that ability is indifferent between investing and not investing given a Walrasian equilibrium in the labor market consistent with that threshold. That is, $\theta^*$ solves $U_1(\theta^*) - U_0(\theta^*) = c(\theta^*)$, where with some abuse of notation we replaced $U_i(w)$ by $U_i(\theta^*)$, $i = 0, 1$. Integrating $U_1 - U_0$ by parts yields

$$\int \int w'(w(x,\alpha,\theta^*))w_x(x,\alpha,\theta^*)\Delta H(x)dxdL(\alpha) = c(\theta^*), \tag{3}$$

where $\Delta H \equiv H_0 - H_1 \geq 0$.

Given any $\theta^*$ that solves (3), we obtain skill distribution $H(\cdot,\theta^*)$, which then yields a Walrasian equilibrium in the matching stage given by $(w(\cdot,\alpha,\theta^*),\mu(\cdot,\theta^*))$ for each realization of shock $\alpha$. In turn, this pins down the utility difference $U_1(\theta^*) - U_0(\theta^*)$ in the investment stage, which is higher than investment cost $c(\theta)$ for $\theta \geq \theta^*$ and lower otherwise. This rationalizes $H(\cdot,\theta^*)$, completing the equilibrium construction.

**Risk-sharing Case (RS).** The steps are similar to NRS, except that now each matched pair solves an optimal risk-sharing problem regarding the shock $\alpha$, which realizes after matching.

Consider the matching stage and a worker-firm pair with attributes $(x,y)$, and assume that the firm must be given a payoff of at least $\pi_0$. The pair’s risk-sharing problem is to find a measurable function $\xi$ that solves $\max_\xi \int_0^1 u(\xi(\alpha))dL(\alpha)$ subject to $\int_0^1 (f(x,y,\alpha) - \xi(\alpha))dL(\alpha) \geq \pi_0$. Since the firm is risk-neutral and the worker strictly risk-averse, optimal risk-sharing calls for the firm

\footnote{At several points in the analysis we will pass the derivative through the integral and use integration by parts, as in (3). To avoid technical detours in the text, we justify these operations in the Online Appendix B.3.
to fully insure the worker and thus the optimal transfer $\xi$ is independent of shock $\alpha$. Since the constraint binds at the optimum because $u$ is strictly increasing, we obtain that the optimal $\xi$ is $E_L[f(x, y, \alpha)] - \pi_0$, where $E_L$ is the expectation with respect to the cdf $L$. The worker’s utility is then $u(E_L[f(x, y, \alpha)] - \pi_0)$.

Since $f$ is strictly spm in $(x, y)$ for each $\alpha$, so is $E_L[f]$, and thus any equilibrium once again exhibits positive sorting, so $H(x, a) = G(\mu(x, a))$ for all $x$ given $a$.

Regarding the wage function that supports $\mu(\cdot, a)$, note that once we have pinned down the optimal risk-sharing, the only difference with the analysis under NRS is that now $\bar{f}$, given by $\bar{f}(x, y) = E_L[f(x, y, \alpha)]$, replaces $f$. Hence, we can follow the same steps as in NRS and obtain $w(x, \alpha, a) = \bar{w}(x, a) = \int_0^x \bar{f}_x(s, \mu(s, a))ds$ for all $x$. The pair $(\bar{w}(\cdot, a), \mu(\cdot, a))$ is the unique Walrasian equilibrium of the matching stage given any investment function $a$. Within any match between a worker with skill $x$ and a firm with attribute $y$, the worker’s payoff is $u(\bar{w}(x, a))$ (minus the sunk investment cost if he invested) and the firm’s expected payoff is $\bar{\pi}(y, a) = E_L[\pi(y, \alpha, a)] = \bar{f}(\mu^{-1}(y, a), y) - \bar{w}(\mu^{-1}(y, a), a)$.

Consider now the investment stage. As before, $a$ is characterized by a threshold ability $\theta^*$ such that a worker with ability $\theta$ invests if and only if $\theta \geq \theta^*$. The equilibrium condition is again $U_1(\theta^*) - U_0(\theta^*) = c(\theta^*)$, which, by integration by parts, is equal to

$$
\int u'(\bar{w}(x, \theta^*)) \bar{w}_x(x, \theta^*) \Delta H(x)dx = c(\theta^*). \quad (4)
$$

Once we solve (4) for $\theta^*$, we can finish the construction of an equilibrium as in NRS.

**Equilibrium Existence and Uniqueness.** Let $R \equiv -u''/u'$ be the coefficient of absolute risk aversion. The following theorem covers both NRS and RS:

**Proposition 1 (Existence and Uniqueness)** An equilibrium exists in both NRS and RS, and all equilibria exhibit $\theta^* \in (0, 1)$. It is unique if agents’ absolute risk aversion $R$ is (uniformly) sufficiently small, or if the cost function $c$ is sufficiently convex.

To see the proof of the first claim, note that $U_1 - U_0$ is strictly positive and continuous for all $\theta^* \in [0, 1]$, and $c$ diverges to infinity when $\theta^*$ goes to zero and it is zero at $\theta^* = 1$. Hence, there is at least one solution to $U_1(\theta^*) - U_0(\theta^*) = c(\theta^*)$ where $U_1 - U_0$ crosses $c$ from below (see Figure 1(i)), and point A in Figure 1(ii)). Moreover, in any equilibrium $\theta^*$ is interior.\(^{10}\)

The proof of uniqueness is in Appendix A.2 For some intuition, note that equilibrium is unique if $U_1 - U_0$ is increasing in $\theta^*$ (Figure 1(i)). This condition is also necessary if we want uniqueness for all $c$’s that satisfy our assumptions. If workers were risk neutral, then $U_1 - U_0$ would be increasing since $w$ and $\bar{w}$ are spm in $(x, \theta^*)$, and hence the left sides of (3)–(4) increase in $\theta^*$. But then, the same holds if workers are not too risk averse.\(^{11}\) Equilibrium is also unique if

\(^{10}\) If we did not assume $\lim_{\theta \to 0} c(\theta) = \infty$, or more generally that $c(0)$ is sufficiently large, then there could be an equilibrium with $\theta^* \equiv 0$, i.e., everyone invests.

\(^{11}\) See Appendix A.2 for a detailed example, which also illustrates that $R$ need not be very small.
Figure 1: (i) The left panel depicts a unique equilibrium. (ii) The right panel depicts multiple equilibria, in which at A an C, $U_1 - U_0$ crosses $c$ from below, and from above at B.

c is sufficiently convex. Graphically, if $c$ is convex enough, its graph will be sufficiently “close to the axes”, and then only the first crossing of $c$ by $U_1 - U_0$ (point A in Figure 1 (ii)) survives. How could multiple equilibria arise? Note that here investment is one-sided, and thus the strategic complementarities that can lead to multiplicity in the two-sided investment case are absent. There are, however, two opposite forces — complementarities in wages and risk aversion — that can lead to multiplicity. To see this consider NRS (RS is similar) and note that $\partial(U_1(\theta^*) - U_0(\theta^*)/\partial\theta^* = \int \int (u''w_\theta w_x + u'w_x\theta)\Delta H dx dL$. If workers’ risk aversion dominates the complementarities of the wage in $(x, \theta^*)$, then $U_1 - U_0$ decreases in $\theta^*$, meaning that the incentives to invest co-move with the measure of workers who invest. This can lead to several solutions to $U_1 - U_0 = c$ and thus to multiplicity: If incentives to invest $U_1 - U_0$ are high, then many workers invest ($\theta^*$ is small), wages fall, and “on average” the marginal utility of income rises, rationalizing the high $U_1 - U_0$. The opposite happens when $U_1 - U_0$ is low, leading to little investment, high wages, low marginal utility, thus confirming the low $U_1 - U_0$. If workers were risk neutral, this risk aversion effect would be absent, and complementarities would cause the incentives to invest and investment to go in opposite directions, precluding multiplicity. Online Appendix B.1 presents an example of multiplicity and describes the construction, which follows the logic just explained.

Of particular interest for our comparative statics results below are equilibria that are stable in the sense that they occur where “$U_1 - U_0$ crosses $c$ from below,” as in A and C in Figure 1 (ii). Proposition 1 shows that there is at least one. They are stable in a natural (tâtonnement) sense: If $\theta^*$ is to the left of the crossing point, then $U_1(\theta^*) - U_0(\theta^*) < c(\theta^*)$ and $\theta^*$ will go up since some

\[ \text{For example, if } c(\theta) = (1/\theta \hat{x}) - 1, \text{ then } \lim_{\theta \to \infty} (1/\theta \hat{x}) - 1 = 0 \text{ for all } \theta \in (0, 1), \text{ and } c(\theta) = -(1/j)(1/\theta \hat{x} + 1) \text{ diverges to } -\infty \text{ as } \theta \text{ goes to } 0. \text{ Equilibrium is unique for sufficiently large } j. \]
workers with \( \theta \) above \( \theta^* \) will not find it optimal to invest. The opposite happens to the right of the crossing point. Equilibria such as \( B \), where the crossing is from above, are unstable. This will be also illustrated in Figure 2 (i) in the Section 4 below.

**Remark 3** The assumption that \( H_i, i = 0, 1 \), are continuously differentiable with positive densities and have common supports conveniently simplifies the equilibrium analysis of the model.\(^{13}\) That is, whether a worker invests or not, \( x \) is drawn from a continuously differentiable cdf with support \([0, 1]\). This implies a strictly increasing and continuously differentiable matching function. Moreover, we do not need to worry about how to match a worker who invests and obtains a skill that is larger (smaller) than the largest (smallest) realization in the support of the skills, and so we do not have to price the skills associated with those off-equilibrium investment choices. This is obvious with a binary investment choice, but it also holds in the continuous case described in Remark 1, unlike the model in Bhaskar and Hopkins (2016), which allows for moving supports. Moreover, our assumption precludes the possibility of gaps or atoms that are common in the pre-match investment literature with deterministic returns (see, for example, Cole, Mailath, and Postlewaite (2001), for an extensive analysis of this issue).

**Remark 4** As is well-known, the Walrasian equilibrium of the matching market we have derived in each of the two cases is stable in the more standard matching sense that we cannot find a worker-firm pair that would like to match but they are not. More precisely, the sum of wage and profit within each match exhausts the available match output (or expected match output), but it is greater than the match output for those not matched with each other. This property follows since \( f \) is spm. For completeness, we prove this property in Appendix A.1.

### 3.2 Efficiency Properties

We now turn to the efficiency properties of equilibrium. We first analyze whether, in NRS and RS, equilibrium is efficient in the sense that there is no alternative investment threshold that, taking into account its effect on the matching and wage functions, leaves everyone as well off as before and some agents strictly better off. We have the following result:

**Proposition 2 (Efficiency)** In both NRS and RS, the equilibrium with the lowest investment threshold \( \theta^*_\ell \) is efficient. Hence, if equilibrium is unique, then it is efficient.

Since we have provided primitives where equilibrium is unique, and since, when there are multiple ones, the equilibrium involving the largest fraction of workers who invest (corresponding to \( \theta^*_\ell \)) is of particular interest, it follows that Proposition 2 covers an interesting class of problems.

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\(^{13}\)We are grateful to a referee who pointed this out to us.
The proof is in Appendix A.3, and it shows that any alternative threshold that can strictly improve the utility of a positive-measure set of workers strictly lowers the profits of firms on a positive-measure set, and vice versa. It then follows that $\theta^*_\ell$ is efficient.

For some intuition, consider the effect of a marginal increase in the investment threshold. Aggregate profits of firms strictly decrease because, since fewer workers invest, the skill distribution worsens, and this strictly increases wages due to more competition for high-skilled workers that are now scarcer. So for $\theta^* \neq \theta^*_\ell$ to strictly increase profits on a positive-measure set of firms, it must be that $\theta^* < \theta^*_\ell$. However, we show that for workers, utility decreases as we reduce $\theta^*$ below $\theta^*_\ell$. This monotonicity in opposite direction between workers’ aggregate utility and firms’ aggregate profits makes it impossible to find a Pareto improvement, leading to efficiency.

**Remark 5** Using the terminology from Nöldeke and Samuelson (2015) and Cole, Mailath, and Postlewaite (2001), what we have analyzed is the efficiency of our “ex-post” equilibrium concept, where matching and wages are realized after investments are sunk and skills are drawn. But a planner could strictly improve welfare if she could also provide skill insurance. For example, fix an equilibrium, and assume that the planner promises a constant transfer to each worker, to be paid once match output is realized. And she uses the same investment threshold as in the assumed equilibrium as well as positive sorting, and she gives firms the same profits as in the equilibrium (this requires some suitable redistribution among firms). Firms are then as well off while workers are strictly better off since they are now fully insured against not only shock $\alpha$ but also skill risk in $x$. And the planner can do even better by also choosing the investment threshold optimally. Hence, in this “ex-ante” sense where the planner can design transfers to insure workers against skill risk before skills are realized, equilibrium is inefficient (see Online Appendix B.2).

### 4 Comparative Statics

We now provide a thorough analysis of the equilibrium comparative statics of the model (in both NRS and RS) when distributions $L$, $H_1$, $G$, or $Q$ undergo a FOSD or an IR shift, or when technology $f$ changes. We will also analyze whether more workers invest in NRS than in RS.

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14 An interesting open question for future research is to analyze ex-ante equilibria, in which firms and workers match pairwise based on $y$ and $\theta$ and then each pair chooses an investment level. In our model, matching is stable in the sense of Remark 4 since it takes place after skills are realized, but workers bear skill risk. This risk could be insured if firms and workers matched based on $\theta$ and $y$. But, in that case, there would be incentives to re-match after skills realize. Thus, a novel trade-off between insurance and stability ensues: If re-matching is not allowed, there are instabilities; and with re-matching, the initial skill insurance does not take place. See also Remark 13 on p.61 of Nöldeke and Samuelson (2014), which discusses the difference between ex-ante and ex-post equilibria once investments have stochastic returns, and the presence of uninsurable risks.

15 An IR shift asserts that a random variable $X \in [0, 1]$ is riskier than a random variable $Y \in [0, 1]$ if $\int_0^1 F_X(s)ds \geq \int_0^1 F_Y(s)ds$ for all $t$, with equality at $t = 1$ (so both have the same mean). Alternatively, this holds if and only if $\mathbb{E}[z(X)] \leq \mathbb{E}[z(Y)]$ for all concave functions $z$. This is called the convex order in the stochastic order literature (see Chapter 1 in Muller and Stoyan (2002)).
that is, we will investigate how investment changes as risk-sharing opportunities change.

A useful taxonomy is to distinguish changes in risk (distributions $L$ and $H_1$) from changes in heterogeneity (distributions $G$ and $Q$) and technology ($f$). Such a distinction will also be relevant in our application in Section 6. We provide conditions on primitives under which a stochastically better or riskier output shock, skill, ability, or firm-productivity distribution or under which a shift in technology yield the intuitive comparative statics results explained in the Introduction. In the distributional shifts, special attention will be given to background risk $L$, since the shock $\alpha$ is at the heart of the uncertainty that workers face, and is insurable in one of our versions. For all shifts, we first focus on the impact of them on $\theta^*$, which determines the measure of workers who invest, and then discuss how this change in investment alters matching and wages.

The overarching theme underlying our results is a fundamental trade-off between risk aversion and complementarities in the equilibrium wage function (inherited from similar properties of match output). If workers were risk-neutral or if the wage function was additively separable, this analysis would be much simpler but also miss two important features of the labor market.

Our results will apply to all stable equilibria in NRS and RS, where $U_1 - U_0$ crosses $c$ from below. Proposition 1 shows that there is at least one equilibrium, and it follows from it that the equilibria with the smallest and largest investment thresholds are stable. Our approach is to index the appropriate distribution by a parameter $t \in [0, 1]$, where an increase in $t$ represents a FOSD or an IR shift. In the case of technological change, an increase in $t$ shifts the worker marginal product. We then consider how $\theta^*$ changes with $t$ by analyzing the equilibrium condition $U_1(\theta^*, t) - U_0(\theta^*, t) = c(\theta^*)$, where for clarity we added $t$ as an argument of $U_i$, $i = 0, 1$.

Figure 2 (ii) reveals that if we want $\theta^*$ to decrease in $t$, so that more workers invest in response to the shift under consideration, then a sufficient condition is that $U_1 - U_0$ increases in $t$ for all values of $\theta^*$. This condition is also necessary for the results to hold for all cost functions $c$ in the class we consider. This is because if $U_1 - U_0$ was strictly decreasing in $t$ on an interval of values of $\theta^*$, then one could find a $c$ in our class such that $U_1 - U_0$ crosses $c$ from below in that interval, and thus $\theta^*$ would increase in $t$ for any such crossing, meaning fewer workers would invest.

### 4.1 Changes in Background Risk

We first investigate how investment threshold $\theta^*$ changes with a change in background risk $\alpha$. For example, if $\alpha$ is an aggregate shock that affects production, then it can be interpreted as the state of the labor market, and we want to assess how the number of workers who invest changes if the market stochastically improves or becomes more uncertain. In turn, if $\alpha$ is an idiosyncratic income shock, we assess how improved or more income risk affects investment behavior. We index $L$ by $t \in [0, 1]$, and assume that $L(\alpha | \cdot)$ is continuously differentiable in $t$ for each $\alpha$.

Our aim is to find weak and economically intuitive conditions on preferences and technology such that $U_1 - U_0$ increases in $t$ for all $\theta^*$. To see the main idea in a unified way, let $\varphi$ be given by
Figure 2: (i) Two stable equilibria, $A$ and $C$, an unstable equilibrium in between, $B$. (ii) Comparative Statics of $U_1 - U_0$ with respect to $t$, which increases investment in stable equilibria $A$ and $C$.

\[ \varphi(x, t) = \int u(w(x, \alpha, \theta^*))dL(\alpha|t) \] in NRS, and by \[ \varphi(x, t) = u(\tilde{w}(x, \theta^*, t)) = u(\int w(x, \alpha, \theta^*)dL(\alpha|t)) \] in RS. Then (3) and (4) can be written simply as

\[ U_1(\theta^*, t) - U_0(\theta^*, t) = \int \varphi_x(x, t)\Delta H(x)dx, \]

and thus $U_1 - U_0$ increases in $t$ (and thus more workers invest) if and only if \[ \int \varphi_{xt}\Delta Hdx \geq 0, \] for which it suffices that $\varphi$ is spm in $(x,t)$ since $\Delta H = H_0 - H_1 \geq 0$.

When is $\varphi$ spm? Consider NRS. Since $\varphi_x = \int u'(w)w_xdL$, under a FOSD shift in $L$ it suffices that $u'(w)w_x$ increases in $\alpha$. Similarly, under an IR shift, it suffices that $u'(w)w_x$ is convex in $\alpha$. Hence, a FOSD shift in $L$ reduces $\theta^*$ in any stable equilibrium — more workers invest — if $u'(w)w_x$ is increasing in $\alpha$; in turn, an IR shift in $L$ induces more investment if $u'(w)w_x$ is convex in $\alpha$. A similar logic applies to RS, only that $L$ is now inside the wage $\tilde{w}$.

For an intuition, consider a FOSD or IR shift in $L$ and a worker with skill $x$. Then an increase in $t$ changes $\varphi$, and the change is larger for higher $x$ since $\varphi$ is spm. Since investment yields stochastically higher skills $x$, the change in utility with investment, $U_1$, dominates the change in utility without investment, $U_0$, and hence an IR shift in $L$ induces more workers to invest.

The task now is to derive the supermodularity of $\varphi$ from conditions on primitives, $u$ and $f$.

**No Risk-sharing Case (NRS).** Consider first a FOSD shift in $L$. That is, $L(\alpha|\cdot)$ is decreasing

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16With an arbitrary $w$ this condition is also necessary for the result to hold for all $H_i$, $i = 0, 1$, ordered by strict FOSD. For if $\varphi_{xt} < 0$ on an interval of $x$, then we can choose $H_0$ and $H_1$ such that $H_1(x) < H_0(x)$ only for $x$ in that set, and the result fails. The wage function in our model, however, also depends on $H_0$ and $H_1$ through $\mu$, so this argument does not apply. But it suggests that it would be hard to relax our conditions.

17These conditions are necessary if we want $\varphi$ spm in $(x,t)$ for all cdf’s $L$ ordered by FOSD or IR.
in \( t \) for all \( \alpha \), so higher shock realizations are more likely. Equivalently, the expectation of any increasing function of \( \alpha \) is increasing in \( t \). The following result provides a class of primitives under which such a shift increases the measure of workers who invest.

**Proposition 3 (NRS: FOSD Shift in \( L \))** In any stable equilibrium in NRS, more workers invest in skills in response to a FOSD shift in \( L \) if either of the following conditions hold:

(i) Absolute risk aversion \( R \) is (uniformly) sufficiently small; or

(ii) For all \( w \), \( R(w) \leq 1/w \), and \( f_x \) is log-spm in \((x, \alpha)\) for each \( y \), and in \((y, \alpha)\) for each \( x \).

The idea of the proof is simple (see Appendix A.4). Recall that \( w = \int_0^x f_x \). We want \( u'(w)w_x \) to increase in \( \alpha \), and its derivative is

\[
u'(w)w_{\alpha}w_x \left( \frac{w_{\alpha}w_x}{w_{\alpha}w_x} \right) \frac{1}{w} - R(w) \right) ,
\]

(6)

When workers are risk neutral, this is positive since \( w_{\alpha} = f_{x\alpha} \geq 0 \), and actually this holds strictly almost everywhere given the assumptions on \( f \). By continuity, it remains positive when risk aversion is small, which yields part (i).\(^{18}\) Regarding (ii), the log-supermodularity conditions on \( f_x \) imply that \( w \) is log-spm in \((x, \alpha)\), and so the term in parentheses is bigger than 1. The result then follows if risk aversion \( R(w) \) is bounded above by \( 1/w \) for each value of \( w \).

For an economic intuition, interpret \( u'(w)w_{\alpha}w_x \) as a substitution effect: Because of the complementarities in the wage between skill \( x \) and shock \( \alpha \), and since \( \alpha \) is now stochastically larger, some agents switch from not investing to investing since investment yields stochastically higher \( x \). In turn, \( u''(w)w_{\alpha}w_x \) resembles an income effect where a stochastically better shock increases all wages — even for those who do not invest — thus reducing the incentives of risk-averse workers to invest. If the substitution effect dominates the income effect, our result obtains.

The conditions in Proposition 3 are weak and easily satisfied. Say that \( f \) is a separable class if it can be written as \( f(x, y, \alpha) = \eta(\alpha)z(x, y) \), with \( \eta \) increasing, convex and log-concave, and twice continuously differentiable on \((0, 1] \), and \( z \) satisfies the rest of the assumptions on \( f \) we made before. This class of match output functions is perhaps the most commonly used in applications.

The following example uses this class of match output functions to illustrate Proposition 3.

**Example 1** Let \( f \) be in the separable class \( f(x, y, \alpha) = \eta(\alpha)z(x, y) \). Part (i) holds for constant absolute risk aversion (CARA) \( u(w) = -e^{-Rw} \), \( R > 0 \), when \( R \) is sufficiently small. For part (ii), note that \( f_x(x, y, \alpha) = \eta(\alpha)z_x(x, y) \), which is trivially log-spm in \((x, \alpha)\) and in \((y, \alpha)\) since \( f \) is

\(^{18}\) A twice continuously differentiable function \( z : \mathbb{R}^2 \rightarrow \mathbb{R} \) is log-spm (log-submodular) in \((x, y)\) if \( \log z \) is smp (submodular), or \( z_{xy}z - z_xz_y \geq 0 \) (\( \leq 0 \)). This is stronger than supermodularity when \( z_xz_y \geq 0 \).

\(^{19}\) More generally, this expression is positive if risk aversion is uniformly bounded above by a positive number that depends on the complementarities in \( w \).
multiplicatively separable in those variables. If utility is CRRA, \( u(w) = (w^{1-\sigma} - 1)/(1 - \sigma) \), then \( R(w) = \sigma/w \), and hence \( R(w) \leq 1/w \) for \( \sigma \in (0, 1) \).[20]

This example also suggests an intuitive counterexample: A separable class \( f \) has the property that \( w_x w_\alpha = w w_{x\alpha} \), and hence our comparative statics result does not hold if \( \sigma > 1 \).[21] Using the intuition above, absent sufficiently strong complementarities in wages, large risk aversion makes the income effect outweigh the substitution effect. It follows that less workers invest.

Consider now an IR shift in \( L \). That is, \( \int_0^\alpha L_t(s|t)ds \geq 0 \) for all \( \alpha \) and \( t \), and \( \int L_t(s|t)ds = 0 \) (the mean remains constant), so that the shock \( \alpha \) becomes riskier. Equivalently, the expectation of any convex function of \( \alpha \) is increasing in \( t \).

Let \( P \equiv -u'''/u'' \) be the coefficient of absolute prudence. The following result provides a large class of primitives under which the intuitive comparative statics obtains.

**Proposition 4 (NRS: IR Shift in \( L \))** In any stable equilibrium in NRS, more workers invest in skills in response to an IR shift in \( L \) if either of the following conditions hold:

(i) Absolute risk aversion is (uniformly) sufficiently small and \( f_{xaa} \geq 0 \) with strict inequality on a set of \((x, y, \alpha)\) of positive measure; or

(ii) For all \( w \), \( P(w) \geq 3/w \) and \( f \) is a separable class.

To see the idea of the proof (see Appendix [A.5]), recall that we seek conditions on primitives such that \( u'(w) w_x \) is convex in \( \alpha \). Differentiating twice with respect to \( \alpha \), rearranging, and using the definition of \( R \) and \( P \) yields

\[
R(w) w_x w_\alpha^2 \left( P(w) - \frac{1}{w} \left( \frac{2 w_{x\alpha} w}{w_x w_\alpha} + \frac{w_{\alpha\alpha} w}{w_\alpha^2} \right) \right) + w_{x\alpha\alpha}. \tag{7}
\]

If workers are risk neutral, \( R = P = 0 \), then this expression is positive if and only if \( w_{x\alpha\alpha} \geq 0 \), and this holds if \( f_{x\alpha\alpha} \geq 0 \). Intuitively, this also holds when risk-aversion is sufficiently small as long as \( w_{x\alpha\alpha} > 0 \) on a set of positive measure, which yields part (i). Regarding part (ii), since \( f \) is a separable class, the term in the inner parentheses involving complementarities and curvature of \( w \) is bounded above by 3. This is because \( w_x w_\alpha = w w_{x\alpha} \) and \( w_{\alpha\alpha} w \leq w_\alpha^2 \) by log-concavity of \( \eta \). Hence, the first term in (7) is positive if \( P(w) \geq 3/w \), which yields part (ii).

Besides making the proof transparent, (7) also provides a clean economic intuition. In line with our common theme, this one also relies on two forces, curvature properties of \( w \) (and thus of \( f \)) and workers’ risk attitudes, summarized by absolute risk aversion and prudence.

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[20] CRRA also satisfies (i) if \( u(w) = ((w + b)^{1-\sigma} - 1)/(1 - \sigma) \), where \( b > 0 \) is a worker’s initial wealth.

[21] When \( \sigma \geq 1 \), \( u \) is not continuous at \( w = 0 \), something we have assumed mainly to ensure a finite reservation utility at zero income. But this is easy to fix by adding an initial wealth level \( b > 0 \) as in the previous footnote, so that total wealth is \( w + b \) and the condition is \( P(w + b) \geq 2/(w + b) \); similarly in each example below with CRRA and \( \sigma \geq 1 \). Since this is just a technicality, we will not repeat it anymore.
Consider part (i). Under risk neutrality \( (R = 0) \), a riskier shock induces more agents to invest if \( w_x \) is convex in \( \alpha \), a condition that implies that the mean wage increases for all workers and especially so for those with a higher skill \( x \) — something that investment yields stochastically. This result extends by continuity to small absolute risk aversion.

Regarding the more interesting part (ii), if workers are sufficiently prudent, then a riskier shock induces more workers to invest in their skills. It is well-known in consumption-saving problems that prudence triggers precautionary savings to insure against bad shock realizations\(^{22}\). Here, sufficient prudence leads to a new precautionary action in our context: precautionary investments.

How large prudence needs to be depends on curvature and complementarity properties of the wage function. Note that if \( w \) were additively separable in \( x \) and \( \alpha \), which is typically not true in this setting, then \( P \geq 0 \) would suffice for the result. But, the nontrivial curvature properties of the wage function lead to a lower bound on prudence. Intuitively, if the wage function is convex in \( \alpha \), a riskier shock generates more upside compared to downside risk. Since prudent agents particularly dislike downside risk, this reduces the incentives to invest and only prudent enough workers would take the precautionary action. Moreover, since the wage function exhibits complementarities in \( x \) and \( \alpha \), a riskier shock exposes workers with high \( x \) even more to this risk than those with low \( x \) (since for high \( x \) the difference in payoffs between high and low shock is much larger), further reducing the incentives for precautionary investment. Note that these forces against investment are tamed but still present if \( w \) is submodular and log-concave. Only under enough prudence the net response to increased background risk is more precautionary investment\(^{27}\).

The conditions of the proposition are weak and easily satisfied in commonly used parameterizations, as the following example illustrates.

**Example 2** Let \( f(x, y, \alpha) = \alpha^\beta xy, \beta > 1 \), which implies that \( w_{x\alpha\alpha} \geq 0 \), that \( w_{x\alpha} w = w_x w_\alpha \), and that \( w \) is log-concave in \( \alpha \). Part (i) then holds for \( u \) CARA when \( R \) is sufficiently small. The condition in part (ii) is \( P(w) \geq (2 + ((\beta - 1)/\beta))/w \). If \( \beta = 2 \) then the bound is \( P(w) \geq 2.5/w \). If \( u \) is CRRA, then \( P(w) = (\sigma + 1)/w \), and thus \( \sigma \geq 2 \) suffices.

In some cases with CRRA we can obtain a sharper characterization. The following example justifies our focus on “small” risk aversion in (i) and on “large” prudence in (ii) of Proposition 4.

**Example 3** Let \( f(x, y, \alpha) = \alpha^\beta z(x, y), \beta > 1 \), and let \( u \) be CRRA. Then \( U_1 - U_0 \) is given by
\[
(\int \alpha^\beta(1-\sigma) dL) \int (\int_0^x z_x ds)^{-\sigma} z_x \Delta H dx.
\]
An IR shift in \( L \) increases investment if and only if \( \alpha^\beta(1-\sigma) \) is convex in \( \alpha \). The sign of second derivative with respect to \( \alpha \) equals the sign of \( (1-\sigma)(1-(1/\beta)-\sigma) \). This is strictly positive if and only if \( \sigma \in [0, 1 - (1/\beta)] \cup (1, \infty) \). Hence, investment increases (\( \theta^* \) decreases) if \( \sigma \) is small (as in part (i) since \( R = \sigma/w \) or large enough (as in part (ii) since \( P =
\]

\(^{22}\)See Lise (2012) for a careful analysis of precautionary savings in a search model.

\(^{27}\)Indeed, we could have written part (ii) without assuming that \( f \) is a separable class: it suffices that \( \varrho = \sup_{\alpha}(2w_{x\alpha} w)/(w_x w_\alpha) + ((w_{\alpha\alpha} w)/w_\alpha^2) \) < \( \infty \) and \( P(w) \geq \varrho/w \) for all \( w \). This subsumes (ii) with \( \varrho = 3 \).
As a small detour of economic interest, we ask the following question: What would happen if the shock \( \alpha \) becomes stochastically worse and more variable? Parts (ii) of Propositions (3)–(4) and intuition suggest that sufficiently risk averse and prudent workers would have more incentives to invest. We confirm in Corollary 1 in Appendix A.6 that these conditions on risk attitude indeed lead to more investment under such a compound shift in \( L \). For an economic application of this result, Bloom, Floetotto, Jaimovic, Saporta-Ekstein, and Terry (2018) uncover the interesting fact that recessions in the U.S. entail both an increase in uncertainty of aggregate shocks to production (a higher variance) and a decrease in their mean. Our model then predicts that we should see educational investment go up in response.

**Risk-sharing Case (RS).** In NRS, workers are not insured against the shock \( \alpha \), so the wage function \( w \) depends on the realization of \( \alpha \) but not on the shock distribution \( L \) itself, which only impacts the worker’s problem through expected utility. But in RS, workers are fully insured against the shock, so the wage \( \bar{w} \) does no longer depend on \( \alpha \). The shock distribution \( L \), however, impacts \( \bar{w} \) directly (and hence \( \bar{w} \) depends on \( t \)). Thus, risk changes still have non-trivial effects.

The analysis of a FOSD shift in \( L \) mimics that in NRS, and hence to avoid repetition we have placed in the appendix (see Proposition 8 in Appendix A.7 for details). The economic intuition is also similar: The substitution effect (driven by complementarities) must outweigh the income effect (driven by risk aversion) for more workers to invest with a FOSD shift in \( L \).

In turn, RS does make a difference compared to NRS when we consider an IR shift in \( L \). Since in RS, firms fully insure workers against shock \( \alpha \), there is no need to use education as a precautionary investment against this risk. Thus, prudence plays no role. But, such a shift still affects in interesting ways the workers’ incentives to invest via \( \bar{w} \) and risk aversion.

Say that \( f = \eta z \) is a weak separable class if \( z \) satisfies the same assumptions as in the separable class but \( \eta \) is only assumed positive and strictly increasing. The following result provides a simple and permissive class of primitives under which more workers invest with an IR shift in \( L \).

**Proposition 5 (RS: IR Shift in \( L \))** In any stable equilibrium in RS, more workers invest in skills in response to an IR shift in \( L \) if either of the following conditions hold:

(i) Absolute risk aversion is (uniformly) sufficiently small and \( f_x \alpha \alpha \geq 0 \) with strict inequality on a set of \((x,y,\alpha)\) of positive measure; or

(ii) Match output function \( f \) is a weak separable class and either \( \eta \) is convex and \( R(w) \leq 1/w \) for all \( w \), or \( \eta \) concave and \( R(w) \geq 1/w \) for all \( w \).

Note that the condition on the risk attitude is less stringent in this example compared to part (ii) of Example 2 since we did not use \( w_x \alpha \alpha \geq 0 \) to ‘help’ with relaxing the bound on prudence.
The proof is in Appendix A.8. Part (i) is analogous to that of Proposition 4 in NRS. For (ii), we need to find conditions under which the integrand in (5), \( \varphi_x = u'(\bar{w})\bar{w}_x \), is increasing in \( t \). Its derivative is

\[
 u''(\bar{w})\bar{w}_t\bar{w}_x + u'(\bar{w})\bar{w}_{xt},
\]

which rearranges to

\[
 u'(\bar{w})\bar{w}_t\bar{w}_x \left( \left( \frac{\bar{w}_{xt}\bar{w}}{\bar{w}_t\bar{w}_x} \right) \frac{1}{\bar{w}} - R(\bar{w}) \right).
\]

Note that the wage function is \( \bar{w} = \int_0^x z_x ds \int \eta dL \) and hence \( \bar{w}_{xt}\bar{w}/\bar{w}_t\bar{w}_x = 1 \). It follows that if \( \eta \) is convex in \( \alpha \), then \( \bar{w}_t \geq 0 \) (income is increasing in risk for all workers, independently of their skill), fueling an income effect that discourages investment. The positive substitution effect stemming from a complementarity of skills and risk, \( \bar{w}_{xt} \geq 0 \), prevails if and only if risk aversion \( R \) is bounded by \( 1/w \) for all values of \( w \). If \( \eta \) is concave, however, then \( \bar{w}_t \leq 0 \) (and also \( \bar{w}_{xt} \leq 0 \)), and so income and substitution effects switch signs: Sufficient risk aversion is now needed for the income effect to prevail and induce more workers to invest.

### 4.2 Changes in Controllable Risk, Initial Heterogeneity and Technology

Equilibrium investment changes with the underlying heterogeneity, that is, with a better distribution of firm productivity (for example, due to increased competitive pressure) or of workers’ ability (for example, due to improved early childhood investment). These changes can be captured by a FOSD shift in firm distribution \( G \) or ability distribution \( Q \). It also changes when the quality of education improves or when dropout risk declines, which can be captured by a FOSD shift in \( H_1 \) (which is the controllable idiosyncratic risk in the model). We will focus on these FOSD shifts here and, for completeness, present the results for IR shifts in Online Appendix B.6.

In the case of the shifts in background risk \( L \), we argued that the natural comparative statics result to seek was that under both shifts a larger measure of workers invests, either to take advantage of stochastically better risk or to use investment as an insurance against the increased risk. In the present case, we contend that the natural comparative statics results are that more workers invest if there is a FOSD shift in \( G \) but fewer invest if there is a FOSD in \( Q \). To see this, note that such a shift in \( G \) improves the matching of all workers since there is a better pool of firms available. Hence, wages increase for workers of all skill levels, but — due to worker-firm complementarities in production — this effect is more pronounced for those with higher skills. So if risk aversion is not too large, then the positive substitution effect dominates the negative income effect (workers earn more even if they do not invest) and more workers invest. The opposite holds with a FOSD shift in \( Q \). Under this improved ability distribution, for any given investment threshold \( \theta^* \), the mass of workers who invest is larger. This reduces the quality of the match for all skill levels, depressing wages, with a more pronounced decrease for those of higher skills. This negative substitution effect prevails if workers are not too risk averse, reducing investment.

The problem is much more complex if there is a FOSD shift in skill risk \( H_1 \). The direct effect
is to incentivize investment since the distribution of skills of those who invest improves relative to those that do not. But there are also equilibrium effects: The matching of workers of all skill levels worsens, depressing wages (leading to an income effect that incentivizes investment), with a more pronounced decrease for those with higher skills (leading to a substitution effect that reduces the incentives to invest). The negative equilibrium effect is tamed if risk aversion is sufficiently large, inducing the direct effect to dominate and thus more workers invest.

The next result confirms these intuitive explanations.

**Proposition 6 (NRS and RS: FOSD G, Q, H₁)** In any stable equilibrium of NRS or RS:

(i) More workers invest in skills in response to a FOSD shift in G if absolute risk aversion is (uniformly) sufficiently small;

(ii) Fewer workers invest in skills in response to a FOSD shift in Q if absolute risk aversion is (uniformly) sufficiently small;

(iii) More workers invest in skills in response to a FOSD shift in H₁ if \( \partial h₁/\partial t < 0 \) at \( x = 0 \), \( wR(w) \) is sufficiently large for all \( w \), and \( f_x(0, 0, \alpha) > 0 \) for all \( \alpha \in [0, 1] \).

The proof is in Appendix A.9 and parts (i)–(ii) are similar to the proof of Proposition 3 (i).25 The proof of part (iii) formalizes the intuitive idea explained above.26 There, we also show that the condition on \( f_x \) can be dispensed with under an alternative premise.

Finally, consider the effect of technological change on investment. Assume that \( t \) is a parameter that shifts upwards both the match output function \( f \) and its derivative \( f_x \). We show in Proposition 9, Appendix A.10, that both in NRS and RS, a technology shift of this kind increases investment if \( f_x \) is log-spm and workers’ risk aversion is small enough \( R(w) \leq 1/w \) for all values of \( w \). Intuitively, an upward-shift in \( f_x \) increases the wage for all skills, and particularly so for high skills given the complementarities in \( f \). So this shift leads to more incentives to invest if risk aversion does not outweigh the resulting complementarities of wages in skill and technical change.

### 4.3 Effects on Equilibrium Matching and Wages

Changes in risk (\( L, H₁ \)), heterogeneity (\( G, Q \)), and technology (\( f \)) also affect equilibrium matching and wages. We will assume that the appropriate sufficient conditions in the propositions above hold. So threshold \( \theta^* \) decreases (more investment) when considering changes in \( L, G, H₁ \), or \( f \) and \( \theta^* \) increases (less investment) when we consider changing \( Q \). More investment improves the workers’ skill distribution \( H \) and lowers the matching function, since a worker with given \( x \) matches with a worse firm \( y \) due to tougher competition for top firms. A lower matching function then implies a lower wage for workers of all skills. The opposite logic applies when investment

---

25There we also provide other classes of primitives that deliver the same comparative statics.

26The only role that the mild regularity conditions on \( f \) and \( h₁ \) play is to ensure that enough risk aversion can overtake the complementarities effect even near \( (x, \alpha) = (0, 0) \).
falls. These indirect effects through investment must be weighed against the direct effect of the shift, and the two effects can go in opposite directions.

**Effects on the Matching Function.** Consider changes in risk. Under both FOSD and IR shifts in the distribution of background risk, $L$, the effect on $\mu$ is unambiguous when our comparative statics result holds: An increase in $t$ reduces $\theta^*$ and thus the derivative of matching function $\mu$ with respect to $t$ is $\mu_t = (H_0 - H_1)q\theta_t^*/g \leq 0$. This is due to the improvement in the endogenous skill distribution $H = (1 - Q(\theta^*))H_1 + Q(\theta^*)H_0$ as more workers invest, reinforcing the competition for firms with better $y$. As a result, any worker with given skill $x$ is matched to a firm with lower productivity $y$. Similarly, if we consider the FOSD shift in skill risk $H_1$, then both the change in $H_1$ (improvement in exogenous skill distribution) and in $\theta^*$ (improvement in endogenous skill distribution) reinforce each other and the total derivative of $\mu$ with respect to $t$ is $\mu_t + \mu_\theta\theta_t^* = (q\theta_t^*(H_0 - H_1) - G_t)/g$, which is negative. Hence, more workers investing implies worse matching outcomes for each worker of given skill $x$.

If we consider technological change, a similar logic applies, where the matching function decreases in response to this shift in $f_x$, meaning any given worker type has a worse match.

Instead, if we consider changes in heterogeneity, the effect on matching can be ambiguous due to opposing direct and equilibrium effects: If there is a FOSD shift in firm productivity $G$, once we take into account the change in $\theta^*$, the derivative of $\mu$ with respect to $t$ is $\mu_t + \mu_\theta\theta_t^* = (q\theta_t^*(H_0 - H_1) - G_t)/g$, which is ambiguous. This is intuitive since on the one hand, the firm distribution directly improves, $G_t \leq 0$, leading to a better match for each skill $x$; but on the other hand, workers’ endogenous skill distribution improves with more investment, $\theta_t^* \leq 0$, and this worsens the match for each $x$. In turn, the effect of a FOSD shift of ability distribution $Q$ on matching is again unambiguous. The total derivative of $\mu$ with respect to $t$ is $\mu_t + \mu_\theta\theta_t^* = 1/g(Q_t(H_0 - H_1) + q\theta_t^*(H_0 - H_1))$, which — after replacing the expression for $\theta_t^*$ that obtains by totally differentiating $U_1(\theta^*, t) - U_0(\theta^*, t) = c(\theta^*)$ — is negative. Thus, an improvement in ability deteriorates the matching from the workers’ point of view.

**Effects on the Wage Function.** As is typical for assignment models, changes in the matching function readily translate into changes of the wage function. Consider a FOSD or an IR shift in shock distribution $L$. In NRS, for each $(x, \alpha)$ the derivative of the wage with respect to $t$ is $w_0\theta_t^* = \int_0^x f_{xy}\mu_\theta\theta_t^* \leq 0$. Since more workers invest, the distribution of their attributes improves, deteriorating the matching and thus decreasing the wage for each $(x, \alpha)$. In RS, the effect on wage $\bar{w}$ is ambiguous, since the direct effect of $L$ can increase $\bar{w}$, while the indirect one via $\theta_t^*$ decreases it.

Under a FOSD shift in $G$, $H_1$, or $Q$, and under NRS, the change in $w$ is given by $\int_0^x f_{xy}(\mu_t + \mu_\theta\theta_t^*)$. The term in parentheses is negative with a FOSD shift in $H_1$ or $Q$, meaning wages unambiguously decrease for all $(x, \alpha)$ if skill risk or ability improves. But wages are ambiguous in response to a FOSD shift in firm heterogeneity $G$. Analogous results hold for $\bar{w}$ in RS.

Finally, the shift of technology $f$ has ambiguous effects on wages as the direct positive effect
(through increase in $f_x$) must be weighed against a worse matching, in both NRS and RS.

4.4 Equilibrium Investment With and Without Risk-Sharing

Finally, we are interested in how risk-sharing opportunities affect investment. Do more workers invest in NRS than in RS? Intuition suggests that, without risk-sharing, wages are more spread than in the risk-sharing case, and workers can use (more) pre-match investment to cope with the higher uncertainty. But, complementarities in the wage function make the result ambiguous without further assumptions.

Fix $\theta^*$; then the incentives to invest are higher in NRS than in RS if and only if

$$\int \int u(w)dLdH_1 - \int \int u(w)dLdH_0 \geq \int u(\bar{w})dH_1 - \int u(\bar{w})dH_0,$$

which is equivalent to

$$\int \left( u(\bar{w}) - \int u(w)dL \right) dH_0 \geq \int \left( u(\bar{w}) - \int u(w)dL \right) dH_1.$$

Since $H_1$ FOSD $H_0$, this holds if $u(\bar{w}) - \int u(w)dL$ decreases in $x$, or $u'(\bar{w})\bar{w}_x - \int u'(w)w_xdL \leq 0$.

We then have the following clear-cut comparison of the investment thresholds:

**Proposition 7 (Investment in NRS and RS)** If $f$ is a separable class and $P(w) \geq 2/w$, then in any stable equilibrium more workers invest in NRS than in RS.

On an intuitive level, the result shows that the higher $x$ is, the smaller is the difference $u(\bar{w}) - \int u(w)dL$, and thus high-skilled workers value full insurance over no insurance less than lower-skilled ones. Since drawing a higher $x$ is more likely when the worker invests, it follows that under the conditions in the proposition, the incentives to invest are higher without risk-sharing (in NRS). Intuitively, more spread out wages in NRS than in RS induce sufficiently prudent workers to make precautionary investments. The lower bound on prudence is easily satisfied: If $u$ is CRRA, then $P(w) = (1 + \sigma)/w$, and so $P(w) \geq 2/w$ if $\sigma \geq 1$.

5 Robustness

Our analysis relies on several assumptions that allow us to focus, in a tractable way, on pre-match investments with stochastic returns and equilibrium comparative statics. We now discuss how we can relax several of them. The technical details are in the Online Appendix B.7–B.10.

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27If the wage was additively separable in $x$ and $\alpha$, which is not typically the case in our model, then the inequality would readily satisfied by Jensen’s inequality, since it reduces to $u'(\bar{w}) - \int u'(w)dL \leq 0$, and $u'$ is convex because $u'' \geq 0$. Note also that we can think of this exercise as a comparative statics result where we index by $t \in \{0, 1\}$ the availability of risk-sharing opportunities, which affects $U_1 - U_0$. 
As is standard, we assume risk-neutral firms. But, since our model is amenable to other interpretations, for completeness we explore the case with two-sided risk aversion. Intuitively, nothing changes in NRS since at the matching stage all the relevant uncertainty has been resolved. In RS, however, efficient risk-sharing now calls for each party to bear some risk, so firms’ risk aversion enters the wage function. Although the general analysis is a challenging open problem beyond the scope of this paper, we provide a complete solution when utility is CRRA, and show that most insights go through, providing a suggestive direction for future research on this topic.

We further assume that either workers are heterogeneous ex-ante but investment is binary, or that they are homogeneous ex-ante and investment can take values on an interval. We also study a version of the model with both ex-ante heterogeneous workers and continuous investments, where investment determines the probability that a worker obtains his skill from the better distribution. This allows us to maintain the convenient features discussed in Remark 3. Equilibrium existence follows from available results in large games, and comparative statics results extend for the average investment level in the smallest and largest equilibria. But if both sides are symmetric (or, equivalently, there is just one population, as in partnership models), then all the results extend.

Finally, we assume that only workers make pre-match investments. If firms could also invest, then — when risk aversion is small — a stable equilibrium exists and the results on changes in $L$ hold essentially as before. But those that operate via matching and wages are ambiguous since they have opposite effects on each side of the market.

6 Economic Relevance

Distinguishing between cross-sectional heterogeneity and labor market risk is crucial for pinning down the determinants of life-time earnings and consumption inequality (Keane and Wolpin [1997], Storesletten, Telmer, and Yaron [2004], Huggett, Ventura, and Yaron [2011]), and for analyzing changes in earnings inequality over time (Cunha and Heckman [2016]).

As evidence of the economic relevance of our model, we now illustrate how it can help understand the sources of rising wage inequality in the U.S. labor market. We focus on two questions. First, through the lens of our model with endogenous educational investment, how much of the increase in U.S. wage inequality is due to changes in risk, heterogeneity of workers and jobs, or technology? Second, how would our conclusions change if we considered education as exogenous?

To this end, we estimate our stylized model, fitting it to U.S. data from an early period (1979-1981) and a later period (2015-2017), which allows us to perform a decomposition of rising inequality into its driving forces. We choose these two points in time since the significant surge of U.S. wage inequality started at the beginning of the 80s (Autor, Katz, and Kearney [2008]).

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28We pool three years in each period to increase the number of observations. We conducted robustness checks regarding which years and the number of years to pool. Our results are robust to different choices.
highlight the importance of accounting for endogenous education, we then contrast our conclusions to those from two alternative approaches that consider education as exogenous — a statistical decomposition of inequality and a structural one based on a model with exogenous investment.

6.1 Estimation

DATA, PARAMETERIZATION AND IDENTIFICATION. Our main dataset is the annual Current Population Survey (CPS). We focus on the years 1979-1981 (‘early period’, before surge in wage inequality) and 2015-2017 (‘later period’, after surge in wage inequality) and focus on full-time, full-year workers in their prime working age 25-50. We use the CPS to compute the wage distributions, wage inequality measures, and how they changed over time. We also assess the college graduation share in our sample where we consider ‘college education’ as the empirical counterpart to our model’s investment. We supplement the CPS by three other datasets. To obtain a measure of productivity of occupations (our empirical counterpart of the model’s job/firm productivity) and hence a proxy for the productivity distribution $G$, we use the DOT (early period) and its follow-up database, O*NET (later period). These datasets give detailed information about occupational skill requirements, where we choose in both periods ‘math skills’ as our proxy for (cognitive) productivity. This choice allows us to ensure comparability of the $G$ distribution across periods. In turn, to measure the ability distribution $Q$ in the data, we use the AFQT test scores assessed in the National Longitudinal Survey of Youth (NLSY) for our early period (NLSY79) and later period (NLSY97). See Appendix C.1 and C.2 for details on data and summary statistics.

We consider NRS and parameterize our model as follows. For utility, we assume CRRA, and for the utility cost $c$ we assume $c(\theta) = \lambda((1/\theta) - 1)$ for all $\theta$, with $\lambda > 0$. The skill distributions are $H_0(x) = x$ and $H_1(x) = x^b$ for all $x$, with $b > 1$ to ensure the strict FOSD order. The match output function $f$ is given by $f(x, y) = qx^{\gamma_1}y^{\gamma_2} + K$ for all $x$ and $y$, where $(q, \gamma_1, \gamma_2, K)$ are TFP, output elasticity parameters, and a constant, respectively. Finally, we interpret the shock $\alpha$ as an idiosyncratic income shock (see Remark 2 and Online Appendix B.4) that is normally distributed with mean $\bar{\alpha}$ and variance $\tau^2$. We consider a multiplicative functional form for the shock, making this specification similar to our separable class $f$ used before, and denote wage income by $i(x, \alpha, \theta^*) \equiv \eta(\alpha)w(x, \theta^*) = e^\alpha w(x, \theta^*)$, where $w$ is the endogenous wage function.

Our model is static while educational decisions in the data are dynamic in the sense that current educational costs are weighed against a future stream of benefits when making the education choice. To bridge data and model, we assume for this exercise that workers are infinitely lived, discount the future with discount factor $\beta$ and obtain an income payment in each period.

We set $\sigma = 1.1$ in the utility function, which is within the standard range the macro literature considers (Chetty 2006, Table 1), normalize $\bar{\alpha} = 0$ (for identification purposes), and set $\beta$ to be a standard monthly discount factor. We then need to identify and estimate the following seven parameters $(\lambda, b, q, \gamma_1, \gamma_2, K, \tau)$. Since we want to keep utility parameters fixed across periods, we
estimate the cost parameter $\lambda$ only in the early period and keep it fixed at that value for the later period. Therefore, we estimate seven parameters in the early and six in the later period.

We provide a formal identification argument in Appendix C.3.2 which informs the moments chosen to pin down the parameters. The first set of moments relates to the monthly wage income distribution (mean, variance, wage percentiles, and within and between worker-type wage variance) and is meant to identify the match output function and the variance of the income shock. The next moment, the skill premium, is meant to identify the skill risk parameter of the skill distribution of the educated. Finally, the college graduation share aims to identify the investment utility cost. See Appendix C.3.3 Table 9 for the detailed list of moments and how they are constructed. We estimate the model by simulated methods of moments, see Appendix C.3.4.

**Parameter Estimates and Fit.** We estimate our model both in the early period (1979-1981) and later period (2015-2017), and summarize the parameter estimates in Table 1. Comparing the estimates across periods (where we call the early period ‘1980’ and the later period ‘2015’ in tables), there are several noteworthy changes. First, there is a sharp increase in parameter $b$ pertaining to the skill risk of educated workers. Second, there is an increase in $\tau$, the income risk. Third, there are notable changes in $f$. It became more elastic with respect to both skill $x$ and productivity $y$. Finally, TFP $q$ increased substantially. We analyze the model fit in Appendix C.4.

<table>
<thead>
<tr>
<th>Parameter Description</th>
<th>Estimate 1980</th>
<th>Estimate 2015</th>
</tr>
</thead>
<tbody>
<tr>
<td>Utility Cost Parameter</td>
<td>$\lambda$ 186.53</td>
<td>186.53 (fixed)</td>
</tr>
<tr>
<td>Skill Risk</td>
<td>$b$ 8.57</td>
<td>16.14</td>
</tr>
<tr>
<td>Standard Deviation of Shock</td>
<td>$\tau$ 0.49</td>
<td>0.59</td>
</tr>
<tr>
<td>TFP</td>
<td>$q$ 6001.10</td>
<td>10433.12</td>
</tr>
<tr>
<td>Elasticity of $f$ in $x$</td>
<td>$\gamma_1$ 0.31</td>
<td>0.36</td>
</tr>
<tr>
<td>Elasticity of $f$ in $y$</td>
<td>$\gamma_2$ 0.28</td>
<td>0.70</td>
</tr>
<tr>
<td>Constant in $f$</td>
<td>$K$ 454.29</td>
<td>698.78</td>
</tr>
</tbody>
</table>

**Changes in Risk, Heterogeneity and Technology.** Parameters $b$ and $\tau$ impact risk in our economy, the skill risk $H_1$ and the income risk $L$. Moreover, distributions $G$ and $Q$ (which we measure directly in the data in both periods) reflect the heterogeneity of jobs and workers.

Figure 3 shows the changes in risk across periods. Income risk has increased (the estimated increase in $\tau$ induced an IR shift in $L$), left panel, while skill risk has improved, right panel (the increase in $b$ induced a FOSD shift of $H_1$ and thus a larger difference between $H_0$ and $H_1$).

Figure 4 shows that heterogeneity also changed significantly, with the ability distribution be-

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29In our model, within/between-worker $x$-type variance is the same as within/between-job $y$-type variance since there is pure matching in $(x,y)$. In the data, we compute this variance decomposition using job types.

30It shows that the model achieves a decent fit despite its parsimony, see Tables 11 and 12 and Figure 8.
coming better in the FOSD sense (to a first-order approximation), left panel, and the occupational productivity distribution displaying an IR shift, indicating more dispersion, right panel. Finally, Figure 9, Appendix C.4 shows a significant upward shift and steepening of technology.

![Figure 3: Changes in Risk: Income Risk (left) and Skill Risk (right)](image)

![Figure 4: Changes in Heterogeneity: Ability (left) and Occupation Productivity (right)](image)

### 6.2 Wage Inequality Changes Through the Lens of our Model

**WAGE INEQUALITY CHANGES IN THE DATA.** U.S. wage inequality significantly increased over the last decades. Figure 7 in Appendix C.2 suggests an IR shift in the wage distribution comparing the blue distribution (early period) with the red distribution (later period), indicative of growing wage dispersion. Table 6 in Appendix C.2 confirms this development: The standard deviation of the wage distribution increased by 18%, the 75/25 wage ratio by 13%, and the skill premium by 20% (defined as the ratio of wages of college-educated workers and non-college educated workers).

31To see the IR shift in $G$, note that the cdf in the early period (blue) is below the one from the later period (red) near zero and above it near one. Moreover, we checked that, as IR requires, $\int_0^y G_1(\tilde{y}|t)d\tilde{y} \geq 0$ for all $y$, which is true up to a numerical error. The IR shift is accompanied by a slight FOSD as well.
Wage Inequality Changes in Our Model. Our estimated model naturally lends itself to a decomposition of the observed change in inequality into the underlying primitives. We focus here on the evolution of the 75/25 wage income ratio (as both of these percentiles are targeted in our estimation) but we note that our conclusions are robust to considering other measures of Table 6.

We are interested in how much of the 13% increase of the 75/25 wage ratio is due to the change in each of our primitives. We use various counterfactual experiments to decompose this change into the shares driven by changes in risk, heterogeneity, and technology. For instance, to study how much of the change in wage inequality is due to changes in income risk alone, we keep all parameters at their 1980-level and only feed the estimated change of L into the model (given by the change in τ); we proceed similarly to isolate the role of changes in heterogeneity and technology.

Table 2 presents our main decomposition. The principal driver of the rise in inequality is increased income risk, followed by changes in technology. Had only income risk L increased, we would have seen an increase in the 75/25 wage ratio by 10%. Had only technology f changed, inequality would have risen by 4%. In turn, improvements of skill risk H and changing heterogeneity of jobs G or workers Q alone would have had little impact on inequality.

A recurring theme in the literature is that the increase of the skill premium or ‘returns to skill’ played a dominant role in the rise of overall wage inequality. Table 13 in Appendix C.4 shows that indeed the skill premium increased by 18% in our model (compared to the 20% in the data, see above). This increase is exclusively driven by an improvement in skill risk (FOSD shift in H), increased dispersion of job productivity (IR shift in G), and a change in technology (change in f). The technology shift alone accounts for around 60% of the total change in skill premium while changes in G and H account for a bit less than 20%. Had only the skill premium changed (that is, had H, G, f changed jointly while all other primitives remained at their 1980-level), wage inequality would have increased by around 7%, see last row of Table 2.

Importantly, especially for what we will show below, the same forces underlying the change in the skill premium (IR shift in G, FOSD shift in H and technological change in f) also explain most of the increase in the college graduation share, captured by an increase in 1 − Q(θ∗) (see Table 14 in Appendix C.4). Educational investment in our model increased by more than 50% (as in the data), of which almost 80% is accounted for by changing G, H and f to their 2015 level (where f was the main driver, followed by H and G). In turn, had only ability Q improved, educational investment would have decreased. All these effects of primitives on investment go in the same direction as the natural comparative statics results in our theory (Propositions 6(ii)–(iii) and 9, and Proposition 10(i) in Online Appendix B.6). Finally, changes in income risk L

---

32 An important role of idiosyncratic income risk in rising U.S. wage inequality is in line with the findings by Cunha and Heckman (2016). But in contrast to their analysis, here workers are risk averse and investment affects wages.

33 One can show either analytically or numerically in our estimated model that our sufficient conditions for the natural comparative statics from Proposition 6(ii) and (iii), and Proposition 9 are satisfied here.
had a quantitatively negligible impact on investment when fed into the model in isolation.\footnote{Based on Proposition 4), one can show that the effect of a change in \( \tau \) in our estimated model is essentially zero.}

While the increase in wage inequality has received much attention in the literature, we believe that our results on the underlying sources of risk, heterogeneity and technology through the lens of an equilibrium model are novel. Moreover, existing decompositions of inequality tend to neglect the \textit{endogenous} education response to the shifts in primitives, which we will show is important.

<table>
<thead>
<tr>
<th>Table 2: Decomposition: Wage Inequality</th>
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<tbody>
<tr>
<td></td>
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<tr>
<td>( w_{75}/w_{25} )</td>
</tr>
<tr>
<td>---------------------</td>
</tr>
<tr>
<td>1980</td>
</tr>
<tr>
<td>2015</td>
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<tr>
<td>( G )</td>
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<tr>
<td>( H_1 )</td>
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<tr>
<td>( Q )</td>
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<tr>
<td>( L )</td>
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<tr>
<td>( f )</td>
</tr>
<tr>
<td>( G, H_1, f ) (Skill Premium)</td>
</tr>
</tbody>
</table>

6.3 Wage Inequality Changes with Exogenous Investment

\textbf{Statistical Decomposition.} A common reduced-form approach to analyze inequality changes is a statistical decomposition that assumes that the skill distribution is \textit{exogenous}. Eika, Mogstad, and Zafar (2018) use this tool to show that the increase in U.S. household income inequality is mostly due to an increase in the returns to skill rather than an increase in positive sorting in marriage. This is an important insight but is based on a decomposition that accounts neither for the primitives behind changes in skill returns nor for their effects on educational investment.

To understand how our conclusions change under this approach, we statistically decompose the change in wage inequality into (i) the change in wages conditional on education (capturing the change in returns to skill/skill premium) and (ii) the change in the education distribution (taken as exogenous and independent of changes in returns to skill (i)). We are interested in how wage inequality would have changed if only (i) or (ii) had changed (see Appendix C.5.1 for details).

The results are in Table 3, columns 2 and 3. The first two rows indicate again how wage inequality has changed in the data (increase of 13\%). If only wages and thus the returns to skill/skill premium had changed (conditional on education), then wage inequality would have increased by 17\%, third row, implying an overshoot. In turn, if only education had changed, then wage inequality according to the 75/25 income ratio would have decreased by 2\%, fourth row. These results qualitatively resemble those by Eika, Mogstad, and Zafar (2018) for household income inequality changes from 1962–2013 (especially regarding the importance of changing returns to skill).
Compared to our model decomposition, this statistical decomposition is silent about the primitives that led to changes in both the skill premium and the educational distribution, and ultimately to changes in wage inequality. It also misses that, apart from changes in the wage structure/skill premium (which according to our model were induced by changes in $f$, $G$ and $H_1$), a crucial driver of inequality has been the rise in income risk, something we found above. Maybe most importantly, we showed in our model that changes in education and changes in the wage structure (skill premium) are driven by the same primitives $f$, $G$ and $H_1$. The ceteris paribus exercise of the statistical decomposition (that is, keeping education fixed while varying the wage structure or vice versa) overlooks this point. As a result, the statistical decomposition overstates the impact of the skill premium on inequality changes. To better understand why this is the case, we now turn to a version of our model with exogenous education.

<table>
<thead>
<tr>
<th>Table 3: Decomposition: Wage Inequality in Counterfactuals</th>
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</thead>
<tbody>
<tr>
<td><strong>Statistical Decomposition</strong></td>
</tr>
<tr>
<td>$w_{75}/w_{25}$</td>
</tr>
<tr>
<td>1980</td>
</tr>
<tr>
<td>2015</td>
</tr>
<tr>
<td>wages</td>
</tr>
<tr>
<td>education</td>
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</table>

**Model with Exogenous Investment.** We re-estimate our model in early and later period with one change: We set educational investment exogenously and in line with the data. This means that $\theta^*$ and thus the skill distribution still differs in both periods but investment is no longer a choice.

The new estimates are in Table 15 in Appendix C.5.2 which reveals a noteworthy change: The estimate of $b$ is considerably higher in the later period under exogenous investment, indicating a much more pronounced FOSD shift in skill risk. This change occurs because in this version, the parameters no longer have to incentivize the ‘right’ growth in educational investment. Recall that $H_1$ was a crucial driver of the change in educational investment with endogenous $\theta^*$ (see Table 14).

We repeat the exercise of decomposing wage inequality shifts into changes of the different primitives. The results are in Table 19, Appendix C.5.2. Compared to our baseline model, this naive model overstates the impact of the change in wage structure (skill premium) on inequality by almost 60% which, as before, is driven by changes in $f$, $G$, and $H_1$ (see Table 18, Appendix C.5.2). As a result, the skill premium becomes the most important driver of wage inequality, overtaking income risk, whose importance has diminished compared to the model with endogenous investment. 

\footnote{In our full model decomposition the skill premium change alone would have increased the 75/25 wage ratio by approximately 7% (Table 2, last row), and by 11% in the model with exogenous education (Table 19, second last row), over-estimating the effect of the skill premium on wage inequality by almost 60%.
By not taking into account that skill premium and educational investment are driven by the same underlying forces \((G, H_1, \text{and } f)\) and that they push wage inequality in opposite directions (compare the last row with the second last row of Table [19]), this exercise shows that the ceteris paribus view of holding education fixed while varying the wage structure is misleading. Indeed, it overstates the impact of the wage structure on inequality and understates the importance of income risk changes. This explains why we find similar results in the statistical decomposition since one could interpret such a decomposition as being based on a model with exogenous educational investment (compare columns 3 and 5, Table 3). According to both approaches, educational changes put downward pressure on wage inequality (2% decrease), while changes in the skill premium conditional on education would have increased wage inequality by 11-17%.

We conclude that the endogenous supply response is crucial for studying the sources of wage inequality. If investment \(\theta^*\) was fixed, then the estimated change in the skill premium (more specifically, its underlying primitives) would have an unattenuated positive effect on wage inequality. But, if as indicated by our estimated model the rise in skill premium goes hand in hand with an endogenous increase in investment, then the direct effect of the skill premium is dampened by equilibrium effects: They put downward pressure on wages for all skills, but most strongly for educated workers, pushing wage inequality down. Hence, caution is warranted when decomposing inequality changes under the assumption of exogenous education, as that approach neglects important equilibrium effects. Moreover, pinning down the underlying cause for the rise in inequality in terms of primitives (for example, shifts in \(G, H_1, Q, L, \text{or } f\)) is important for designing effective redistributive policies. Depending on the main driving force, policy interventions should be directed at providing insurance against initial conditions (worker heterogeneity), technological change (either in form of a changing technology or of changes in firm heterogeneity) or shocks (risk). The results of our application provide some guidance into this issue.

7 Concluding Remarks

This paper develops a matching model with heterogeneous workers and firms, with the key feature of pre-match investment under stochastic returns. We analyze two versions that depend on the timing of when uncertainty is resolved, which we call the no-risk-sharing and the risk-sharing cases.

We first analyze equilibrium existence, uniqueness, and efficiency. We then provide a thorough and novel analysis of the equilibrium comparative statics of the model. In particular, we ask how changes in risk, heterogeneity and technology affect investment, matching, and wages. Underlying our results is a fundamental trade-off between complementarities in the equilibrium wage function and workers’ risk attitudes, which provides a clear economic intuition for all of our results. We derive weak conditions on risk preferences and on match output under which the measure of workers who invest changes in what we assert is the intuitive direction. We also show how this
change in investment translates into matching and wages.

To illustrate the relevance of our theory for applied work, we use our model for a stylized quantitative assessment of the rise in U.S. wage inequality. Our estimates reveal pronounced shifts in risk, heterogeneity as well as technology over the past few decades. Guided by our comparative statics results, we decompose the rise in wage inequality into the changes of these primitives. We find that changes in income risk have been the main driver of the surge in inequality. Importantly, we show that alternative approaches, which treat education as exogenous, arrive at different conclusions regarding the main driver of inequality. We argue that our model with endogenous educational investment — the central feature of our framework — leads to a more reliable decomposition of the rise in U.S. wage inequality.

A Appendix

A.1 Proof of Walrasian Equilibrium

Consider NRS. We will show that, given an investment function \(a\) and a shock realization \(\alpha\), the pair \((w(\cdot, \alpha, a), \mu(\cdot, a))\) is a unique Walrasian equilibrium of the labor market, where \(\mu(x, a) = G^{-1}(H(x, a))\) for all \(x, w(x, \alpha, a) = \int_0^x f_x(s, \mu(s, a), \alpha)ds\) for all \(x\). That is, the market clears, and agents behave optimally. By construction, \(\mu(\cdot, a)\) clears the market.

Consider a firm with attribute \(y\). It solves

\[
\max_{x \in [0,1]} \left( f(x, y, \alpha) - \int_0^x f_x(s, \mu(s, a), \alpha)ds \right),
\]

and from the first-order condition (FOC) we obtain \(f_x(x, y, \alpha) = f_x(x, \mu(x, a), \alpha)\) and hence \(y = \mu(x, a)\) or \(x = \mu^{-1}(y, a)\). To show that this is a global optimum, note that for any \(x' < x\)

\[
f(x, y, \alpha) - \int_0^x f_x(s, \mu(s, a), \alpha)ds \geq f(x', y, \alpha) - \int_0^{x'} f_x(s, \mu(s, a), \alpha)ds
\]

if and only if

\[
f(x, y, \alpha) - f(x', y, \alpha) \geq \int_x^{x'} f_x(s, \mu(s, a), \alpha)ds.
\]

Since \(y = \mu(x, a)\) and since \(f(x, y, \alpha) - f(x', y, \alpha) = \int_x^{x'} f_x(s, y, \alpha)ds\), it follows that (8) is equivalent to \(\int_x^{x'} (f_x(s, \mu(x, a), \alpha) - f_x(s, \mu(s, a), \alpha))ds \geq 0\), and this holds from the supermodularity of \(f\) and from \(\mu(x, a) \geq \mu(s, a)\) for all \(s \in [x', x]\). Hence, firm \(y\) prefers \(x\) over \(x' < x\). A similar argument holds for \(x' > x\), and hence choosing \(x\) is a global optimum for \(y\).

In turn, a worker with attribute \(x\) obtains \(u(w(x, \alpha, a)) \geq u(0)\). Let \(\pi(y, \alpha, a)\) be the profit of a firm with attribute \(y\). If \(y\) is matched with \(x\), we have \(\pi(y, \alpha, a) = f(\mu^{-1}(y, a), y, \alpha) - w(\mu^{-1}(y, a), \alpha, a)\). By the Envelope Theorem, we can rewrite it as \(\pi(y, \alpha, a) = \int_y^y f_y(\mu^{-1}(s, a), s, \alpha)ds\),
where we omitted $\pi(0, \alpha, a)$ since it is zero (recall that $f(0, 0, \alpha) = 0$).

The above analysis implies that workers behave optimally too, that is, each worker with skill $x$ maximizes his payoff at $y = \mu(x, a)$. Such $x$ solves

$$\max_{y \in [0, 1]} u \left( f(x, y, \alpha) - \int_0^y f_y(\mu^{-1}(s, a), s, \alpha) ds \right).$$

Since the objective function is a strictly increasing transformation of $f(x, y, \alpha) - \pi(y, \alpha, a)$, the FOC holds if and only if $f_y(x, y, \alpha) = f_y(\mu^{-1}(y, a), y, \alpha)$, so $x = \mu^{-1}(y, a)$ or $y = \mu(x, a)$. The proof of global optimality of this choice is analogous to the firms’ case above and is omitted. Note that both $w(\cdot, \alpha, a) \geq 0$ and $\pi(\cdot, \alpha, a) \geq 0$, and thus no agent prefers to remain unmatched.

We have shown uniqueness of the Walrasian equilibrium when $w$ is differentiable. But dispensing with differentiability does not alter this result. Under the stated assumptions, there exists a unique optimal assignment $\mu$, which is deterministic, which follows from Theorem 10.28 in Villani (2009). Regarding $w$, note that it must be strictly increasing to ensure that if $x > x'$ then $x$ chooses a strictly higher $y$ than $x'$, so $w$ is almost-everywhere differentiable. Also, $w$ is continuous since any jumps would lead to some profitable deviation by some $x$ near the jump. Indeed, the existence of a unique wage function (up to an additive constant, which in our case is zero) follows from Theorem 10.28 and Remark 10.30 in Villani (2009), and it is the one we have pinned down.

Finally, let us show that the outcome of the Walrasian equilibrium is stable in the sense of Remark 3. For each $a$ and $\alpha$, by construction $w(x, \alpha, a) + \pi(y, \alpha, a) = f(x, y, \alpha)$ if a worker with $x$ is matched to a firm with $y$, or $y = \mu(x, a)$.

Assume that $y$ is not matched with $x$ in equilibrium, so that $y \neq \mu(x, a)$. Let $x'$ be the skill of the worker matched in equilibrium with $y$, or $x' = \mu^{-1}(y)$. If $x' < x$ (so that $(x, y)$ is below the graph of $\mu(\cdot, a)$), then

$$w(x, \alpha, a) + \pi(y, \alpha, a) = w(x, \alpha, a) + f(x', y, \alpha) - w(x', \alpha, a)$$

$$= f(x', y, \alpha) + \int_{x'}^x f_x(s, \mu(s, a), \alpha) ds$$

$$> f(x', y, \alpha) + \int_{x'}^x f_x(s, \mu(x', a), \alpha) ds$$

$$= f(x', y, \alpha) + \int_{x'}^y f_x(s, y, \alpha) ds$$

$$= f(x, y, \alpha),$$

where the inequality follows from $\mu(\cdot, a)$ strictly increasing and $f$ strictly quasi-continuous, and the last equality from the Fundamental Theorem of Calculus. If $x' > x$, then the worker with skill $x$ is matched in equilibrium with a firm with attribute $y'$ such that $y' = \mu(x, a) < y$ (so that the pair $(x, y)$ is above the graph of $\mu(\cdot, a)$), and the proof follows along the same lines using $y'$, $\pi(y', x, a)$, and $\mu^{-1}$.
instead. This shows that any pair \((x, y)\) that is not matched together cannot block the allocation since the sum of the payoffs they are obtaining is strictly bigger than the match output upon rematching. Hence, the outcome of the Walrasian equilibrium is stable.

To prove the same results in RS, replace \(f\) by \(\bar{f}, f_x\) and \(f_y\) by \(\bar{f}_x, \bar{f}_y, w\) and \(\pi\) by \(\bar{w}\) and \(\bar{\pi}\), and follow the same steps. For completeness, we provide some of the details.

Fix an investment function \(a\) and consider a pair \((x, y)\) when \(y\) must be given \(\pi_0\). The pair’s problem can be written as

\[
\phi(x, y, \pi_0) = \max_{\xi} \int_0^1 u(\xi(\alpha))dL(\alpha)
\]

\[
s.t. \int_0^1 (f(x, y, \alpha) - \xi(\alpha))dL(\alpha) \geq \pi_0.
\]

As explained in the text, the solution is a constant \(\xi\) given by \(E_L[f] - \pi_0 = \bar{f} - \pi_0\). Hence \(\phi\) is given by \(\phi(x, y, \pi_0) = u(\bar{f}(x, y) - \pi_0)\). Since this is a strictly increasing transformation of \(\bar{f} - \pi_0\) and \(\bar{f}\) is strictly spm in \((x, y)\), we obtain that any equilibrium exhibits positive sorting.  

Let \(\bar{\pi}(\cdot, a)\), given by \(\bar{\pi}(y, a)\) for all \(y\), be the payoff of firms in a Walrasian equilibrium. Consider the problem of a worker with characteristic \(x\) who takes as given \(\bar{\pi}(\cdot, a)\) and chooses his match optimally. He solves \(\max_{\pi} (\bar{f}(x, y) - \pi(y, a))\), and the first-order condition yields \(\bar{\pi}_y(y, a) = \bar{f}_y(x, y)\) for all \(y\). In equilibrium, \(y = \mu(x, a)\), and thus from \(\bar{\pi}_y(y, a) = \bar{f}_y(\mu^{-1}(y), y)\), we obtain

\[
\bar{\pi}(y, a) = \int_0^y \bar{f}_y(\mu^{-1}(s, a), s)ds,
\]

for all \(y\), where we have set \(\bar{\pi}(0, a) = 0\) since the pair \((0, 0)\) produces 0 for all \(\alpha\) and \(a\). That \(y = \mu(x, a)\) for all \(x\) is a global optimum follows as before.

Since the wage that a worker receives is given by \(\bar{f} - \bar{\pi}\), we obtain, after an application of the Fundamental Theorem of Calculus and a change of variables,

\[
\bar{w}(x, a) = \int_0^x \bar{f}_x(s, \mu(s, a))ds,
\]

for all \(x\). Since \(\bar{w}(\cdot, a) \geq 0\), each worker prefers to be matched than to remain unmatched.

If firms take \(\bar{w}(\cdot, a)\) as given, then as in NRS, one can show that each will optimally choose \(x = \mu^{-1}(y, a)\). And since \(\bar{\pi}(\cdot, a) \geq 0\), each firm prefers to be matched than to remain unmatched.

In short, we have shown that \(u(\bar{w}(x, a)) = \max_y \phi(x, y, \bar{\pi}(y, a)) = \phi(x, \mu(x, a), \bar{\pi}(\mu(x, a), a))\) for all \(x\), and \(\bar{\pi}(y, a) = \max_x (\bar{f}(x, y) - \bar{w}(x, a)) = \bar{f}(\mu^{-1}(y, a), y) - \bar{w}(\mu^{-1}(y, a), a)\) for all \(y\). Hence, \(\bar{w}(\cdot, a)\) (and \(\bar{\pi}(\cdot, a)\)) and \(\mu(\cdot, a) = G^{-1}(H(\cdot, a))\) is a Walrasian equilibrium. The proof that

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\[\text{Alternatively, we can apply the differential version of Legros-Newman’s condition for sorting in Chade, Eck-}
\]

---

\[\text{hout, and Smith (2017): } \phi_{xy} \geq (\phi_y/\phi_x)\phi_{xx}. \text{ Differentiating yields } u''\bar{f}_y\bar{f}_x + u'\bar{f}_x \geq \frac{u'\bar{f}_x}{u''}(-u''\bar{f}_x), \text{ which rearranges to } \bar{u}'\bar{f}_x \geq 0, \text{ and this holds strictly since } u' > 0 \text{ and } \bar{f}_x > 0.\]
it is unique and is stable follow exactly as before (with \( \bar{w}, \bar{\pi}, \) and \( \bar{f} \) instead of \( w, \pi, \) and \( f \)). \( \square \)

### A.2 Proof of Proposition [1]

Existence is proven in the text. Regarding uniqueness, we first show that \( U_1 - U_0 \) is increasing in \( \theta^* \) when \( R(w) = -u''(w)/u'(w) \) is sufficiently small (uniformly in \( w \)). To this end, let \( r \in [0,1] \) be a parameter indexing \( u \), so that \( u(w,0) = w \) (risk neutral), and as \( r \) increases this leads to a strictly increasing and concave transformation of \( u \) (so if \( r > r' \), then \( R(w,r) \geq R(w,r') \) for all \( w \)). Since \( R \) cannot be uniformly bounded in \( w \) if it is unbounded above, we will assume that \( R \) is uniformly bounded above for all \( r \), and also that \( u, u', \) and \( u'' \) are continuous in \( r \) for each \( w \).

Consider the no-risk-sharing case. Differentiating \( U_1 - U_0 = \int u'w_x \Delta H dx dm \) w.r.t. \( \theta^* \) yields

\[
\int \int (u''w_xw_{\theta^*} + u'w_{\theta^*}) \Delta H dx dm = \int \int u'(w,r)(w_{\theta^*} - R(w,r)w_{\theta^*}) \Delta H dx dm \\
\geq \int \int u'(w,r)(w_{\theta^*} - R(w,r)\max_{x,\theta^*,\alpha}(w_{\theta^*})) \Delta H dx dm \\
= \int \int u'(w,r)(w_{\theta^*} - R(w,r)\gamma) \Delta H dx dm,
\]

where we have set \( \gamma \equiv \max_{x,\theta^*,\alpha}(w_{\theta^*}) > 0 \), which is positive and finite since \( w_x \) and \( w_{\theta^*} \) are positive almost everywhere and continuous on \([0,1]^3\).

At \( r = 0 \), \( R(w,0) = 0 \) and \( u'(w,0) = 1 \) for all \( w \), so the last expression above becomes \( \int \int w_{\theta^*} \Delta H dx dm \), which is positive since \( w_{\theta^*} \) is positive almost everywhere. Hence, equilibrium is unique in the risk neutral case. From the continuity of \( u'(w,\cdot) \) and \( R(w,\cdot) \) for all \( w \), it follows that \( \int \int (u''w_xw_{\theta^*} + u'w_{\theta^*}) \Delta H dx dm \) is also positive for \( r \) sufficiently small, that is, when the coefficient of absolute risk aversion is uniformly small in \( w \), and equilibrium uniqueness follows.

In the risk-sharing case, we have \( \bar{w} \) instead of \( w \) and thus \( U_1 = U_0 = \int u'(\bar{w}) \Delta H dx \). Then as above \( \int \int (u''\bar{w}_x\bar{w}_{\theta^*} + u'\bar{w}_{\theta^*}) \Delta H dx \geq \int \int u'(\bar{w},r)(\bar{w}_{\theta^*} - R(\bar{w},r)\bar{\gamma}) \Delta H dx \), where \( \bar{\gamma} = \max_{x,\theta^*}((\bar{w}_x\bar{w}_{\theta^*}) \). The same continuity argument on \( r \) applies and yields the result.

Finally, to see that equilibrium is also unique if \( c \) is sufficiently convex, index \( c \) by \( j \in \mathbb{R}_+ \), and assume that \( c(\cdot,j) \) is convex, strictly decreasing, and differentiable on \((0,1)\), with \( c(1,j) = 0 \) and \( \lim_{\theta \to 0} c(\theta,j) = +\infty \) for all \( j \). Also, assume that \( c(\theta,\cdot) \) is strictly decreasing in \( j \) and converges to 0 for all \( \theta \in (0,1) \) as \( j \to \infty \).

We will focus on the first (lowest) crossing between \( U_1 - U_0 \) and \( c \), and show that for \( j \) sufficiently large it becomes the unique crossing and thus the unique equilibrium. To this end, note that since \( U_1 - U_0 > 0 \) for all \( \theta^* \in [0,1] \) and continuous in \( \theta^* \), \( \beta = \min_{\theta^* \in [0,1]}(U_1(\theta^*) - U_0(\theta^*)) > 0 \).

We first show that the first crossing converges to 0 as \( j \) goes to infinity. Let \( \theta^*_j(j) = \inf\{\theta^* | U_1(\theta^*) - U_0(\theta^*) = c(\theta^*,j)\} \) be the lowest equilibrium threshold given \( j \). We claim that \( \theta^*_j(j) > 0 \) for all \( j \) (since \( c \) is unbounded near 0 and \( U_1 - U_0 \) is finite). We claim that as \( j \) increases
\( \theta^*_j \) strictly decreases and converges to 0. To see this, note that \( c \) strictly decreases in \( j \) for each \( \theta \in (0, 1) \) and so does the first crossing, and this implies that \( \theta^*_j \) strictly decreases in \( j \). Now, given \( \epsilon > 0 \) there exists \( N_1 \) such that \( 0 < c(\theta^*, j) < \beta \) for all \( j \geq N_1 \) and \( \theta^* \geq \epsilon \). Hence, any equilibrium has \( \theta^* \in (0, \epsilon) \) for all \( j \geq N_1 \). Since \( \epsilon > 0 \) was arbitrary, \( \lim_{j \to \infty} \theta^*_j(j) = 0 \).

Now, \( U_1 - U_0 \) continuously differentiable in \( \theta^* \) on \([0, 1]\) implies that \( |\partial(U_1 - U_0)/\partial \theta^*| < M < \infty \). In turn, the derivative of \( c \) is unbounded near \( \theta^* = 0 \) for all \( j \). To see this, fix \( j \) and let \( \theta^*_j < \theta^* \); by the Mean Value Theorem, there exists \( c \) and consider \( [0, \theta^*_j] \) such that \( c(\theta^*_j, j) - c(\theta^*_j, j) = c_0(\theta^*_j, j)(\theta^*_j - \theta^*_j) \). As \( \theta^*_j \) goes to 0, the left side goes to \( -\infty \) and hence so does \( c_0(\theta^*_j, j) \).

We now show that there exists an \( N \) such that for all \( j \geq N \) equilibrium is unique. To see this, note that there exists an \( N_2 \) such that for all \( j \geq N_2 \), \( c_0(\theta^*_j, j) < -M < \partial(U_1(\theta^*_j) - U_0(\theta^*_j))/\partial \theta^* \), and the same holds in a neighborhood of \( \theta^*_j(j) \) for all \( j \geq N_2 \). Take any \( j_0 \geq N_2 \) and consider \([0, \theta^*_j(j_0)]\). There exists an \( N > N_2 \) such that \( c(\theta^*, j) < \beta \) for all \( \theta^* \geq \theta^*_j(j_0) \) and \( j \geq N \), and such that there is a unique crossing below \( \theta^*_j(j_0) \). The last point follows because \( c_0(\theta^*, j) < -M < \partial(U_1(\theta^*(j)) - U_0(\theta^*(j))/\partial \theta^* \) for all \( j \geq N \) and \( \theta^* \in (0, \theta^*_j(j_0)] \), so there cannot be more than one crossing in that interval, and there is at least one, and hence equilibrium is unique. Since the construction is in terms of \( c \) and \( U_1 - U_1 \), it applies to both the risk-sharing and the no-risk-sharing cases, and we are done.

Regarding the example mentioned in footnote \( \Box \) after Proposition \( \Box \) let \( f = \alpha xy \), so \( f_x = \alpha y \) and \( f_{xy} = \alpha \). Also, assume that \( g = 1 \) and \( \mathbb{E}[\alpha] > 0 \). Finally, set \( R_u = \sup_w R(w, 1) \), which is an upper bound for risk aversion for all \((w, r)\). By replacing above \( R(w, r) \) by \( R_u \) (which preserves the inequality), we obtain that \( \int \int (u''w_xw_{x^\alpha} + u'w_{x^\alpha}) \Delta H dx dL \geq 0 \) if

\[
R_u \leq \int \int \frac{u'(w_x, r)w_{x\alpha} \Delta H dx dL}{\gamma \int \int u'(w_x, r) \Delta H dx dL},
\]

where we recall that \( \gamma = \max_{x, \theta^*} \alpha w_xw_{\theta^*} > 0 \). Let \( w_u = w(1, 1, 1) = \int H_0(x) dx \) be the largest possible wage. Since \( u'(\cdot, r) \) is decreasing in \( w_u \), it suffices to show that

\[
R_u \leq \frac{u'(w_u, r)}{u'(0, r)} \int \int w_{x\alpha} \Delta H dx dL \gamma \int \int \Delta H dx dL.
\]

Since \( w_x = \alpha \mu(x, \theta^*) \leq 1 \), with equality at \( x = \alpha = 1 \), and \( w_{x\alpha} = \alpha \int_0^x \mu_{t\alpha}(s, \theta^*) ds = \alpha \int_0^x \Delta H ds \leq \int \Delta H dx \), with equality at \( x = \alpha = 1 \), we have \( \gamma = \int \Delta H dx \). Also, \( w_{x\theta^*} = \alpha \mu_{t\alpha}(x, \theta^*) = \alpha \Delta H \), and thus \( \int \int w_{x\theta^*} \Delta H dx dL = \mathbb{E}[\alpha] \int \int \Delta H dx dL \). Finally, from the unique solution on \([0, w_u] \) (for each \( r \)) to the differential equation \(-u''(w, r)/u'(w, r) = R(w, r) \), we obtain 

\[
u'(w_u, r)/u'(0, r) = e^{-\int_0^{w_u} R(s, r)ds} \geq e^{-R_u w_u}.
\]

Thus, \( \int \int (u''w_xw_{x^\alpha} + u'w_{x^\alpha}) \Delta H dx dL \geq 0 \) if

\[
e^{R_u H_0 dx} R_u \leq \mathbb{E}[\alpha] \int \int (\Delta H)^2 dx dL \gamma \int \int \Delta H dx dL.
\]

35
which holds for $R_u$ small, which makes the coefficient of risk aversion uniformly small. The resulting bound is determined by the primitives $L$, $H_0$, and $H_1$. Note that the second term on the right side is bigger than one, and that higher $\mathbb{E}[\alpha]$ relaxes the upper bound on $R$.

A.3 Proof of Proposition 2

Consider the no-risk-sharing case. Let $(\theta_\ell^*, \mu(\cdot, \theta_\ell^*), w(\cdot, \alpha, \theta_\ell^*))$ be the equilibrium with the lowest investment threshold, denoted by $\theta_\ell^*$. We will show that any alternative threshold that can strictly improve the utility of a positive-measure set of workers strictly lowers the profit of firms on a positive measure set, and vice-versa. It then follows that $\theta_\ell^*$ is efficient.

The aggregate profits obtained by firms in equilibrium is

$$\Pi(\theta_\ell^*) \equiv \int \int (f(x, \mu(x, \theta_\ell^*), \alpha) - w(x, \theta_\ell^*, \alpha))dH(x, \theta_\ell^*)dL(\alpha).$$

Consider $\int (f-w)dHdL$ a function of $\theta^*$. Differentiating with respect to $\theta^*$ yields

$$\int \int (f_y \mu_y - w_{\theta^*})dHdL + \int \int (f-w)qHdxdL.$$

Integration by parts allows us to write the second term as $-\int \int q (f_x + f_y \mu_x - w_x) \Delta H dxdL = -\int \int q f_y \mu_x \Delta H dxdL$. And since $\mu_{\theta^*} h = q \mu_x \Delta H$, we obtain

$$\frac{\partial}{\partial \theta^*} \int \int (f-w)dHdL = -\int \int w_{\theta^*} dHdL < 0.$$

Hence, aggregate profits of firms are strictly higher than $\Pi(\theta_\ell^*)$ for $\theta^* < \theta_\ell^*$, and strictly lower than $\Pi(\theta_\ell^*)$ for $\theta^* > \theta_\ell^*$. Hence, we can restrict attention to the interval $[0, \theta_\ell^*]$, since above $\theta_\ell^*$ a positive measure set of firms is strictly worse off than in the equilibrium with $\theta_\ell^*$.

In turn, the aggregate utility of workers is given by

$$U(\theta_\ell^*) \equiv \int_0^1 \int_0^1 u(w(x, \theta_\ell^*, \alpha))dH(x, \theta_\ell^*)dL(\alpha) - \int_{\theta_\ell^*}^1 c(\theta)dQ(\theta).$$

The derivative of $\int \int u(w)dHdL - \int_{\theta^*}^1 c(\theta)dQ(\theta)$ with respect to $\theta^*$ is

$$\int \int u' w_{\theta^*} dHdL - \int \int u' w_x q \Delta H dxdL + c(\theta^*) q(\theta^*),$$

Note that the first term in (9) is strictly positive for all $\theta^* \in [0, 1]$, and also that $-\int \int u' w_x \Delta H dxdL + c(\theta^*) = 0$ at $\theta^* = \theta_\ell^*$ (since it reduces to the equilibrium condition (3)).

Since $\theta_\ell^*$ is stable, $U_1 - U_0$ crosses $c$ from below at that point. And since $\theta_\ell^*$ is the lowest equilibrium, $U_1(\theta^*) - U_0(\theta^*) < c(\theta^*)$ for all $\theta^* < \theta_\ell^*$. This is equivalent to $-\int \int u' w_x q \Delta H dxdL +
\[ c(\theta^*)q(\theta^*) > 0 \] for all \( \theta^* < \theta_\ell^* \). Finally, since \( \int u'w_\theta^* dHdL \) is strictly positive, \[ \text{(9)} \] is strictly positive for all \( \theta^* < \theta_\ell^* \), which implies that on \([0, \theta_\ell^*]\) aggregate utility for workers is maximum at \( \theta_\ell^* \).

Hence, the equilibrium with the lowest investment threshold equilibrium is efficient. It is clear that the same is true if equilibrium is unique, since it is stable.

The proof for the risk-sharing case is analogous: simply replace \( \Pi(\theta_\ell^*) \) and \( \mathcal{U}(\theta_\ell^*) \) by
\[ \bar{\Pi}(\theta_\ell^*) \equiv \int_0^1 (\bar{f}(x, \mu(x, \theta_\ell^*))) - \bar{w}(x, \theta_\ell^*) \, dH(x, \theta_\ell^*) \]
and \( \bar{\mathcal{U}}(\theta_\ell^*) \equiv \int_0^1 u(\bar{w}(x, \theta_\ell^*)) \, dH(x, \theta_\ell^*) - \int_{\theta_\ell^*}^1 c(\theta) \, dQ(\theta), \)
and follow exactly the same steps as above. \[ \Box \]

A.4 Proof of Proposition 3

We first provide conditions on absolute risk aversion \( R \) and the wage function \( w \) under which \( \int \varphi_{x \ell} \Delta H dx \geq 0 \), and then we pin down the result from primitives.

**Lemma 1** In any stable equilibrium of the no-risk-sharing case, more workers invest in skills in response to a FOSD shift in \( L \) if either of the following conditions hold:

(i) Absolute risk aversion is (uniformly) sufficiently small and \( w_{x \alpha} > 0 \) almost everywhere; or

(ii) For all values of \( w \), \( R(w) \leq 1/w \), and \( w \) is increasing and log-spm in \((x, \alpha)\).

**Proof.** Using integration by parts and the definition of \( \varphi \) we obtain
\[ \int \varphi_{x \ell} \Delta H dx = \int \int u'(w) [w_{x \alpha} - R(w)w_xw_\alpha] (-L_\ell) \Delta H d\alpha dx, \]
where \( L_\ell \leq 0 \) for all \( \alpha \). If it is 0 for all \( \alpha \), then there is nothing to prove (there is no FOSD shift), so we will assume that it is strictly negative on a set of \( \alpha \)’s of positive measure.

Part (i) follows as in the uniqueness part of the proof of Proposition \[ \Box \] with \( w_\alpha \) and \( w_{x \alpha} \) instead of \( w_\theta^* \) and \( w_{x \theta^*} \). It also subsumes the case where workers are risk neutral, so that \( u'(w) = 1 \) and \( R(w) = 0 \) for all values of \( w \), and so \( w \) spm in \((x, \alpha)\) implies that \( \int \varphi_{x \ell} \Delta H dx \geq 0 \).

Finally, to prove part (ii) it suffices to show that the term in the square brackets in the integrand is positive. But since \( R(w) \leq 1/w \) for all values of \( w \), we obtain \( w_{x \alpha} - R(w)w_xw_\alpha \geq w_{x \alpha} - \frac{w_xw_\alpha}{w} \geq 0 \), where the inequality follows from the log-supermodularity of \( w \) in \((x, \alpha)\). \[ \Box \]

**Proof of Proposition 3.** Part (i) is immediate since the assumptions on \( f \) imply that the \( w \) is strictly increasing and \( w_{x \alpha} > 0 \) for almost all values of \((x, \alpha)\) and \( \theta^* \). Hence, the result follows from Lemma \[ \Box \] (i) if \( R \) sufficiently small uniformly in \( w \).
Consider part (ii). Rewrite the equilibrium wage function as as follows:

$$w(x, \alpha, \theta^*) = \int_{[0,x]} I(s, \mu(s, \theta^*), \alpha) ds. \tag{10}$$

It is well-known (Karlin and Rinott [1980]) that $w$ is log-spm in $(x, \alpha)$ if the integrand is log-spm in $(x, s, \alpha)$ for each $\theta^*$, and this holds if $I$ is log-spm in $(x, s)$ and $f$ is log-spm in $(s, \alpha)$.

It is easy to verify that $f$ in the integrand is log-spm in $(s, \alpha)$ if and only if

$$(f_{xx\alpha}f_x - f_{x\alpha}f_{xx}) + (f_{xy\alpha}f_x - f_{x\alpha}f_{xy}) \mu_x \geq 0,$$

which holds if the parentheses are positive, i.e., if $f_x$ is log-spm in $(x, \alpha)$ and $(y, \alpha)$.

Similarly, if we take any two pairs $(s, x)$, $(s', x')$, it is easy to check that

$$I_{[0,x\lor x']} (s \lor s') I_{[0,x\land x']} (s \land s') \geq I_{[0,x]} (s) I_{[0,x']} (s'),$$

and thus $I$ is log-spm in $(x, s)$.

Hence, $w$ is log-spm in $(x, \alpha)$ for each $\theta^*$ and thus $w_{x\alpha} w_{x\alpha} - 2w_{\alpha x} w_{\alpha} + w_{x\alpha} w_{\alpha} - w_{x} w_{\alpha} \geq 0$. It follows that $R(w) \leq (1/w)(w_{x\alpha} w_{x\alpha} - 2w_{\alpha x} w_{\alpha} + w_{x\alpha} w_{\alpha} - w_{x} w_{\alpha})$ if $R(w) \leq 1/w$. Therefore, the conditions in (ii) of Lemma 1 hold and a FOSD shift in $L$ increases $U_1 - U_0$, thereby decreasing the equilibrium value of $\theta^*$.

□

### A.5 Proof of Proposition 4

We first provide conditions on absolute risk aversion and prudence, $R$ and $P$, and on the wage function $w$ under which $\int \varphi_{x\ell} \Delta H dx \geq 0$, and then pin down the result from primitives.

#### Lemma 2

In any stable equilibrium, more workers invest in skills in response to an IR shift in $L$ if either of the following conditions hold:

(i) Absolute risk aversion is (uniformly) sufficiently small and $w_{x\alpha} \geq 0$, with strict inequality on a set of $(x, \alpha)$ of positive measure; or

(ii) For all $w$, $P(w) \geq 3/w$, with $w_{x\alpha} \geq 0$, and $w$ is increasing in $(x, \alpha)$, log-concave in $\alpha$ for all $x$ and log-submodular in $(x, \alpha)$.

Proof. Using integration by parts twice on $\varphi_x = \int u'(w)w_x dL$ we obtain

$$\int \varphi_{x\ell} \Delta H dx = \int \int \left( \frac{\partial}{\partial \alpha} (u''(w)w_{\alpha}w_x + u'(w)w_{x\alpha}) \right) \left( \int_0^\alpha L ds \right) \Delta H d\alpha dx$$

$$= \int \int u'(w) \left[ R(w) \left( P(w)w_{x}w_{\alpha}^2 - 2w_{\alpha x}w_{\alpha} - w_{x}w_{\alpha\alpha} \right) + w_{x\alpha} \right] \left( \int_0^\alpha L ds \right) \Delta H d\alpha dx,$$

A function $z$ is log-concave in $x$ if log $z$ is concave in $x$. 

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37A function $z$ is log-concave in $x$ if log $z$ is concave in $x$. 

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where \( \int_0^\alpha L_t ds \geq 0 \) for all \( \alpha \) by IR. If it is 0 for all \( \alpha \), then there is nothing to prove (there is no IR shift), so we will assume that it is strictly positive on a set of \( \alpha 's \) of positive measure.

Assume that workers are risk neutral, so \( u'(w) = 1 \) and \( R(w) = P(w) = 0 \) for all \( w \). Then \( w_{xaa} \geq 0 \) implies that \( \int \varphi_{xt} \Delta Hdx \geq 0 \). As in Proposition 1, the same holds for absolute risk aversion that is uniformly small in \( w \). This is because the condition on \( w_{xaa} \) guarantees that \( \int w_{xaa} \left( \int_0^3 L_t ds \right) \Delta Hdx > 0 \), and thus the inequality remains strict if \( R \) is sufficiently small.

To prove part (ii), it suffices to show that the first term inside the square brackets in the integrand is positive under the premises. Note that

\[
P(w)w_xw_\alpha^2 - 2w_\alpha w_\alpha - w_xw_{xaa} \geq 3 \frac{w_xw_\alpha^2}{w} - 2w_\alpha w_\alpha - w_xw_{xaa} \\
= 2 \left( \frac{w_xw_\alpha}{w} - w_\alpha \right) w_\alpha + \left( \frac{w_\alpha^2}{w} - w_{xaa} \right) w_x \\
\geq 0,
\]

where the first inequality uses the assumption \( P(w) \geq 3/w \) for all \( w \), and the second uses \( w \) submodular \((x, \alpha)\) and \( w \) log-concave in \( \alpha \) for each \( x \), which imply that both terms in parentheses are positive. Hence, since the integrand is positive it follows that \( \int \varphi_{xt} \Delta Hdx \geq 0 \).

**Proof of Proposition 1** The proof of part (i) follows from Lemma 2 (i), since \( w_{xaa} = f_{xaa} \geq 0 \), strictly so on a set of \((x, \alpha) \) of positive measure. Thus, the last term in (7) is positive.

Consider part (ii). Under the conditions on \( \eta \) and \( z \), the last term of (7) is positive (it is given by \( w_{xaa} = \eta''z_x \geq 0 \)), and \( w_{xao}w/w_xw_\alpha = (z_x\eta''\eta_0^x z_x)/(\eta_0^x \eta'') = 1 \). Thus, the expression in parentheses in the first term of (7) is positive if and only if

\[
P(w) \geq \frac{1}{w} \left( 2 + \frac{\eta''(\alpha)\eta(\alpha)}{(\eta'(\alpha))^2} \right).
\]

If \( \eta \) is log-concave in \( \alpha \), \( \eta''/\eta^2 \leq 1 \), then it suffices that \( P(w) \geq 3/w \) for all values of \( w \).

**A.6 Decreasing Convex Order Shift in \( L \)**

We now consider a shift on \( L \) called called a decreasing convex order (DCO), which both reduces the mean and increases the riskiness of \( \alpha \).\(^{38}\)

**Corollary 1 (NRS: DCO Shift in \( L \))** If \( f \) is a separable class, \( R(w) \geq 1/w \) and \( P(w) \geq 3/w \) for all \( w \), then a DCO shift in \( L \) increases the measure of workers who invest in NRS.

\(^{38}\)Random variables \( X \in [0, 1] \) and \( Y \in [0, 1] \) are ordered by DCO if \( \int_0^t F_X(s) ds \geq \int_0^t F_Y(s) ds \) for all \( t \) (so it holds for \( t = 1 \) and thus \( E[X] \leq E[Y] \)). Alternatively, this holds if and only if \( E[z(X)] \leq E[z(Y)] \) for all decreasing and convex functions \( z \) (see Shaked and Shanthikumar (2007) p.182). Intuitively, \( X \) has more spread and lower mean than \( Y \), so only decision makers with a convex and decreasing utility function prefer \( X \) to \( Y \). Note the similarity with second-order stochastic dominance or increasing convex order, except for the reverse monotonocity in \( z \).
Proof. As in the proof of Lemma 2 twice integrating by parts \( \varphi_x = \int u'(w)w_x dL \) yields

\[
\int \varphi_x \Delta H dx = -\int \left( u''(w)w_\alpha w_x + u'(w)w_{x\alpha} \right) \left( \int_0^\alpha L_t ds \right) |_{\alpha=0}^{\alpha=1} \Delta H dx \\
+ \int \left[ \frac{\partial}{\partial \alpha} (u''(w)w_\alpha w_x + u'(w)w_{x\alpha}) \right] \left( \int_0^\alpha L_t ds \right) \Delta H d\alpha dx \\
= -\int \left( u''(w)w_\alpha w_x + u'(w)w_{x\alpha} \right) |_{\alpha=1} \left( \int_0^\alpha L_t ds \right) \Delta H dx \\
+ \int \int u'(w) \left[ R(w) (P(w)w_x^2 - 2w_\alpha w_\alpha - w_x w_{x\alpha}) + w_{x\alpha} \right] \left( \int_0^\alpha L_t ds \right) \Delta H d\alpha dx
\]

where the first term reflects the change in the mean via \( \int L_t ds = -\partial(\int \alpha dL)/\partial t \geq 0 \) since \( \int \alpha dL \) decreases in \( t \), and the second term the change in riskiness through \( \int_0^\alpha L_t ds \geq 0 \) for all \( \alpha \).

As in the proof of Proposition 4, the term in square brackets in the last term is positive since \( f \) is a separable class and \( P(w) \geq 3/w \) for all \( w \).

Regarding the first term, note \( \int L_t ds \geq 0 \), and since \( w_\alpha w_x = w w_{x\alpha} \) and \( R(w) \geq 1/w \), \( u''(w)w_\alpha w_x + u'(w)w_{x\alpha} \leq 0 \). Along with the minus sign we obtain that the first term is positive. Hence, a DCO shift in \( L \) increases the measure of workers who invest (\( \theta^* \) decreases).

\[\square\]

A.7 FOSD Shift in \( L \) in RS

Proposition 8 (RS: FOSD Shift in \( L \)) In any stable equilibrium in RS, more workers invest in skills in response to a FOSD shift in \( L \) if either of the following conditions hold:

(i) Absolute risk aversion \( R \) is (uniformly) sufficiently small; or

(ii) For all values of \( w \), \( R(w) \leq 1/w \), \( f_x \) is log-spm in \( (x,\alpha) \) for each \( y \), and in \( (y,\alpha) \) for each \( x \), and \( L \) has a log-spm density \( l \).

Proof. For (i), the assumptions on \( f \) imply that the \( \tilde{w} \) is strictly increasing and \( \bar{w}_{xt} > 0 \) for almost all values of \( x \) and \( \theta^* \). Hence, as in the proof of Lemma 1(i) but with \( \tilde{w} \) instead of \( w \) and \( \bar{w}_{xt} \) instead of \( w_{x\alpha} \), the result follows if \( R \) sufficiently small uniformly in \( w \).

To prove (ii), rewrite the wage function as follows:

\[
\bar{w}(x,\theta^*,t) = \int \int \mathbb{I}_{[0,x]}(s)f_x(s,\mu(s,\theta^*),\alpha)l(\alpha|t)ds d\alpha.
\]

Then \( \bar{w} \) is log-spm in \( (x,t) \) if the integrand is log-spm in \( (x,s,\alpha,t) \) for each \( \theta^* \), which holds since \( \mathbb{I} \) is log-spm in \( (x,s) \), \( f_x \) is log-spm in \( (s,\alpha) \) as in the proof of Proposition 3, \( l \) is log-spm in \( (\alpha,t) \), the product of log-spm functions is log-spm, and so is their integral.

Hence, \( \bar{w}_{xt}\bar{w}/\bar{w}_{xt}\bar{t} \geq 1 \), and \( R(w) \leq (1/\bar{w})(\bar{w}_{xt}\bar{w}/\bar{w}_{x}\bar{t}) \) if \( R(w) \leq 1/w \) for all values of \( w \). And since \( \bar{w}_{t} \geq 0 \), it follows as in the proof of Proposition 3 that a FOSD shift in \( L \) increases \( U_1 - U_0 \), thereby decreasing the equilibrium value of \( \theta^* \). \[\square\]
Differentiating $\int u'(\bar{w})\bar{w}_x\Delta Hdx$ with respect to $t$ yields

$$\int (u''\bar{w}_t\bar{w}_x + u'\bar{w}_{xt})\Delta H(x)dx = \int u'(\bar{w})\bar{w}_t\bar{w}_x \left(\frac{\bar{w}_{xt}\bar{w}}{\bar{w}_t\bar{w}_x} \right) \frac{1}{\bar{w}} - R(\bar{w}) \Delta H(x)dx$$

$$= \int u'(\bar{w})\bar{w}_t\bar{w}_x \left(\frac{1}{\bar{w}} - R(\bar{w}) \right) \Delta H(x)dx,$$  \hspace{1cm} (11)

where the second equality uses that $f = \eta z$ and hence $\bar{w} = \int_0^z \zeta ds \int_0^1 \eta dL$. If $\eta$ is convex in $\alpha$ then $\bar{w}_t \geq 0$ since $\int \eta dL$ increases in $t$, and thus $R \leq 1/w$ for all $w$ implies that (11) is positive. If $\eta$ is concave, then $\bar{w}_t \leq 0$, and $R \geq 1/w$ for all $w$ implies that (11) is positive. In either case an IR shift in $L$ reduces the threshold $\theta^*$ and hence increases the measure of workers who invest. \hfill \Box

**A.9  Proof of Proposition 6**

(i) Consider first the no-risk-sharing case. Fix $\theta^*$ and consider $U_1 - U_0 = \int \int u'w_x\Delta Hdx dL$; we want to show that it is increasing in $t$. The derivative of $U_1 - U_0$ with respect to $t$ is

$$\int \int (u''w_xw_t + u'w_{xt})\Delta Hdx dL = \int \int u'(w_{xt} - Rw_xw_t)\Delta Hdx dL.$$  \hspace{1cm} (12)

Note that $w_t = \int_0^x f_{xy} \mu_t ds \geq 0$ and $w_{xt} = f_{xy} \mu_t \geq 0$, since $\mu_t = -G_t/g \geq 0$, and strictly so on a positive-measure set. Hence, if $R = 0$, then $U_1 - U_0$ strictly increases in $t$ and $\theta^*$ decreases, and the same holds for $R$ uniformly small in $w$, following analogous steps as in the proof of Proposition 1.

By replacing $w$ by $\bar{w}$ we obtain the same result for the risk-sharing-case.

(ii) If the FOSD shift is in $Q$, then $\mu_t = Q_t \Delta H/g \leq 0$, strictly so on a positive measure set. It follows from (12) that $U_1 - U_0$ strictly decreases in $t$ if $R = 0$, and the same holds for $R$ uniformly small. As a result, $\theta^*$ increases in this shift. This holds both in the no-risk-sharing and the risk-sharing cases.

(iii) Consider the no-risk-sharing case. The derivative of $U_1 - U_0$ with respect to $t$ can be written, after integration by parts and using the definition of $R$, as follows

$$\int \int u'w_x \left(-\frac{\partial H_1}{\partial t}\right) dx dL + \int \int u'(R(-w_t)w_x - (-w_{xt}))\Delta Hdx dL.$$  \hspace{1cm} (13)

Assume that $wR(w) \geq b$ for all $w$, where $b \geq 0$. We will show that under the assumptions on $f$ and $h_1$, this derivative of $U_1 - U_0$ with respect to $t$ is positive if $b$ is large enough.

Fix $\alpha \in [0, 1]$ and $b \geq 0$, and write the integral with respect to $x$ as follows:

$$\int u' \Delta H \left(\frac{w_x}{\Delta H} \frac{\partial H_1}{\partial w} + R(-w_t)w_x - (-w_{xt})\right)dx.$$  \hspace{1cm} (14)
Since \( w_t = \int_0^x f_{xy} \mu_t \, ds \), \( w_{xt} = f_{xy} \mu_t \) and \( \mu_t = (1 - Q)(\partial H_1 / \partial t) / g \leq 0 \), the first two terms in the big parentheses are positive, while the last one is negative. Also, since \( R \geq b/w \), this integral is bigger than
\[
\int u' \Delta H \left( w_x \frac{\partial H_1}{\Delta H} + b \frac{(-w_t)w_x}{w} - (-w_{xt}) \right) \, dx.
\]
Consider the term in the big parenthesis which, using the wage function, is equal to
\[
\frac{-\partial H_1}{\Delta H} \left( f_x - \Delta H f_{xy} \frac{1 - Q}{g} \right) + b \frac{\left( \int_0^x f_{xy} \frac{1-Q}{g} \left( \frac{-\partial H_1}{\partial t} \right) \right) f_x}{\left( \int_0^x f_x \, ds \right)}.
\]
Let us examine the behavior of (15) near \( x = 0 \). Note that
\[
\lim_{x \to 0} \frac{-\partial H_1}{\Delta H} = \lim_{x \to 0} \frac{\partial h_1}{\partial t} = \frac{\partial h(0)}{\Delta h(0)} > 0,
\]
where the first equality follows from L'Hospital's rule applied to the indeterminate form 0/0 and the second uses the continuous differentiability of \( h_1 \), the premise that its derivative with respect to \( t \) at \( x = 0 \) is strictly negative, and that \( h(0) = h_1(0) \) for all \( t \) by FOSD. Regarding the term in parenthesis in the first term of (15), if \( f_x(0,0,\alpha) > 0 \), then the entire first term is positive at \( x = 0 \) since \( \partial H_1 / \partial t = 0 \) at \( x = 0 \). Consider now the second term in (15) at \( x = 0 \), which is positive but of the form 0/0. Applying L'Hospital's rule once, we obtain that if \( f_x(0,0,\alpha) > 0 \), then the limit is zero. Hence, (15) is strictly positive and finite at \( x = 0 \) for each \( \alpha \).

By continuity, for each \( \alpha \) and \( b \geq 0 \), there exists a \( \hat{x}(\alpha,b) > 0 \) such that the integrand in (14) is strictly positive for all \( x \in (0, \hat{x}(\alpha,b)) \), where \( \hat{x} \) can be taken uniformly in \( \theta^* \) since the inequalities are strict and bounded away from zero for all \( \theta^* \in [0,1] \). Note that \( \hat{x}(\alpha,0) > 0 \) (the strict sign is driven by the first term in the big parenthesis, which is independent of \( b \)), and hence we can assume that \( \hat{x} \) does not depend on \( b \). Moreover, \( \hat{x} \) is strictly positive and bounded away from zero for all \( \alpha \) given the premises, and thus the infimum with respect to \( \alpha \in [0,1] \), which with some abuse we denote also by \( \hat{x} \), is a strictly positive number.

It follows that the double integral (13) when \( \alpha \) runs from zero to one and \( x \) from 0 to \( \hat{x} \) is positive. And if \( x \in [\hat{x},1] \), the first term in (15) is bounded, and dominated by the second term for \( b \) large enough. The bound on \( b \), \( \hat{b} \), can be made uniform in \( (x, \alpha, \theta^*) \), so that for all \( b \geq \hat{b} \) the integral (14) is strictly positive for all \( \alpha \). Integrating now with respect to \( \alpha \) completes the proof.

---

\(^{39}\)If we do not want to assume that \( f_x(0,0,\alpha) > 0 \), we can proceed as follows. Suppose \( f_x(0,0,\alpha) = 0 \), then the term in the first parenthesis vanishes at \( x = 0 \). Its derivative at \( x = 0 \) is \( f_{xx} + (f_{xy}/g)(Qh_0(0) + (1 - Q)h_1(0) - \Delta h(0)(1 - Q)) \). This is strictly positive if either \( f_{xx}(0,0,\alpha) > 0 \) or \( f_{xy}(0,0,\alpha) > 0 \) and \( h_1(0) > h_0(0)/2 \) since \( Qh_0(0) + (1 - Q)h_1(0) - \Delta h(0)(1 - Q) > 0 \) if \( h_1(0) > h_0(0)/2 \). Consider the second term. If \( f_x(0,0,\alpha) = 0 \), then it is a 0/0 form, and by L'Hospital the numerator vanishes at \( x = 0 \) while the denominator is strictly positive if either \( f_{xx}(0,0,\alpha) > 0 \) or \( f_{xy}(0,0,\alpha) > 0 \), since \( Qh_0(0) + (1 - Q)h_1(0) - \Delta h(0)(1 - Q) > 0 \) if \( h_1(0) > h_0(0)/2 \). Hence, (15) is strictly positive and finite at \( x = 0 \) for each \( \alpha \) under these weaker conditions as well.
The proof for the risk-sharing case is similar\footnote{In fact, one can relax the assumption on $f_x$, $f_{xx}$, and $f_{xy}$ in the statement of the proposition, and assume that the expectation with respect to $\alpha$ of either of them satisfies the strict positivity assumption at $x = y = 0$.} Replacing $w$ by $\bar{w}$ in (14) provides the expression we need to show is positive. As above, one can show that, for each $b \geq 0$, the expression is strictly positive near $x = 0$ under the premises, find $\hat{x}$, and then the bound $\hat{b}$. \hfill \Box

As mentioned in the text, by restricting the class of $f$ and $G$ considered we can relax the uniformly small risk aversion condition in part (i). Indeed, $U_1 - U_0$ increases in $t$ if $f_x$ is log-spm in $(x, y)$ for each $\alpha$ and log-convex in $y$ for each $(x, \alpha)$, $G_t/g$ is decreasing in $y$, and $R(w) \leq 1/w$ for all $w$. To prove this claim, consider the no risk sharing case and rewrite (12) as follows:

$$\int \int u'(w)w_x w_t \left( \left( \frac{w_x w_t}{w_x w_t} \right) \frac{1}{w} - R(w) \right) \Delta H(x) dxdL(\alpha).$$

Since $w_t \geq 0$, it is clear that this is positive if $R(w) \leq 1/w$ and $w$ is log-spm in $(x, t)$ for each $\alpha$ and $\theta^*$ (since the ratio in parenthesis will be bigger than one). From $w = \int I_{[0,x]} f_s ds$, it suffices that the integrand is log-spm. Now, $I_{[0,x]}(s)$ is log-spm in $(x, s)$, and we will show that the assumptions on $f_x$ imply that it is log-spm in $(s, t)$ for each $\alpha$ and $\theta^*$. Since the product of log-spm functions is log-spm and this property is preserved by integration, the result will follow.

Consider $\log f_x(x, \mu(x, \theta^*), \alpha)$. Differentiating with respect to $x$ and then with respect to $t$ reveals that the sign of the cross-partial $\partial^2 \log f_x/\partial x \partial t$ is the same as the sign of

$$(f_{xxy} f_x - f_{xx} f_{xy}) \mu_t + (f_{xyy} f_x - f_{xy}^2) \mu_t \mu_x + f_{xy} f_x \mu_{xt}.$$  

If $f_x$ is log-spm in $(x, y)$ for each $\alpha$ and log-convex in $y$ for each $(x, \alpha)$, then $f_{xxy} f_x - f_{xx} f_{xy} \geq 0$ and $f_{xyy} f_x - f_{xy}^2 \geq 0$, so the first two terms are positive. Regarding the third term, $f_{xy} f_x \mu_{xt}$, note that $f_x$ and $f_{xy}$ are positive, and some algebra reveals that $\mu_{xt} \geq 0$ if and only if $G_{ty} g - G_{t} g' \leq 0$, which holds if $G_t/g$ is decreasing in $y$. Hence, under the premises $f_x$ is log-spm in $(x, t)$ for each $\alpha$ and $\theta^*$, and thus $w$ is log-spm in $(x, t)$, completing the proof of the claim. Replacing $w$ by $\bar{w}$ and following the same steps reveal that the claim also holds in the risk sharing case.

### A.10 Shift in Technology

**Proposition 9 (NRS and RS: Shift in $f_x$)** Let $f_x$ be log-spm and $R(w) \leq 1/w$ for all $w$.

(i) In any stable equilibrium of NRS, more workers invest when $f_x$ increases;

(ii) In any stable equilibrium of RS, more workers invest when $f_x$ increases if in addition $L$ has a log-spm density $l$.

**Proof.** (i) We must show that under the stated conditions $U_1 - U_0 = \int \int u'(w) w_x \Delta H dxdL$
increases in $t$ for each $\theta^*$. Differentiating with respect to $t$ and rearranging yields
\[
\int \int (u''w_tw_x + u'w_xt) \Delta H(x) dx dL(\alpha) = \int \int u'(w)w_tw_x \left( \frac{w_xtw}{w_tw_x} \right) \frac{1}{w} - R(w) \Delta H(x) dx dL(\alpha).
\]

The conditions on $f_x$ ensure that $w$ is log-spm in $(x, t)$ for each $\alpha$ and $\theta^*$, and hence $w_xtw/(w_tw_x) \geq 1$ almost everywhere. Since $R(w) \leq 1/w$, it follows that the inner integral is positive for almost all values of $\alpha$, and so is the integral with respect to $\alpha$.\footnote{Note that this part only uses that $f_x$ is log-spm in $(x, t)$ for each $\alpha$ and $y$, and in $(y, t)$ for each $\alpha$ and $x$.}

(ii) The derivative of $U_1 - U_0$ with respect to $t$ is now
\[
\int \left( u''\bar{w}_t\bar{w}_x + u'\bar{w}_xt \right) \Delta H(x) dx = \int u'(\bar{w})\bar{w}_t\bar{w}_x \left( \frac{\bar{w}_xt\bar{w}}{\bar{w}_t\bar{w}_x} \right) \frac{1}{\bar{w}} - R(\bar{w}) \Delta H(x) dx.
\]

Since $f_x$ and $l$ are log-spm, $\bar{w}$ is log-spm in $(x, t)$ for each $\theta^*$, and hence $\bar{w}_xt\bar{w}/(\bar{w}_t\bar{w}_x) \geq 1$ almost everywhere. But then $R(w) \leq 1/w$ for all values of $w$ implies that the integral is positive. \hfill \square

A.11 Proof of Proposition 7

We will show that $U_1(\theta^*) - U_0(\theta^*)$ is larger for all $\theta^*$ in NRS than in RS, from which the result will follow. Since $f$ is a separable class, $w = \eta \int_0^x z_x$. Then $u'(\bar{w})\bar{w}_x - \int u'(w)w_xt dL$ is equivalent to $z_x(x, \mu(x, \theta^*))$, which is positive, times
\[
u' \left( \int \eta(\alpha) dL(\alpha) \int_0^x z_x(s, \mu(s, \theta^*))ds \right) \int \eta(\alpha) dL(\alpha) - \int u'(\eta(\alpha) \int_0^x z_x(s, \mu(s, \theta^*))ds) \eta(\alpha) dL(\alpha),
\]
and hence it suffices to show that this expression is negative. Let $I \equiv \eta([0, 1])$ be the range of the function $\eta$, where $I$ is an interval since $\eta$ is monotone. Consider the function $m(\cdot, x, \theta^*) : I \rightarrow \mathbb{R}$ given by $m(\eta, x, \theta^*) = u'(\eta \int_0^x z_x)\eta$ for all $\eta \in I$. As a function of $\eta$, $m(\cdot, x, \theta^*)$ is convex for each $(x, \theta^*)$ if and only if $m_{\eta\eta} \geq 0$, or, after algebra, if and only if
\[-u''\eta \left( \int_0^x z_x \right)^2 \left( \frac{\eta \int_0^x z_x}{\eta \int_0^x z_x} \right) - \frac{2}{\eta} \left( \frac{\eta \int_0^x z_x}{\eta \int_0^x z_x} \right) \geq 0.
\]
This condition is satisfied if $P(w) \geq 2/w$ for all $w$. But then, by Jensen’s inequality, $\int m(\eta(\alpha), x, \theta^*) dL(\alpha) \geq m(\int \eta(\alpha) dL(\alpha), x, \theta^*)$, which is what we wanted to prove. \hfill \square

References


\footnote{Note that this part only uses that $f_x$ is log-spm in $(x, t)$ for each $\alpha$ and $y$, and in $(y, t)$ for each $\alpha$ and $x$.}


B  Online Appendix: Theory

B.1 Example with Multiplicity

At the end of Section 3, we discussed the mechanism by which risk aversion and complementarities in the wage function between \( x \) and \( \theta^* \) can lead to multiple equilibria. We now present a simple but robust example where, by a suitable choice of \( c \), there are multiple equilibria.

Let \( Q(\theta) = \theta, G(y) = y, H_0(x) = x, H_1(x) = x^\beta, \beta > 1, f(x, y) = xy \) (we omit the shock \( \alpha \) in this example but it can easily be introduced), \( u(w) = ((b + w)^{1-s} - 1)/(1-s), b > 0, \) and \( c(\theta) = \max\{-k \log \theta, \lambda(1-\theta)\}, \) with \( k > 0 \) and \( \lambda > 0 \).

It follows that \( \mu(x, \theta^*) = G^{-1}(\theta^*H_0(x) + (1-\theta^*)H_1(x)) = \theta^*x + (1-\theta^*)x^\beta \) and, since \( f_s(x, \mu(x, \theta^*)) = \mu(x, \theta^*) \), the wage function \( w \) is given by

\[
w(x, \theta^*) = \int_0^x \mu(s, \theta^*)ds = \theta^*x^2/2 + (1-\theta^*)x^{1+\beta}/\beta.
\]

Since \( U_1 - U_0 = \int u'w_2 \Delta H dx \), where \( \Delta H(x) = x - x^\beta \) and \( w_2(x, \theta^*) = \mu(x, \theta^*) \), we have

\[
U_1(\theta^*) - U_0(\theta^*) = \int_0^1 \frac{\theta^*x + (1-\theta^*)x^\beta}{(b + \theta^*x^2/2 + (1-\theta^*)x^{1+\beta}/\beta)} \sigma(x - x^\beta) dx,
\]

and hence the equilibrium condition \( U_1(\theta^*) - U_0(\theta^*) = c(\theta^*) \) is

\[
\int_0^1 \frac{\theta^*x + (1-\theta^*)x^\beta}{(b + \theta^*x^2/2 + (1-\theta^*)x^{1+\beta}/\beta)} \sigma(x - x^\beta) dx = \max\{-k \log \theta^*, \lambda(1-\theta^*)\}.
\]

As we discussed in the text, once \( U_1 - U_0 \) fails to be globally increasing in \( \theta^* \), then there can be multiple equilibria for some choice of \( c \). Indeed, computing the left side with Mathematica shows that for \((\beta, \sigma, b)\) in a neighborhood of \((3.8, 2.8, 0.01)\), \( U_1 - U_0 \) is strictly decreasing in \( \theta^* \) on \([0, 1]\). Since \( c \) is also strictly decreasing and goes to infinity as \( \theta^* \) vanishes and to 0 as \( \theta^* \) goes to 1, one can find values for \( k \) and \( \lambda \) such that multiple crossings occur. Indeed, Figure 5 produced with Mathematica, shows that when \((k, \lambda)\) is in a neighborhood of \((7, 1.7)\) million, then there are three solutions to the equilibrium equation (with the pattern stable/unstable/stable), all in the interval \([0, 0.2]\).

B.2 Ex-Ante Inefficiency

We now analyze the solution of the planner’s problem discussed at the end of Section 3.2, where he can design transfers not only to ensure workers against the output shock but also against skill.

\(\text{Note that } c \text{ is differentiable everywhere except at the unique } \theta \text{ where the two elements in the max are equal. Since differentiability is not used for existence and multiplicity, this does not pose any problem.}\)
Figure 5: **Multiple Equilibria.** The figure depicts three equilibria in the parametrization described above. The blue curve is $U_1 - U_0$ while the yellow one is $c$.

risk. The planner chooses $\theta^*_p$ and a function $s$ that maps $(x, \alpha)$ into $\mathbb{R}$, namely, the consumption or “share” of the output $f$ given to a worker of characteristic $x$ when the shock is $\alpha$. Given the complementarities in the problem, the planner uses the positive assortative matching and hence $\mu(x, \theta^*_p) = G^{-1}(H(x, \theta^*_p))$, where as usual $H(x, \theta^*_p) = Q(\theta^*_p)H_0(x) + (1 - Q(\theta^*_p))H_1(x)$. We analyze the following problem:

$$\max_{\theta^*_p, s} \int_0^1 \int_0^1 u(s(x, \alpha))dH(x, \theta^*_p)dL(\alpha) - \int_{\theta^*_p}^1 c(\theta)dQ(\theta)$$

s.t. $\int_0^1 \int_0^1 (f(x, \mu(x, \theta^*_p), \alpha) - s(x, \alpha))dH(x, \theta^*_p)dL(\alpha) \geq \pi_0,$

where $\pi_0$, the aggregate profits that firms must at least obtain, parameterizes the Pareto frontier.

Let $\nu$ be the Lagrange multiplier of the constraint. It is clear that the constraint is binding since $u$ is strictly increasing. Optimizing point-wise with respect to $s$ for each $(x, \alpha)$ yield $u'(s) - \nu = 0$, $\forall (x, \alpha)$, and thus the planner wants full insurance for workers, as intuition suggests since firms are risk neutral. It then follows from the binding constraint that

$$\bar{s}(\theta^*_p, \pi_0) = \int_0^1 \int_0^1 f(x, \mu(x, \theta^*_p), \alpha)dH(x, \theta^*_p)dL(\alpha) - \pi_0.$$ 

Before continuing solving the problem, consider an equilibrium with threshold $\theta^*$ and aggregate
profits $\Pi(\theta^*)$. The planner can choose $\theta^*_p = \theta^*$, leave firms indifferent, and strictly improve worker’s utility since $\bar{s}$ provides both skill and shock insurance. This way the planner can strictly improve welfare, and even more so if $\theta^*_p$ is chosen optimally.

The optimal choice of $\theta^*_p$ given any $\pi_0$ solves

$$
\max_{\theta^*_p \in [0,1]} u \left( \int_0^1 \int_0^1 f(x, \mu(x, \theta^*_p), \alpha) dH(x, \theta^*_p) dL(\alpha) - \pi_0 \right) - \int_{\theta^*_p}^1 c(\theta) dQ(\theta).
$$

The first-order condition (for an interior solution) is

$$
u'(\bar{s}(\theta^*_p, \pi_0)) \left( \int \int f_y \mu \cdot dH \cdot dL + \int \int f q \Delta H dxdL \right) + c(\theta^*_p) q(\theta^*_p) = 0.
$$

Now, by integration by parts and using $\mu_x = h/g$, we obtain

$$
\int \int f q \Delta H dxdL = - \int \int f_x q \Delta H dxdL - \int \int f_y q \Delta H dxdL.
$$

Also, $\mu_{\theta^*} = q \Delta H/g$, and thus, after cancelling $q$ from both sides, the-first order condition becomes

$$
u'(\bar{s}(\theta^*_p, \pi_0)) \int_0^1 \int_0^1 f_x(x, \mu(x, \theta^*_p), \alpha) \Delta H(x) dxdL(\alpha) = c(\theta^*_p).
$$

which can be written as

$$
u' \left( \mathbb{E}[f(x, \mu(x, \theta^*_p), \alpha)] - \pi_0 \right) \int_0^1 \mathbb{E}_L \left[ f_x(x, \mu(x, \theta^*_p), \alpha) \right] \Delta H(x) dx = c(\theta^*_p),
$$

Comparing the planner’s first-order condition with the equilibrium condition

$$
\int_0^1 u' \left( \int_0^x \mathbb{E}_L[f_x(s, \mu(s, \theta^*), \alpha)] ds \right) \mathbb{E}_L[f_x(x, \mu(x, \theta^*), \alpha)] \Delta H(x) dx = c(\theta^*),
$$

we see that the main difference is that the planner provides also skill insurance. As we saw above, equilibrium is inefficient, driven by the absence of insurance provision. Intuitively, this difference disappears if workers are risk neutral ($u' = 1$), in which case the two conditions are the same.

As we mentioned in the last paragraph of Section 3.2, a trade-off between insurance and stability of matches emerge in some intuitive decentralized versions of an ex-ante equilibrium. Assume that workers and firms match based on $y$ and $\theta$, and then each pair decides whether to invest. Equilibrium is described by two thresholds and two wages. The thresholds are $(\theta^*, y^*)$, so that firms with $y \geq y^*$ match with workers with $\theta \geq \theta^*$ who invest, and firms with $y < y^*$ match with workers with $\theta < \theta^*$ who do not invest. The two wages are $\bar{w}$ for those workers who invest and $w$ for those who do not invest. The system of equations that define an equilibrium are
(i) the indifference condition for \( y^* \), which equalizes the payoff from matching with a worker who invest and the payoff from matching with one who does not, which after integration by parts is 
\[
\int \int f_x(x, y^*, \alpha) \Delta HdxL = \bar{w} - w, 
\]
(ii) the indifference condition for \( \theta^* \), given by \( u(\bar{w}) - u(w) = c(\theta^*) \), and (iii) the market clearing condition \( Q(\theta^*) = G(y^*) \). Note that we can fix \( w \) and solve the system for \( (\bar{w}, \theta^*, y^*) \). For each \( w \in [0, \int \int f(x, 1, 1)dH_0(x)] \), there is a unique solution for the other three unknowns.\(^{[43]}\) Although workers are fully insured, once skills are realized, there are incentives on both sides to rematch to exploit the complementarities in the match output. Hence, there are blocking pairs and thus this ex-ante equilibrium with skill insurance is not stable.

### B.3 Ex-Ante Homogeneous Workers

Instead of assuming that each worker has an ability level \( \theta \) and that investment is binary, we asserted in Remark\([I]\) that we could assume that all workers are homogeneous ex-ante and investment is a continuous variable \( a \in [0, 1] \), with disutility cost function \( \kappa \) given by \( \kappa(a) \) for all \( a \), with \( \kappa(0) = \kappa'(0) = 0, \kappa' > 0, \kappa'' > 0 \) for \( a > 0 \), and \( \lim_{a \to 1} \kappa'(a) = \infty \). Now investment \( a \) indexes the distribution \( H(\cdot|a) \), which has a common support for all \( a \), and with \( H(x|\cdot) \) decreasing in \( a \) so that higher investment shifts \( H \) in the FOSD sense. We will show that this variation with ex-ante homogeneous workers and continuous investment yields essentially the same insights as our binary case with ex-ante heterogeneous workers.

We will consider the no-risk-sharing case in detail and describe the changes needed for the risk-sharing case. Since all workers are ex-ante homogeneous, we focus on symmetric equilibria where all of them invest the same amount. Thus, for each \( a \) there is a unique competitive equilibrium \((\mu, w)\), with \( \mu \) given by \( \mu(x, a) = G^{-1}(H(x|a)) \) for all \( x \), and \( w(x, \alpha, a) = \int_x^1 f_x(s, \mu(s, a), \alpha)ds \).

Consider the first stage when all workers conjecture that everyone invests \( a \). Then any given worker chooses \( \hat{a} \) to maximize
\[
\max_{\hat{a}} \int_0^1 \int_0^1 u(w(x, \alpha, a))dH(x|\hat{a})dL(\alpha) - \kappa(\hat{a}).
\]

The first-order condition for an interior solution is \( \int \int udH_a dL - c'(\hat{a}) = 0 \), which by integration by parts can be written as \( \int \int u'w_x(-H_0)dxdL = \kappa'(\hat{a}) \). To make the objective function concave in \( \hat{a} \), we assume that \( H_{aa} \geq 0 \), which holds if \( H(x|a) = (1 - a)H_0(x) + aH_1(x) \) and thus \( H_a = -\Delta H \).\(^{[44]}\)

The equilibrium investment level \( a^* \) solves
\[
\int_0^1 \int_0^1 u'(w(x, \alpha, a^*))w_x(x, \alpha, a^*) (-H_a(x|a^*)) dxdL(\alpha) = \kappa'(a^*).
\]

\(^{[43]}\)For example, if \( \bar{w} = 0 \), then from (ii)–(iii) we obtain \( \bar{w} = u^{-1}(c(\theta^*)) = u^{-1}(c(Q^{-1}(G(y^*))) \), which is decreasing in \( y^* \). Since the left side of (i) is increasing in \( y^* \), it follows that \( \int \int f_x(x, y^*, \alpha) \Delta HdxL = u^{-1}(c(Q^{-1}(G(y^*))) \) has a unique solution, which then pins down \( \theta^* \) and \( \bar{w} \).

\(^{[44]}\)Much weaker assumptions involving also \( \kappa \) and \( u \) are possible (see Chade and Swinkels (2020)).
Existence of a stable equilibrium, where now the left side crosses \( \kappa' \) from above, follows as in the main text, given the boundary conditions on \( \kappa' \) and the continuity of the left side in \( a^* \). As before, there can be multiple equilibria, which now can happen if the left side is not globally decreasing in \( a^* \). Finally, uniqueness obtains if \( R \) is uniformly small or if \( \kappa' \) is sufficiently convex.

Consider efficiency. The derivative of aggregate profits \( \int \int (f - w) dH dL \) with respect to \( a \) is
\[
- \int \int w_a dH dL > 0 \text{ (since } w_a < 0) \text{.}
\]
Take any equilibrium \( a^* \) and let \( \bar{H}(a^*) = \int \int (f(x, \mu(x, a^*)), \kappa - w(x, \alpha, a^*)) dH(x|a^*) dL(\alpha) \). Then to improve firms’ aggregate profits one needs \( a \geq a^* \). Regarding workers’ aggregate utility, \( \int \int udHdL - \kappa(a) \), its derivative with respect to \( a \) is
\[
\int \int u' w_a dH dL + \int \int u' w_x (-H_a) dxdL - \kappa'(a). \tag{16}
\]
Now, the first term is always negative, while the last two terms vanish at \( a^* \) (since this is the equilibrium condition in the competitive equilibrium). Thus, at \( a^* \) workers’ aggregate utility is strictly decreasing in \( a \). It then suffices to show that \( \int \int u' w_x (-H_a) dxdL - \kappa'(a) \leq 0 \) for all \( a > a^* \), because then (16) is strictly negative for \( a > a^* \), and efficiency ensues.

If equilibrium is unique, then \( \int \int u' w_x (-H_a) dxdL - \kappa'(a^*) = 0 \) and \( \int \int u' w_x (-H_a) dxdL - \kappa'(a) < 0 \) for \( a > a^* \), and similarly if \( a^* \) is the equilibrium with the highest investment. Hence, equilibrium is efficient in both of these cases, exactly as before.

Consider the comparative statics analysis. Note that the equilibrium condition \( \int \int u' w_x (-H_a) dxdL = \kappa' \) has the same functional form as in the text, except for \(-H_a\) instead of \( \Delta H \), which would be the case if \( H \) is linear in \( a \), and also for the presence of \( \kappa' \) instead of \( c \). Hence, the same conditions for the shifts in \( L, G, \) or \( f_x \) apply to this case as well (Propositions 3–6 and 9). Regarding the shift in \( H_1 \), consider the special case where \( H(x|a, t) = (1 - a) H_0(x) + a H_1(x|t) \), which is analogous to the binary case. Then \( \partial^2 H/\partial a \partial t = \partial H_1/\partial t \leq 0 \). So assuming that \( H \) is submodular in \( (a, t) \) for each \( x \) allow us to adapt Proposition 6(iii) to this case.

In the risk-sharing case, the only difference up to this point is that \( \bar{f} \) and \( \bar{w} \) replace \( f \) and \( w \). The same remarks as above apply to existence, uniqueness, efficiency, and comparative statics.

### B.4 Reinterpretation of \( \alpha \) as Idiosyncratic Income Shock

Suppose that \( \alpha \) is now idiosyncratic, uninsurable, multiplicative, and realized after the agent matches with a firm. A worker with \( x \) obtains a wage \( w(x, \theta^*) = \int_0^x f_x(s, \mu(s, \theta^*)) ds \) and his actual income is \( \omega(x, \theta^*, \alpha) = w(x, \theta^*) \eta(\alpha) \), where \( \eta \) is positive and strictly increasing.

The analysis of existence and uniqueness are unaffected by this change since the driving forces in Proposition 1 are risk aversion and complementarities of \( w \) in \( x \) and \( \theta^* \), which are present here as well. The same holds for the efficiency analysis, which does not depend on whether the shock enters production or income. Note that now firm profits are, for each \( y = \mu(x) \), given by \( f(x, \mu(x, \theta^*)) - w(x, \theta^*) \), which do not depend on \( \alpha \).
Comparative statics results are similar to those in the no-risk-sharing case with a separable class \( f = \eta z \). Consider a FOSD shift in \( L \). From \( U_1 - U_0 = \int \int u'\omega x \Delta H dx dL \), where \( \omega_x = w_x \eta \), we obtain that a sufficient condition is that \( u'\eta \) is increasing in \( \alpha \), or

\[
\begin{align*}
u''w\eta' + u'\eta' &= u'\eta'(1 - Rw\eta) = u'\eta'(1 - R) \geq 0.
\end{align*}
\]

It is clear that this holds if \( R \) is uniformly small or if \( R \leq 1/\omega \).

Consider an IR shift in \( L \). Then a sufficient condition is \( u'\eta \) convex in \( \alpha \), or

\[
\begin{align*}
u'''w^2\eta'\eta'^2 + u''w\eta\eta'' + u''w(\eta')^2 + u'\eta'' \geq 0.
\end{align*}
\]

This can be written as follows

\[
Ru'w(\eta')^2\omega \left( P - \frac{1}{\omega} \left( 2 + \frac{\eta''}{(\eta')^2} \right) \right) + u'\eta'' \geq 0.
\]

Hence, \( P \geq 3/\omega \) and \( \eta \) convex and log-concave suffice for the IR comparative statics.

The comparative statics with respect to \( H_1 \) and \( G \) are the same as before as well.

### B.5 Remarks on Integration and Differentiation

At several points we have interchanged differentiation and integration, and used integration by parts. We now discuss the assumptions on primitives that justify these operations.

Note that if \( u \) were three times continuously differentiable for all values of \( w \), then a continuous differentiability condition on \( H_1 \) with respect to \( t \) on \([0, 1]^2\) and the same for \( L \) with respect to \( t \) on \([0, 1]^2\) would suffice to justify all the instances where we use integration by parts and differentiate through the integral (this follows from straightforward applications of Theorems 16.8 (ii) and 18.4 in [Billingsley (1995)]

But if the derivatives of \( u \) diverge at \( w(0, \alpha) = 0 \), which we allow for so as to cover cases like CRRA, then some discussion is in order.

For each \((\alpha, \theta^*) \in [0, 1]^2, u(w(-, \alpha, \theta^*)) \) is absolutely continuous on \([0, 1]\). This is because of the assumptions on \( u \) and the fact that the wage function \( w \) given by \( w(x, \alpha, \theta^*) = \int_0^x f_x(s, \mu(s, \theta^*), \alpha) ds \), is continuously differentiable in \( x \). Hence, we can apply integration by parts (Billingsley [1995] Theorem 18.4) to obtain \( \int u(w)dH_1 - \int u(w)dH_0 = \int u'(w)w_x \Delta H dx \), which we used repeatedly in the text. And if \( L \) has a density \( l \) (so \( L \) is absolutely continuous on \([0, 1]\)), we can also integrate by parts \( \int u(w)dL \) to obtain \( u(w)|_{\alpha=1} - \int u'(w)w_\alpha dL \). Then it is straightforward to justify the interchange of differentiation and integration in most of our results, including those for uniformly small risk aversion since we assume that \( R \) is bounded and thus integrable.

Also, when we pass the derivative through the integral and the integral contains the derivatives of \( u \) and \( w \), we can justify it with Theorem 16.8 (ii) in [Billingsley (1995)], whose conditions only need to hold outside a set of measure zero. Since we allow \( u', u'' \), and \( u''' \) to diverge to infinity
at \( w(0, \alpha) \), this occurs on a set of values of \((x, \alpha)\) of measure zero (given our assumptions on \( H_i \), \( i = 0, 1, \) and \( L \)), and outside that set the derivatives with respect to \( \alpha \) or \( \theta^* \) or \( t \) (depending on the case under consideration) exist, and their integrability with respect to the probability measure over \((x, \alpha)\) can be fulfilled. In turn, the derivatives of the wage function are all integrable under our assumptions. For instance, \( w_{\theta^*} = \int_0^x f_{xy}(\Delta H/g)ds \) is finite as \( f_{xy} \) is continuous and thus bounded, \( \Delta H \leq 1 \), and \( g \) is positive on \([0, 1]\) and thus bounded too. Similarly for \( w_\alpha \) and the cross partials \( w_{x\theta^*} \) and \( w_{x\alpha} \). In cases where \( \partial H_1/\partial t \) appears, as in \( w_t \) and \( w_{xt} \), it suffices that \( H_1 \) be continuously differentiable in \( t \) on \([0, 1]\) for it to be integrable. Similarly for \( L_t, G_t, \) and \( Q_t \), which appear in a couple of instances in the comparative statics analysis.

### B.6 IR Shift in \( G, Q, H_1 \)

In some applications it may be of interest to analyze an increase in firm productivity dispersion, or in the variance of college completion risk across educational institutions. These questions call for IR shifts in \( G \) or \( H_1 \), which are more difficult to sign due to the changes in sign of the relevant change in the distribution when \( t \) changes.

We will focus on \( G \) and \( H_1 \) since an IR shift in \( Q \) is ambiguous and can increase or decrease \( \theta^* \) depending on where the initial equilibrium is located, so it is unclear what the natural comparative statics result is. All one can say is that if \( c \) is sufficiently convex so that \( \theta^* \) is close to zero, then an IR shift in \( Q \) decreases \( \theta^* \) if risk aversion is small enough.\(^{45}\)

The natural comparative statics result to seek in the case of \( G \) and \( H_1 \) is that an IR shift increases the number of workers who invest. In the case of \( G \) we will see that \( w_{xt} \) is strictly positive for large \( x \), which increases the incentives to invest. And in the case of \( H_1 \), we will see that the direct effect is positive under mild conditions, and we would expect that the indirect equilibrium effect via the matching will not offset it. This is the content of the next proposition, which provides primitives under which these comparative statics results hold.

**Proposition 10 (IR Shift in \( G, H_1 \))** In any stable equilibrium of either the NRS case or the RS case, the following results hold:

1. More workers invest in skills in response to an IR shift in \( G \) if \( f_{xy} \) is increasing in \( x \) and \( y \) for each \( \alpha \) and strictly so on a positive-measure set, either \( \min_{x,y,\alpha}(f_{xxy}/f_{xy}) \) or \( \min_{x,y,\alpha}(f_{xyy}/f_{xy}) \) is sufficiently large, \( h_0(0) > h_1(0) \), and \( R \) is (uniformly) sufficiently small;
2. More workers invest in skills in response to an IR shift in \( H_1 \) if \( f_{xx} \geq 0, f_{xy} \) decreases in \( x \) and \( y \) for each \( \alpha \), \( h_1(0) > h_0(0)/2, \min_y(g'/g^2) \) is sufficiently large, and \( R \) is (uniformly) sufficiently small.

\(^{45}\)To see the general ambiguity, note that when differentiating \( U_1 - U_0 \) with respect to \( t \), \( Q_t \) drops out of the integral (it does not depend on \( x \) or \( \alpha \)), and by definition of IR, it changes sign starting from \( Q_t \geq 0 \) near \( \theta^* = 0 \).
Proof. (i) Consider NRS. The result follows if under the stated conditions \( \int (u''w_x w_t + u'w_{xt}) \Delta H \, dx \geq 0 \) for each \( \alpha \). Since we will focus on \( R \) sufficiently small, we can ignore the term involving \( u'' \) and show that \( \int u'w_{xt} \Delta H \, dx = \int u'f_{xy}(\Delta H/g)(-G_t) \, dx > 0 \), where we have used \( w_{xt} = f_{xy} \mu_t \), and then appeal to the usual continuity argument.

Dividing and multiplying by \( \mu_x = h/g > 0 \), and integrating by parts we obtain

\[
\int u' f_{xy} \frac{\Delta H}{g} (-G_t) \, dx = \int u' f_{xy} \frac{\Delta H}{h} (-G_t \mu_x) \, dx = \int \left( \frac{d}{dx} \left( u' f_{xy} \frac{\Delta H}{h} \right) \right) \left( \int_0^x G_t \mu_x \, ds \right) \, dx,
\]

where the lead term in the integration by parts in the last expression is zero since \( \Delta H = 0 \) at \( x = 0, 1 \). Expanding the derivative in the integrand we obtain

\[
\int u' \left( -R f_x f_{xy} \frac{\Delta H}{h} + (f_{xy} + f_{xxy} \mu_x) \frac{\Delta H}{h} + f_{xy} \frac{1}{h^2} (h \Delta h - \Delta H(Qh_0' + (1 - Q)h_1')) \right) \left( \int_0^x G_t \mu_x \, ds \right) \, dx,
\]

which can be rewritten as follows:

\[
\int \frac{u' f_{xy}}{h} \left( -R f_x \Delta H + \left( f_{xy} + f_{xxy} \mu_x \right) \Delta H + \Delta h - \frac{\Delta H(Qh_0' + (1 - Q)h_1')}{h} \right) \left( \int_0^x G_t \mu_x \, ds \right) \, dx.
\]

Now, by IR, \( \int_0^y G_t(y|t) \, ds \geq 0 \) for all \( y \). Hence, by a change of variables we obtain

\[
0 \leq \int_0^y G_t(s|t) \, ds = \int_{\mu(0, \theta^*)}^{\mu(x, \theta^*)} G_t(s|t) \, ds = \int_0^x G_t(\mu(\tau, \theta^*)|t) \mu_x(\tau, \theta^*) \, d\tau,
\]

and hence the last term in the integrand above is positive for all \( x \) and is zero at \( x = 1 \).

Let \( h_1 = \max_{(x, \theta^*) \in [0, 1]^2} (Qh_0 + (1 - Q)h_1') \), which is finite since \( h_0' \) and \( h_1' \) are continuous. Let

\[ X = \{ x \in [0, 1] \mid \Delta h - (\Delta H h_1'/h) > 0 \}, \]

which is nonempty since \( h_0(0) > h_1(0) \) so it contains a right-neighborhood of \( x = 0 \). Define \( X^c = [0, 1]/X \).

Note that, by definition of \( X \),

\[
\int_X \frac{u' f_{xy}}{h} \left( \left( f_{xy} + f_{xxy} \mu_x \right) \Delta H + \Delta h - \frac{\Delta H(Qh_0' + (1 - Q)h_1')}{h} \right) \left( \int_0^x G_t \mu_x \, ds \right) \, dx > 0,
\]

where we ignored the term involving \( R \) since it will be made uniformly small. Since this is bounded away from zero for all \( \theta^* \), the minimum with respect to \( \theta^* \), denoted by \( \delta(\alpha) \), is strictly positive.

Since all the terms in the integrand are uniformly bounded and \( \int_0^1 G_t \mu_x \, ds = 0 \), it follows that there exists \( \varepsilon(\delta) > 0 \) such that

\[
C \int_{[1-\varepsilon(\delta), 1]/X} \left( \int_0^x G_t \mu_x \, ds \right) \, dx < \delta(\alpha), \quad (17)
\]
where
\[ C = \max_{x, \alpha, \theta^*} \left| \frac{u' f_{xy}}{h} \left( \left( \frac{f_{xxy}}{f_{xy}} + \frac{f_{xyy}}{f_{xy}} \mu_x \right) \Delta H + \Delta h - \frac{\Delta H(Qh'_0 + (1 - Q)h'_1)}{h} \right) \right| . \]

Hence, even if the left side of (17) (without the absolute values) is negative, it will not be less than \(-\delta(\alpha)\).

Finally, consider \([0, 1 - \varepsilon(\delta)]/X\). Then
\[
\int_{[0, 1 - \varepsilon(\delta)]/X} u' \frac{f_{xy}}{h} \left( \left( \frac{f_{xxy}}{f_{xy}} + \frac{f_{xyy}}{f_{xy}} \mu_x \right) \Delta H + \Delta h - \frac{\Delta H(Qh'_0 + (1 - Q)h'_1)}{h} \right) \left( \int_0^x G_t \mu_x ds \right) dx
\]
can be made positive if either \(\min(f_{xxy}/f_{xy})\) or \(\min(f_{xyy}/f_{xy})\) are sufficiently large, and this can be made uniform on \(x\) and \(\theta^*\). Hence, for every \(\alpha\), we can make the integral with respect to \(x\) strictly positive. Integrating over \(\alpha\) yields the result. For RS, replace \(w\) by \(\bar{w}\), delete the integral with respect to \(L\) (in the last step), and follow the same steps.

(ii) Consider NRS. The derivative of \(U_1 - U_0\) with respect to \(t\) is
\[
\int \int u'(w_{xt} - Rw_x w_t) \Delta H dxdL + \int \int u' w_x \left( -\frac{\partial H_1}{\partial t} \right) dxdL. \tag{18}
\]
Fix \(\alpha\) and ignore the integral with respect to \(L\). Also, ignore the term involving \(R\) since we will make it arbitrarily small at the end. Integration by parts and using that \(\int (\partial H_1/\partial t) dx = 0\) (constant mean) yields
\[
\int u' w_x \left( -\frac{\partial H_1}{\partial t} \right) dx = \int (u''w_x^2 + u'w_{xx}) \left( \int_0^x \frac{\partial H_1}{\partial t} ds \right) dx
\]
\[
= \int u' \left( f_{xx} + f_{xy} \frac{Qh_0 + (1 - Q)h_1}{g} \right) \left( \int_0^x \frac{\partial H_1}{\partial t} ds \right) dx
\]
\[
- \int u' Rf_x^2 \left( \int_0^x \frac{\partial H_1}{\partial t} ds \right) dx, \tag{19}
\]
where the first term in the last expression is positive since \(f_{xx} \geq 0\) and, by IR, \(\int_0^x (\partial H_1/\partial t) ds \geq 0\) for all \(x\). We will ignore the last term in (19) since it involves \(R\) and can be made arbitrarily
small. Similarly, if we integrate by parts \( \int u' w_{xt} \Delta H dx \) in the first term in (18), we obtain

\[
\int u' w_{xt} \Delta H dx = (1 - Q) \int \left( u' f_{xy} \frac{\Delta H}{g} \right) \frac{\partial H_1}{\partial t} dx
\]

\[= (1 - Q) \int u' R f_{xy} \frac{\Delta H}{g} \left( \int_0^x \frac{\partial H_1}{\partial t} ds \right) dx
\]

\[= (1 - Q) \int u' \left( f_{xyy} + f_{xy} \frac{Q h_0 + (1 - Q) h_1}{g} \right) \frac{\Delta H}{g} \left( \int_0^x \frac{\partial H_1}{\partial t} ds \right) dx
\]

\[- (1 - Q) \int u' f_{xy} \frac{\Delta h - \Delta H \frac{g' \Delta h}{g^2}}{g^2} \left( \int_0^x \frac{\partial H_1}{\partial t} ds \right) dx,
\]

where the second term after the last equality is positive by the premises and the first depends on \( R \) and thus can be made arbitrarily small.

It follows that for (18) to be positive, it suffices that

\[
\int u' f_{xy} \left( Q h_0 + (1 - Q) h_1 - (1 - Q) \left( \Delta h - \Delta H (Q h_0 + (1 - Q) h_1) \frac{g' \Delta h}{g^2} \right) \right) \left( \int_0^x \frac{\partial H_1}{\partial t} ds \right) dx
\]

is positive (where we pull together the second part of the second line of (19) with the last term in (20)), for which it is enough that the middle term of the integrand is positive, which after some algebra can be written as

\[(2Q - 1) h_0 + 2(1 - Q) h_1 + (1 - Q) \Delta H (Q h_0 + (1 - Q) h_1) \frac{g'}{g^2}.
\]

Note first that this is strictly positive for all \( \theta^* \) such that \( Q(\theta^*) \geq 1/2 \), or \( 0 \leq \theta^* \leq Q^{-1}(1/2) < 1 \). Now, at \( x = 0 \), \((2Q - 1) h_0(0) + 2(1 - Q) h_1(0) \geq -h_0(0) + 2h_1(0) > 0\), where the strict inequality follows by assumption. Hence, there is an \( x_0 \) such that \((2Q - 1) h_0(x) + 2(1 - Q) h_1(x) > 0\) on \([0, x_0]\) for all \( \theta^* \). At \( x = 1 \), since \( h_1(1) \geq h_0(1) \), we have that \((2Q - 1) h_0(1) + 2(1 - Q) h_1(1) \geq -h_0(1) + 2h_1(1) > 0\), and hence there is an \( x_1 \) such that \((2Q - 1) h_0(x) + 2(1 - Q) h_1(x) > 0\) on \([x_0, 1]\) for all \( \theta^* \). Letting \( \min_{x \in [x_0, x_1]} (Q h_0 + (1 - Q) h_1) \geq \max_{x \in [x_0, x_1], \theta^* \in [0, Q^{-1}(1/2)]} (|(2Q - 1) h_0 + 2(1 - Q) h_1|/((1 - Q) \Delta H (Q h_0 + (1 - Q) h_1))) \), and making \( R \) uniformly sufficiently small completes the proof. For RS, replace \( w \) by \( \bar{w} \), delete the integral with respect to \( L \) in (18), and follow the same steps. □

### B.7 Two-Sided Risk Aversion with CRRA

In main text we solved two benchmark cases of our model, one without risk-sharing and the other one with risk-sharing. Given the natural assumption of risk neutral firms, in the risk-sharing case firms fully insure workers against the shock \( \alpha \). As a robustness exercise, in this section we allow both sides of the market to be strictly risk averse and derive similar insights as in the main text. To make the analysis tractable, we assume that firms and workers share the same CRRA utility
function with parameter $\sigma$. This affords a closed form solution for the risk-sharing problem within a match, which is a key building block in the equilibrium derivation.

**Risk-sharing.** The risk-sharing problem between a worker with $x$ and a firm with $y$ when the firm must be given a level of utility $\bar{v}$ is

$$
\phi(x, y, \bar{v}) = \max_w \int_0^1 \frac{(w(\alpha))^{1-\sigma} - 1}{1-\sigma} dL(\alpha)
$$

subject to

$$
\int_0^1 \frac{(f(x, y, \alpha) - w(\alpha))^{1-\sigma} - 1}{1-\sigma} dL(\alpha) \geq \bar{v}.
$$

Let $\lambda$ be the multiplier of the constraint. Pointwise optimization and the binding constraint yields the following optimality conditions, whose unknowns are $\hat{w}$ for each $\alpha$, and $\lambda$:

$$
\frac{\hat{w}^{-\sigma}}{(f - \hat{w})^{-\sigma}} = \lambda \quad \forall \alpha
$$

$$
\int_0^1 \frac{(f - \hat{w})^{1-\sigma} - 1}{1-\sigma} dL = \bar{v}.
$$

These conditions yield

$$
\hat{w}(\alpha, x, y, \lambda) = \frac{1}{\lambda^{\sigma} + 1} f(x, y, \alpha) \quad \forall \alpha, \quad \lambda^* = \left( \frac{\frac{\bar{v}(1-\sigma) + 1}{f(x, y, \alpha)^{1-\sigma} dL(\alpha)}^{\frac{1}{1-\sigma}}}{1 - \left( \frac{\bar{v}(1-\sigma) + 1}{f(x, y, \alpha)^{1-\sigma} dL(\alpha)} \right)^{\frac{1}{1-\sigma}}} \right)^{\frac{1}{1-\sigma}},
$$

from which we obtain

$$
w^*(\alpha, x, y, \bar{v}) = \hat{w}(\alpha, x, y, \lambda^*(x, y, \bar{v})) = \left( 1 - \left( \frac{\bar{v}(1-\sigma) + 1}{f(x, y, \alpha)^{1-\sigma} dL(\alpha)} \right)^{\frac{1}{1-\sigma}} \right) f(x, y, \alpha),
$$

and hence (after simple algebra)

$$
\phi(x, y, \bar{v}) = \frac{1}{1-\sigma} \left( \left( \int (f(\alpha, x, y))^{1-\sigma} dL \right)^{\frac{1}{1-\sigma}} - (\bar{v}(1-\sigma) + 1)^{\frac{1}{1-\sigma}} \right)^{\frac{1-\sigma}{1-\sigma}} - 1 \right).
$$

**Sorting.** Set $F = (\int f^{1-\sigma} dL)^{\frac{1}{1-\sigma}}$ and $\bar{\pi} = (\bar{v}(1-\sigma) + 1)^{\frac{1}{1-\sigma}}$; then it is easy to see that the problem is TU-representable and hence positive sorting obtains if and only if $F$ is spm in $(x, y)$. Henceforth, we will assume that $(\int f^{1-\sigma} dL)^{\frac{1}{1-\sigma}}$ is spm in $(x, y)$, which can be justified from primitives. For example, if $f$ is a separable class $f(x, y, \alpha) = \eta(\alpha) z(x, y)$, then

$$
\left( \int (f(x, y, \alpha))^{1-\sigma} dL(\alpha) \right)^{\frac{1}{1-\sigma}} z(x, y) \left( \int (\eta(\alpha))^{1-\sigma} dL(\alpha) \right)^{\frac{1}{1-\sigma}},
$$

11
which is spm in \((x, y)\) if and only if \(z\) is.

**Walrasian Equilibrium.** Let \(\bar{v}(y, \theta^*)\) for all \(y\), be the utility of firms that workers take as given in the market. Then \(x\) solves \(\max_y \phi(x, y, \bar{v}(y, \theta^*))\), or

\[
\max_y \frac{1}{1-\sigma} \left( \left( \int (f(x, y, \alpha))^{1-\sigma} dL(\alpha) \right)^{\frac{1}{1-\sigma}} - (\bar{v}(y, \theta^*)(1-\sigma) + 1)^{\frac{1}{1-\sigma}} - 1 \right)
\]

\[
= \max_y \frac{1}{1-\sigma} \left( (F(x, y, \sigma) - \bar{\pi}(y, \theta^*))^{1-\sigma} - 1 \right)
\]

\[
= \max_y F(x, y, \sigma) - \bar{\pi}(y, \theta^*),
\]

where \(\bar{\pi}\) is the amount of profit that solves \(\bar{v}(y, \theta^*) = ((\bar{\pi}(y, \theta^*))^{1-\sigma} - 1)/(1-\sigma)\) for all \(y\). It follows by standard arguments that

\[
\bar{\pi}(y, \theta^*) = \int_0^y F_y(\mu^{-1}(s, \theta^*), s, \sigma) ds,
\]

where \(\mu\) is given by \(\mu(x, \theta^*) = G^{-1}(H(x, \theta^*))\) for all \(x\), and \(\mu^{-1}\) is its inverse.

Using the same argument as in the full insurance case, it follows that if we solve the problem of \(y\) taking as given \(\bar{w}(x, \theta^*)\) for all \(x\), then

\[
\bar{w}(x, \theta^*) = \int_0^x F_x(s, \mu(s, \theta^*), \sigma) ds,
\]

and hence, for each \(\theta^* \in [0, 1]\), \((\bar{w}(\cdot, \theta^*), \bar{\pi}(\cdot, \theta^*), \mu(\cdot, \theta^*))\) is the unique Walrasian equilibrium.

**Comparative Statics.** We will explore conditions for a FOSD or IR shift in \(L\) to increase the measure of workers who invest. As in the main text, it suffices to show that \(U_1(\theta^*) - U_0(\theta^*) = c(\theta^*)\), and existence and uniqueness follows as in Proposition 1 in the text.

\[\text{As usual, the equilibrium condition is } U_1(\theta^*) - U_0(\theta^*) = c(\theta^*), \text{ and existence and uniqueness follows as in Proposition 1 in the text.}\]

\[\text{Comparative Statics. We will explore conditions for a FOSD or IR shift in } L \text{ to increase the measure of workers who invest. As in the main text, it suffices to show that } U_1 - U_0 \text{ increases in } t \text{ for each } \theta^*. \text{ By integration by parts,}\]

\[
U_1 - U_0 = \int_0^1 \left( \int_0^x F_x ds \right)^{-\sigma} F_x \Delta H dx.
\]

\[\text{A minor point is that the utility is unbounded at } 0 \text{ if } \sigma \geq 1 \text{ and hence is not continuous at that point. As in footnote 21, this can easily be fixed by adding an initial wealth } b > 0 \text{ to each agent on each side of the market.}\]
It suffices that $F_x / (\int_0^x F_x ds)^\sigma$ increases in $t$. Using the expression for $F$ one can show that

$$
\frac{F_x}{(\int_0^x F_x ds)^\sigma} = \left( \frac{\int f^{1-\sigma} dL}{\int_0^x f_x ds} \right)^{\frac{\sigma}{\alpha}} \int f^{-\sigma} f_x dL
$$

(22)

Note that if $\sigma = 0$ then the denominator is equal to one and the numerator reduces to $\int f_x dL$, and hence it increases in $t$ if $f_x$ has the right properties as a function of $\alpha$ (increasing or convex depending on the shift). If (22) strictly increases for $\sigma = 0$, then the same holds for $\sigma$ sufficiently small, so the small risk aversion results in FOSD or IR shifts of $L$ hold in this case as well.

For a sharper result, let $f$ be a separable class $f = \eta z$. Then, inserting $f = \eta z$ into the right side of (22) and manipulating yields

$$
\frac{F_x}{(\int_0^x F_x ds)^\sigma} = \frac{z_x}{(\int_0^x z_x ds)^\sigma} \int \eta^{1-\sigma} dL,
$$

(23)

and the comparative statics result reduces to signing how $\int \eta^{1-\sigma} dL$ depends on $t$.

If there is a FOSD shift in $L$, then more workers invest if $\sigma \in (0,1)$ since $\eta' > 0$. Note that this is the same condition as in the case with risk neutral firms under the separable $f$, since the log-supermodularity conditions on $f$ in Proposition (ii) are satisfied when $f$ is a separable class.

If there is an IR shift in $L$, however, two-sided risk aversion makes a difference. First, note that we are done if $\eta^{1-\sigma}$ is convex in $\alpha$. Since the sign of the second derivative of $\eta^{1-\sigma}$ equals the sign of $(1 - \sigma)(\eta'' \eta - \sigma(\eta')^2)$, it follows that the result holds if either (a) $\sigma \in (0,1)$ and $\eta$ log-convex, or (b) $\sigma > 1$ and $\eta$ is log-concave (since $1 - \sigma < 0$ and $\eta'' \eta - (\eta')^2 \leq 0$ implies $\eta'' \eta - (\eta')^2 < 0$ when $\sigma > 1$). Result (b) requires that workers be sufficiently prudent as before, while result (a) is new and owes to risk aversion of firms, which now affects the properties of the wage function and provides more incentives to invest even with low levels of prudence.

**Remark.** Assume $t$ shifts $L$. In the no-risk-sharing case with risk neutral firms, $w$ is independent of $t$ but depends on $\alpha$. Hence, only the direct effect of $t$ on $L$ is relevant. In the risk-sharing case with risk neutral firms, $w$ is independent of $\alpha$ but depends on $t$. Hence, only the indirect effect of $t$ on $L$ via $w$ is relevant. In the two-sided risk averse case, both effects are relevant, and comparative statics are in general intractable if the model is not TU-representable (in that case $t$ enters $w$ via $L$ and also via the utility given to a partner). In the CRRA case (with common risk aversion parameter), however, the model is TU-representable, and $t$ enters $L$ directly, and indirectly via $w$ (but utility of a partner does not enter $w$). Moreover, the expressions simplify drastically and one can conduct the analysis in a tractable way as above. The conjecture is that this holds more generally in any TU-representable problem (for example, in the class of hyperbolic

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47 One can show that we need to sign $F_{xt} \int_0^t F_x - \sigma F_x \int_0^t F_{xt}$; at $\sigma = 0$, $F_{xt} > 0$ under suitable conditions on $f_x$, and hence by continuity it remains positive for $\sigma$ sufficiently small.
absolute risk aversion, HARA, with common parameters for workers and firms).

Regarding the effects of a FOSD shift in $G$ or $Q$, Proposition 6 easily extends to this case. Consider for simplicity the separable class: it follows from (23) that a shift in $G$ or in $Q$ affects the ratio \( z_x/(\int_0^x z_x ds)^\sigma \) via the effect on the matching function. Differentiating this ratio with respect to $t$ reveals that the sign of the derivative equals the sign of

\[
z_{xy}\mu_t - \frac{\sigma}{\int_0^x z_x ds} z_x \int_0^x z_{xy} \mu_t ds,
\]

which has the same functional form as in the case analyzed in Proposition 6. Under a FOSD shift in $G$, $\mu_t \geq 0$ and strictly so on a set of positive measure. Note that the expression is positive for $\sigma = 0$, and strictly so on a positive-measure set. By continuity it holds for $\sigma > 0$ and sufficiently small. Similarly with a FOSD shift in $Q$ but with $\mu_t \leq 0$.

A FOSD shift in $H_1$ increases the measure of workers who invest under the same conditions as in Proposition 6. To see this, note that the analogue of (15) is

\[
-\frac{\partial H_1}{\partial t} \left( z_x - \Delta H z_{xy} \frac{1 - Q}{g} \right) + \sigma \frac{\left( \int_0^x z_{xy} \frac{1 - Q}{g} \left( -\frac{\partial H_1}{\partial t} \right) \right) z_x}{\left( \int_0^x z_x ds \right)}.
\]

where the only difference with (15) is that the derivatives of $f$ are replaced by those of $z$ and $b$ is replaced by $\sigma$. Following the same steps as in Proposition 6 proves the result.

Regarding the shift in $f$, the same conditions as in Proposition 9 work in this case. To see this, index $z$ by $t$ and consider the derivative of the ratio $z_x/(\int_0^x z_x ds)^\sigma$ with respect to $t$. It has the same sign as

\[
\frac{z_{xt} \int_0^x z_x ds}{z_x \int_0^x z_{xt} ds} - \sigma,
\]

and thus under the log-spm assumptions in the proposition, the first term is bigger than one, and hence an upward shift in $z$ increases the number of workers who invest if $\sigma \in (0, 1)$.

**Efficiency.** Consider the efficiency analysis of equilibrium, in which given the wage and matching function we check whether there is an alternative $\theta^*_\ell$ that can improve efficiency. As before, let $\theta^*_\ell$ be the lowest equilibrium threshold (the one with the largest measure of workers investing), and let $\mathcal{U}(\theta^*_\ell)$ and $\mathcal{V}(\theta^*_\ell)$ be the aggregate utility of workers and firms, respectively.

For any threshold $\theta^*$, the aggregate utility of firms is

\[
\int_0^1 \left( F(x, \mu(x, \theta^*), \sigma) - \int_0^x F_x(s, \mu(s, \theta^*), \sigma) ds \right) \frac{1 - \sigma}{1 - \sigma} dH(x, \theta^*),
\]

which, for simplicity, we write it as $\int v(F - \bar{w})dH$. Differentiating with respect to $\theta^*$, integrating
by parts, and using $\bar{w}_x = F_x$, $\mu_{\theta^*} = q\Delta H / g$, and $\mu_x = h / g$, yields

$$
\int v'(F_y \mu_{\theta^*} - \bar{w}_{\theta^*})dH + \int vq\Delta h dx = \int v'(F_y \mu_{\theta^*} - \bar{w}_{\theta^*})dH - \int v'(F_x + F_y \mu_x - F_x)q\Delta H dx
$$

$$
= -\int v'\bar{w}_{\theta^*}dH < 0.
$$

Hence, only $\theta^* < \theta^*_L$ can strictly improve the welfare of firms over $\mathcal{V}(\theta^*_L)$.

Consider now the aggregate utility of workers for any threshold $\theta^*$:

$$
\int_0^1 \left( \frac{\int_0^1 F_x(s, \mu(s, \theta^*), \sigma)ds}{1 - \sigma} \right)^{1-\sigma} dH(x, \theta^*) - \int_0^1 c(\theta)dQ(\theta),
$$

which, for simplicity, we write it as $\int u(\bar{w})dH - \int_0^1 c(\theta)dQ(\theta)$. Differentiating this expression with respect to $\theta^*$ and integrating by parts yields

$$
\int u'\bar{w}_{\theta^*}dH - \int u'\bar{w}_x q\Delta H dx + c(\theta^*)q(\theta^*).
$$

The first term is strictly positive, while the last two vanish at $\theta^*_L$. Moreover, $-\int u'\bar{w}_x \Delta H dx + c(\theta^*) \leq 0$ for all $\theta^* < \theta^*_L$, and thus workers’ aggregate utility strictly increases in $\theta^*$ on $[0, \theta^*_L]$. Hence, the equilibrium at $\theta^*_L$ is efficient, and the same holds if equilibrium is unique.

Consider now the planner’s problem in which the planner can provide both skill and shock insurance by choosing $\theta^*_p$ and a function $s$ that maps $(x, \alpha)$ into $\mathbb{R}$. The planner again uses positive sorting and hence $\mu(x, \theta^*_p) = G^{-1}(H(x, \theta^*_p))$, where as usual $H(x, \theta^*_p) = Q(\theta^*_p)H_0(x) + (1 - Q(\theta^*_p))H_1(x)$. We analyze the following problem:

$$
\max_{\theta^*_p, s} \int_0^1 \int_0^1 \left( \frac{s(x, \alpha)}{1 - \sigma} \right)^{1-\sigma} dH(x, \theta^*_p)dL(\alpha) - \int_0^1 c(\theta)dQ(\theta)
$$

$$
\text{s.t. } \int_0^1 \int_0^1 \left( \frac{f(x, \mu(x, \theta^*_p), \alpha) - s(x, \alpha)}{1 - \sigma} \right)^{1-\sigma} dH(x, \theta^*_p)dL(\alpha) \geq v_0,
$$

where $v_0$, the aggregate profits that firms must at least obtain, parameterizes the Pareto frontier.

Let $\nu$ be the Lagrange multiplier of the constraint, which binds at the optimum. Pointwise maximization yields, for each $(x, \alpha)$, the optimality condition $s^{-\sigma}/(f - s)^{-\sigma} = \nu$. Using this condition and the binding constraint, we obtain after some algebra, the optimal sharing rule

$$
s(x, \alpha, v_0, \theta^*_p) = \left(1 - \frac{v_0(1 - \sigma) + 1}{\int \int (f(x, \mu(x, \theta^*_p), \alpha))^{1-\sigma} dH(x, \theta^*_p)dL(\alpha)} \right)^{\frac{1}{1-\sigma}} f(x, \mu(x, \theta^*_p), \alpha).
$$
Inserting into the objective function and simplifying yields

$$\max_{\theta_p^* \in [0,1]} \frac{1}{1 - \sigma} \left( \left( \int \int (f(x, \mu(x, \theta_p^*), \alpha))^{1-\sigma} dH(x, \theta_p^*)dL(\alpha) \right)^{\frac{1}{1-\sigma}} - \pi_0 \right)^{1-\sigma} - 1 - \int_{\theta_p^*}^1 c(\theta)dQ(\theta),$$

where $\pi_0 = (v_0(1 - \sigma) + 1)^{1/(1-\sigma)}$.

Before obtaining the first-order condition that pins down $\theta_p^*$, consider an equilibrium with threshold $\theta^*$ and aggregate firms’ utility $V(\theta^*)$. The planner can choose $\theta_p^* = \theta^*$, leave firms indifferent, and strictly improve worker’s utility since $\bar{s}$ provides both skill and shock insurance. This way the planner can strictly improve welfare, and even more so if $\theta_p^*$ is chosen optimally. In this ex-ante sense, any equilibrium is inefficient.

Differentiating with respect to $\theta_p^*$ yields

$$\frac{1}{1-\sigma} \left( \int \int f^{1-\sigma} dHdL \right)^{\frac{\sigma}{1-\sigma}} \left( \int \int (1 - \sigma) f^{-\sigma} f_y \mu_\theta dHdL + \int \int f^{1-\sigma} q \Delta dx dL \right) + c(\theta_p^*)q(\theta_p^*) = 0$$

Integrating by parts $\int \int f^{1-\sigma} \Delta dx dL$ as before reveals that the expression in the big parentheses in the first term equals $-\int \int (1 - \sigma) f^{-\sigma} f_x q \Delta dx dL$ and hence we obtain

$$\left( \int \int f^{1-\sigma} dHdL \right)^{\frac{1}{1-\sigma}} - \pi_0 \left( \int \int f^{1-\sigma} dHdL \right)^{\frac{\sigma}{1-\sigma}} \left( \int \int (1 - \sigma) f^{-\sigma} f_x q \Delta dx dL \right) = c(\theta_p^*)q(\theta_p^*).$$

Using $\int f^{1-\sigma} dL = F^{1-\sigma}$ on the left side of this expression, we obtain

$$\frac{\left( \int F^{1-\sigma} dH \right)^{\frac{\sigma}{1-\sigma}}}{\left( \int F^{1-\sigma} dH \right)^{\frac{\sigma}{1-\sigma}} - \pi_0} \int F^{-\sigma} F_x q \Delta H dx = c(\theta_p^*)q(\theta_p^*), \tag{24}$$

which differs from the equilibrium one

$$\int \left( \int_0^x F_x ds \right)^{-\sigma} F_x \Delta H dx = c(\theta^*).$$

For an illustration, assume that $\pi_0 = 0$, so that the planner essentially cares only about workers’ welfare. Then the first term on the left side of (24) is one. Also, note that $F(0, \mu(0, \theta^*), \sigma) = 0$ and thus by the Fundamental Theorem of Calculus

$$F(x, \mu(x, \theta^*), \sigma) = \int_0^x (F_x + F_y \mu_x) ds \geq \int_0^x F_x ds,$$

with strict inequality for $x > 0$. Thus, given any $\theta^*$, $\int F^{-\sigma} F_x \Delta H dx < \int (\int_0^x F_x ds)^{-\sigma} F_x \Delta H ds$, which is implied by (24).
and so $\theta_0^* > \theta^*$. That is, the efficient threshold with skill insurance is higher than the equilibrium one, so there is overinvestment. This is intuitive since risk averse workers without skill insurance have more incentives to invest. Things are more complex when $\pi_0 > 0$ since in this case the first term on the right side is bigger than one, and over or underinvestment can ensue.

This completes the solution of the two-sided risk aversion case with CRRA utility function and common risk aversion parameter. Most of the insights derived using our two benchmark cases (risk-sharing and no-risk-sharing) with risk neutral firms go through here.

### B.8 Two-Sided Investments

A natural extension is to allow firms to invest as well prior to matching. For variety, we will explore this extension using the continuous investment model with ex-ante homogeneous workers of Section B.3, suitably extended to allow for firms’ investments. Let $e \in [0, 1]$ denote a firm’s investment level, $\xi(e)$ the disutility of $e$, with $\xi$ having the same properties as $\kappa$ (see Section B.3). Investment yields a stochastic return $y$, and we denote by $G(y|e)$ the probability of drawing an attribute less than $y$ if a firm invests $e$, where $G(y|\cdot)$ is twice continuously differentiable, with $G_e \leq 0$ and strictly so on a set of positive measure, and $G_{ee} \geq 0$.

The analysis is tractable for small risk aversion, so we will focus on this case. In fact, with some abuse, we will present the analysis for the risk neutral workers’ case, and then appeal to the usual argument by continuity, which extends the results for uniformly small absolute risk aversion. This will be true in both the no-risk-sharing and the risk-sharing cases.

The equilibrium conditions are:

\[
\int_0^1 \int_0^1 w_x(x, \alpha, a^*, e^*)(-H_a(x|a^*))dxdL(\alpha) = \kappa_a(a^*)
\]

\[
\int_0^1 \int_0^1 \pi_y(y, \alpha, a^*, e^*)(-G_e(y|e^*))dydL(\alpha) = \xi_e(e^*)
\]

(25) (26)

where $w_x = f_x$ and, from the definition of $\pi$, we obtain that $\pi_y = f_y$ (the terms $f_x \mu_y^{-1}$ and $w_x \mu_y^{-1}$ cancel out). Thus, if an equilibrium exists in this case, it is actually efficient in the risk neutral case (with risk neutral workers risk-sharing is irrelevant), since these are the first-order conditions of $\max_{a,e}(f - \kappa - \xi)$, which are necessary and sufficient. With small risk aversion it is efficient if equilibrium is unique.

Equation (25) defines a “best response” function $r_w$ given by $a^* = r_w(e^*)$ for workers, and (26) defines a similar function $r_f$ given by $e^* = r_f(a^*)$ for firms. Any intersection of $r_w$ and $r_f$ is an equilibrium; it is stable if $\partial r_w^{-1}/\partial a^* \geq \partial r_f/\partial a^*$ when they cross.

To show that a stable equilibrium exists, first note that, by differentiating these expressions,
At a stable equilibrium we obtain the following slopes:

\[
\begin{align*}
\frac{\partial r_w}{\partial e^*} &= -\frac{\int w_{xe}(-H_a)dxL}{\int w_{xa}(-H_a)dxL + \int w_x(-H_{aa}) - \kappa_{aa}} > 0 \\
\frac{\partial r_f}{\partial a^*} &= -\frac{\int \pi_y(-G_e)dydL}{\int \pi_y(-G_e)dydL + \int \pi_y(-G_{ee})dydL - \xi_{ee}} > 0,
\end{align*}
\]

where the strictly positive sign follows since the denominators are negative \((w_{xa} \leq 0, \kappa_{aa} \geq 0, \pi_{ye} \leq 0, \text{ and } \xi_{ee} \geq 0)\) and from the positive cross-partial \(w_{xe}\) and \(\pi_y\) in the numerators. Second, note that if \(e^* = 0\), then \(\int w_x(-H_a)dxL = \kappa_a\) has a positive solution \(a^* \in (0, 1)\). Thus, \(r_w^{-1}\) starts a positive value \(a^* > 0\). In turn, if \(a^* = 0\), then \(\int \pi_y(-G_e)dydL = \xi_e\) has a positive solution \(e^* \in (0, 1)\) and hence \(r_f\) has a positive intercept. Graphically, \(r_f\) is above \(r_w^{-1}\) at \(a^* = 0\). At the other end, if \(e^* = 1\), then \(a^* < 1\) given our assumptions on \(\kappa_a\). Similarly, if \(a^* = 1\), then \(e^* < 1\). Graphically, \(r_w^{-1}\) is above \(r_f\) at \(a^* = 1\). Since both best responses are continuous functions, by the Intermediate Value Theorem there is an equilibrium where \(r_w^{-1}\) crosses \(r_f\) from below, and hence a stable equilibrium exists.

Regarding comparative statics, let us differentiate \((25) - (26)\) with respect to \(t\) to obtain

\[
\begin{align*}
a_i^* &= -\frac{\partial}{\partial t} \left[ w_{xa}(-H_a)dxL B + \frac{\partial}{\partial t} \left[ \pi_y(-G_e)dydL \right] \right] A \\
e_i^* &= -\frac{\partial}{\partial t} \left[ \pi_y(-G_e)dydL \right] A + \frac{\partial}{\partial t} \left[ w_{xe}(-H_a)dxL \right] \frac{\int \pi_y(-G_e)dydL}{C},
\end{align*}
\]

where \(A, B, C\) are given by

\[
\begin{align*}
A &= \int w_{xa}(-H_a)dxL + \int w_x(-H_{aa}) - \kappa_{aa} < 0 \\
B &= \int \pi_y(-G_e)dydL + \int \pi_y(-G_{ee})dydL - \xi_{ee} < 0 \\
C &= AB - \int w_{xe}(-H_a)dxL \int \pi_y(-G_e)dydL.
\end{align*}
\]

At a stable equilibrium \(C \geq 0\) since it is equivalent to \(\partial r_w^{-1}/\partial a^* \geq \partial r_f/\partial a^*\). Thus, the signs of \(a_i^*\) and \(e_i^*\) depend on the sign of the numerators of \((27) - (28)\). Note that in the numerator of \((27)\), \(B < 0\) and \(\int w_{xe}(-H_a)dxL > 0\), while the numerator of \((28)\) \(A < 0\) and \(\int \pi_y(-G_e)dydL > 0\).

Consider a FOSD in \(L\). Since \(w_{xa} = f_{xa} \geq 0\) and \(\pi_y = f_ya \geq 0\) (both inequalities are strict on a set of positive measure), the numerator of \((27)\) is strictly positive and thus \(a_i^* > 0\), so workers invest more upon a FOSD shift in \(L\). Similarly, the numerator of \((28)\) is strictly positive and thus \(e_i^* > 0\), so more firms invest with a FOSD shift in \(L\). The same holds for an IR shift in \(L\) if \(f_{xaa} \geq 0\) and \(f_{yaa} \geq 0\), with strict inequality on a set of positive measure. Clearly, this holds also if absolute risk aversion is (uniformly) sufficiently small.
In turn, a FOSD in $G$ or $H$ has ambiguous effects and the reasons are similar to the ones discussed in the one-sided case (direct effect positive while the indirect effect via $\mu$ is negative).

### B.9 Continuous Investments

We now consider an extension to continuous investment and ex-ante heterogeneous agents. A worker can invest in any amount $a \in [0, 1]$, in which case $x$ is drawn from $H(\cdot | a)$, given by $H(x|a) = aH_1(x) + (1 - a)H_0(x)$, with $H_1 \leq H_0$ with strict inequality on a positive-lengthed interval. If a worker invests $-\mu$ discussed in the one-sided case (direct effect positive while the indirect effect via $H$ has ambiguous effects and the reasons are similar to the ones below.

An equilibrium reduces to finding a measurable $\hat{a}$ that satisfies the usual best response property. We will show that an equilibrium exists using an existence theorem for large games in Rath (1992). We will focus on the no-risk-sharing case since the proof is the same for the risk-sharing case by replacing $w$ by $\bar{w}$ and eliminating the outer integral with respect to $L$ below.

An equilibrium reduces to finding a measurable $\hat{a}$ that satisfies the usual best response property. That is, a measurable function $\hat{a}$ is an equilibrium if

$$U(\theta, \hat{a}(\theta), q(\hat{a})) \geq U(\theta, a, q(\hat{a})) \quad \forall a \in [0, 1], \quad \text{for almost all } \theta \in [0, 1].$$
We will show that this large game satisfies the hypothesis of the existence theorem in Rath (1992), and hence a pure strategy Nash equilibrium exists, which implies existence of an equilibrium in our continuous investment setting.

Before proving existence, note that there are two important cases subsumed by this formulation. One is the linear disutility case 

$$d(a, \theta) = ac(\theta),$$

with 

$$a \in [0, 1].$$

Since for any \( \hat{a} \) the function 

$$U(\theta, \cdot, q(\hat{a}))$$

is linear in \( a \), the optimal choice for all \( \theta \) except for one is either \( a = 1 \) or \( a = 0 \). The only exception is \( \theta^* \), who is indifferent. The second one is the strictly convex disutility case, with 

$$d(\cdot, \theta)$$

twice continuously differentiable in \( a \) and \( d_{aa} > 0 \) for all \( a \in (0, 1) \). In this case – and assuming interior solutions for all types – each agent has a unique best response for any investment strategy profile (pure or mixed).

The following result shows that an equilibrium exists in this setting.

**Proposition 11 (Continuous Investments)** An equilibrium exists in both NRS and RS.

**Proof.** Since for each \( a \), a Walrasian equilibrium exists in the second stage, it suffices to show that an equilibrium in the investment game among the workers exists. Let 

$$C([0, 1]^2)$$

be the set of continuous functions endowed with the sup norm, and let 

$$B(C([0, 1]^2))$$

be the Borel sigma field on that metric space. We will verify the conditions for the application of Theorem 2 in Rath (1992).

First, we have a continuum of players uniformly distributed in 

$$[0, 1]$$

(indexed by \( \theta \)).

Second, the set of actions available to each agent is compact, given by 

$$[0, 1].$$

Third, the payoff function of each player \( \theta \), 

$$U(\theta, \cdot, \cdot) : [0, 1]^2 \to \mathbb{R},$$

is continuous in \((a, q)\). To see this, fix \( \theta \) and notice from (29) that we can write 

$$U(\theta, a, q) = \gamma(a, q) - d(a, \theta),$$

where \( \gamma \) is the first term on the right side of (29). Since \( d(\cdot, \theta) \) is continuous, it suffices to show that \( \gamma \) is continuous in \((a, q)\). But \( \gamma \) is the sum of products of continuous functions (each term is a linear function in \( a \) times an integral that — by the Lebesgue Dominated Convergence Theorem — is continuous in \( q \)). Hence, \( \gamma \) is continuous.

Finally, we will show that the function 

$$z : [0, 1] \to C([0, 1]^2)$$

that defines our game, given by 

$$z(\theta) = U(\theta, \cdot, \cdot),$$

is measurable. Indeed, we will prove that it is continuous, from which measurability follows. Given \( \theta \) and \( \theta' \) in 

$$[0, 1],$$

we have

$$|U(\theta', a, q) - U(\theta, a, q)| = |d(a, \theta') - d(a, \theta)| \leq \max_{a \in [0, 1]} |d(a, \theta') - d(a, \theta)| = \nu(\theta', \theta).$$

It follows from the Theorem of the Maximum that \( \nu \) is continuous and converges to zero as \( \theta' \to \theta \). Hence, 

$$||U(\theta', a, q) - U(\theta, a, q)|| \to 0$$

as \( \theta' \to \theta \), proving continuity of \( z \).

We have verified all the conditions for the application of Theorem 2 in Rath (1992) hold. It follows that there exists an equilibrium in our model. □

\(^{48}\)For this argument to work we cannot allow \( d \) to diverge to infinity as \( \theta \) goes to zero. But this is not important, since all that matters in the binary case is that workers with \( \theta \) close to zero do not invest, and this can be ensured with a large but finite bound for \( d \) as \( \theta \) approaches 0.
Regarding the comparative statics of the model, let us sketch how we can extend our results to this case for the average equilibrium level of investment. Let $U$ depend on $t$, where $t$ can shift any of the primitives of the model analyzed in Section 4. Given $q$, our sufficient conditions for comparative statics ensure that $U_{tt} \geq 0$, so that a higher $t$ induces each worker to choose a higher $a$ given $q$. Moreover, if $\hat{a}'(\cdot) \geq \hat{a}(\cdot)$, then $q(\hat{a}') \geq q(\hat{a})$. Finally, for each $t$, one can show that there is a smallest and a largest equilibrium average level of investment, denoted by $q(t)$ and $q_a(t)$, and that each of them increases in $t$.

B.10 The Partnership Model

A variation of our model is the one-population case, where a continuum of agents from a given population match in pairs. This is commonly used in matching models that analyze partnership formation (e.g., Kremer (1993), Kremer and Maskin (1996), and Legros and Newman (2002)). It can also be interpreted as a two-sided matching problem with two identical populations.

The model is as before except that there are no firms, only a unit measure of risk averse agents with ability $\theta \in [0, 1]$ distributed according to $Q$ and, for each $\alpha$, the match output function is symmetric in the partners’ characteristics, so $f(x, x', \alpha) = f(x', x, \alpha)$ for all $x, x' \in [0, 1]$.

Since $f$ is strictly supermodular, there will be positive sorting in the second stage for any investment function $a$, so the matching function $\mu$ is given by $\mu(x) = x$ for all $x$. Note the crucial feature that $\mu$ does not depend $a$. This will drastically simplify the analysis.

Consider the no-risk-sharing case. Intuitively, partners in equilibrium split match output in half, so $w(x, \alpha) = f(x, x, \alpha)/2$, which is also independent of $a$ since $\mu$ is. As usual, in the first stage an agent with characteristic $\theta$ invests if and only if $U_1 - U_0 \geq c(\theta)$. Following the same steps as in the Section 3 and using the expression for $w$ we obtain the equilibrium condition:

$$\int \int u'(f(x, x, \alpha)/2) f_x(x, x, \alpha)/2 \Delta H dx dL(\alpha) = c(\theta^*)$$

Since the left side does not depend on $\theta^*$, it trivially follows that there is a unique equilibrium, and hence it is efficient. Regarding the comparative statics results, note first that the left side is independent of $Q$, and hence a FOSD shift in $Q$ does not affect $\theta^*$. All it does is to increase the measure of agents with $\theta \geq \theta^*$ who invest (since now $Q$ puts more mass on higher $\theta$'s). Also, since $\mu$ is independent of $H$, a FOSD shift in $H$ has only a direct effect on the left side, which increases as a result and hence $\theta^*$ decreases, so more agents invest. Furthermore, and similar to Proposition 3, a FOSD shift in $L$ increases the measure of agents who invest if either risk aversion is sufficiently small or if $R(w) \leq 1/w$ for all $w$ and $f$ is log-spm in its first and third coordinate for each value of the second one (which implies that $w$ is log-spm in $(x, \alpha)$). Finally, an IR shift in $L$ increases the measure of agents who invest under exactly the same conditions as in Proposition 4. Finally, the comparative statics in $f$ follows if $R \leq 1/w$ and $f$ is log-spm.
Consider now the risk-sharing case, and assume as in Section B.7 that agents have CRRA utility function with parameter \( \sigma \). This is a special case of that analysis with symmetric populations on both sides. The wage function \( \bar{w} \) is now \( \bar{w}(x) = \int_0^x F_x(s, s, \sigma)ds \), which is independent of \( \theta^* \), and the equilibrium condition is \( U_1 - U_0 = c(\theta^*) \), where \( U_1 - U_0 \) is given by

\[
\int_0^1 \left( \int_0^x F_x(s, s, \sigma)ds \right)^{-\sigma} F_x(x, x, \sigma) \Delta H(x)dx.
\]

Since this expression is independent of \( \theta^* \), equilibrium is unique and thus efficient (same proof as in Section B.7 applies). To analyze the comparative statics of this case, assume \( f \) is a separable class. Then the comparative statics with respect to \( L \) are the same as in Section B.7. Also, since \( Q \) does not appear in \( \mu \), nothing changes with a change in \( Q \). Finally, a FOSD shift in \( H_1 \) increases the measure of workers who invest if \( z_x/(\int_0^x z_x)^{\sigma} \) increases in \( x \), which holds if and only if, for all \( x \),

\[
(z_{xx}(x, x) + z_{xy}(x, x)) \int_0^x z_x(s, s)ds - \sigma(z_x(x, x))^2 \geq 0,
\]

which can be checked for any given \( \sigma \) since it involves the primitive \( z \). In particular, if \( \int_0^x z_x \) is log-convex in \( x \), then (30) holds for \( \sigma \in [0, 1) \), since by log-convexity this expression is positive at \( \sigma = 1 \), and hence for any lower \( \sigma \). More generally, (30) holds if

\[
0 \leq \sigma \leq \min_{x \in [0, 1]} \frac{(z_{xx}(x, x) + z_{xy}(x, x)) \int_0^x z_x(s, s)ds}{(z_x(x, x))^2},
\]

where one can show by L’Hospital’s rule that the expression on the right side is one at \( x = 0 \) and strictly positive and bounded away from zero for \( x > 0 \), and thus the minimum is strictly positive.

Note that all these results carry over to the continuous investment case (with homogeneous or heterogeneous agents). To see this, replace \( c(\theta^*) \) above by \( da(\hat{a}(\theta), \theta) \). Since the left side is independent of \( \hat{a} \), existence and uniqueness are trivial, and efficiency and the comparative statics results follow exactly as in the binary case.
C Online Appendix: Quantitative Assessment

C.1 Data Sources and Variable Construction

We combine four data sources for our analysis.

C.1.1 CPS

Our main data source is the Current Population Survey (CPS), which is the primary source of information about the United State’s labor force and was initiated in the 1940s. It is administered monthly by the U.S. Census Bureau to over 65,000 households, henceforth referred to as the Basic Monthly CPS, and includes information on education, employment, demographics and other aspects of the U.S. population. The CPS follows a particular rotation pattern: Participants in the CPS are surveyed for four consecutive months, left out for the following eight months and are then again surveyed for the following four consecutive months. For confidentiality reasons and protection of individuals with extremely high incomes, the Census Bureau has applied different non-disclosure methods that can be divided into different phases: Top-coding from 1962 to 1995, replacement values from 1996 to 2010 and rank proximity swapping from 2011 to the present. For instance, in 1980 any individual with an annual income value above $50,000 is assigned said maximum value. Or, in 2015 income values above the annual swap value are ranked from lowest to highest and then systematically swapped amongst one another. This procedure preserves the distribution of values above the threshold, as well as the privacy of the individual.

For our analysis, we rely on the Annual Social and Economic Supplement (ASEC) of the CPS. This is annual data containing rich information on employment, health insurance and taxes, among others. From 1955 to 1975, the ASEC was administered every year during March and included only households from the March Basic Monthly CPS, described above. From then on, the ASEC has expanded. It now includes all March Basic Monthly CPS participants, but adds participants from other months who are not scheduled to receive the March Basic Monthly CPS. In particular, the ASEC consists of two oversamples, which are not part of the March Basic Monthly CPS. First, in 1976, an oversample of hispanic households was introduced. The second oversample was introduced in 2001 and consists of low-income households with children who do not have health insurance. The motivation for this expansion was to produce reliable estimates of the effects of the State Children’s Health Program (SCHIP) on these low-income households. For these two reasons, the ASEC is larger in each year than the March Basic Monthly CPS with greater differences when the SCHIP oversample was introduced in 2001. The data is available every year from 1962 to the present.

49 About the CPS, see: [https://cps.ipums.org/cps/intro.shtml](https://cps.ipums.org/cps/intro.shtml).
50 See [https://cps.ipums.org/cps/topcodes_tables.shtml](https://cps.ipums.org/cps/topcodes_tables.shtml) for CPS topcodes, replacement and swap values.
51 This is the reason why the ASEC is commonly referred to as the 'March Supplement'.
52 See Flood and Pacas (2016) for more details on the CPS and the ASEC.
In our empirical application, we compare outcomes in two periods (one with relatively little and one with significant wage inequality): an early period including the years 1979-1981 and a later period spanning 2015-2017. We pool three years in each period to reduce the chances of picking an outlier year and to increase the number of observations (we conducted robustness pooling for different years). To make the two periods consistent, we do not include observations that are part of the SCHIP oversample, as this oversample is only part of the later period. We focus on employed workers, who work full-time-full-year, earn a positive wage and are in their prime working age (between 25 and 50 years old). The main variables we use from the CPS are the following: We use total pre-tax wage and salary income for the previous year, adjusted for inflation using the consumer price index in 1999, where we divide by 12 to get monthly earnings. We trim the top 1% of wage income. Furthermore, regarding the education choice, we assign agents to two groups: Everybody with at least 4 years of college education is characterized as college educated, and everybody below as non-college educated.

C.1.2 DOT and ONET

Our analysis requires a measure of productivity on the demand side, where we use ‘occupations’ as the model’s counterpart to ‘firms’. We proxy the productivity of an occupation by their cognitive skill requirement, which we obtain from two different data sources (one for each time period) that we make comparable: The Dictionary of Occupational Titles (DOT) for the early period and the Occupational Information Network (O*NET) for the later period.

Early period. For the years 1979-1981 we draw information from the 4th [1977] edition of the U.S. Department of Labor’s Dictionary of Occupational titles (DOT). The DOT provides useful occupational information. It is based on job examiners’ evaluations of more than 10,000 highly detailed occupations along different criteria such as the repetitiveness or the physical or cognitive requirements to perform a particular occupation. It was first published in 1938 and updated various times (in 1949, 1965, 1977 and 1991), without significant alteration to its structure. In our analysis, we focus on the 1977 edition, as the timing aligns most closely with our early period. We rely on the aggregation of these detailed occupations, the selection of the variables and the crosswalk to around 490 census 1990 occupations provided by Autor, Levy, and Murnane (2003).

We focus on the DOT variable MATH (mathematical reasoning) as it best proxies the ‘cognitive skill requirement’ of an occupation. Also, this choice facilitates the comparison with the O*NET for our later period. Since the units of this task measure have no clear interpretation, we use the rank of this measure instead and use it as our proxy for jobs’ productivity $y$ in our

53 Besides the variable MATH, there are other occupational descriptors in the DOT (see also Autor, Levy, and Murnane (2003)): for instance, DCP (direction, control and planning), STS (set limits, tolerances or standards), FINGDEX (finger dexterity) and EYEHAND (eye-hand finger coordination) – measures less related to the cognitive analytic content of an occupation.
model (to do so we compute the percentiles of MATH for each occupation). At this stage, every occupation has a $y$ and we merge it to the occupations held by workers in our early CPS sample using the 1990 census occupational codes as the merging variable. We then construct the demand-side distribution $G$ for 1979-1981 in the CPS.

**Later period.** For the years 2015-2017 we draw information from the Occupational Information Network (O*NET). The O*NET is today’s primary source of occupational information and replaced the DOT with continuously updated information as an online database in the 1990s. Similar to the DOT, the O*NET database contains numerous occupation specific descriptors (around 300) rating almost 1,000 occupations. Those descriptors are organized into ten broad categories: Abilities, Skills, Knowledge, Work Activities, Work Context, Education Requirements, Job Interests, Work Values, Work Styles, and Tasks. In contrast to the DOT that emphasizes the industrial economy, the O*NET reflects the shift in the economy to information and services. We use information from the O*NET instead of the DOT for our later period to account for changes at the intensive, as well as at the extensive margin of occupational skill requirements over time.

For our analysis, we retain descriptors from the O*NET files ‘Abilities’ and ‘Skills’ as descriptors contained in the other files are less directly interpretable in terms of skills. We select the O*NET variables ‘Mathematical Reasoning’, ‘Number Facility’ and ‘Mathematics’, as these three variables match most closely the selected DOT descriptor (see Table 4) and clearly indicate the cognitive skill content of an occupation. In line with our model that features one-dimensional heterogeneity of jobs, we need to reduce the dimension of the cognitive task content from three to one. To this end, we perform a principal component analysis (PCA) on these three descriptors. Table 5 shows the results of the PCA and Figure 6 plots the corresponding eigenvalues. We see that the first principal component explains around 96% of the variance in our data, either directly from column three in Table 5 or from the first principal component’s eigenvalue which is 2.8911 and hence explains $(2.8911/3)$ 96% of the variance in the data. Thus, the first principal component captures very well the variation of the underlying data and we focus on it as a proxy for the model’s job productivity $y$. Again, because the units of the principal component are not readily interpretable and because we want to make this measure comparable to the one from our early period, we compute the percentile of the principal component for each occupation, which we then use as our $y$. Finally, we use the crosswalk from the Standard Occupational Classification of the O*NET to approximately 490 census 1990 occupations, and merge our constructed job productivity to the occupations held by individuals in the CPS 2015-2017. We then take the empirical distribution of the constructed $y$’s as our $G$ in the model.

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54The O*NET was developed by the North Carolina Department of Commerce and sponsored by the U.S. Department of Labor. More information is available on [https://www.onetcenter.org](https://www.onetcenter.org) or on the related site of the Department of Labor [https://www.doleta.gov/programs/onet/eta_default.cfm](https://www.doleta.gov/programs/onet/eta_default.cfm).

55In our early period there was no need to do that since there was only one dimension, MATH, indicating the cognitive task content.
Table 4: O*NET and DOT Description

<table>
<thead>
<tr>
<th>DOT Variable</th>
<th>DOT Definition</th>
<th>ONET Variable</th>
<th>ONET Definition</th>
<th>ONET Database</th>
<th>Task Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>MATH</td>
<td>General educational development, mathematics</td>
<td>Mathematical Reasoning</td>
<td>The ability to choose the right mathematical methods or formulas to solve a problem</td>
<td>Abilities</td>
<td>Non-routine, analytic</td>
</tr>
<tr>
<td></td>
<td>Number Facility</td>
<td>Number Facility</td>
<td>The ability to add, subtract, multiply, or divide quickly and correctly</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mathematics</td>
<td>Mathematics</td>
<td>Using mathematics to solve problems</td>
<td>Skills</td>
<td></td>
</tr>
</tbody>
</table>

Table 5: PCA Later Period

<table>
<thead>
<tr>
<th>Component</th>
<th>Eigenvalue</th>
<th>Proportion</th>
<th>Cumulative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comp 1</td>
<td>2.8911</td>
<td>0.9637</td>
<td>0.9637</td>
</tr>
<tr>
<td>Comp 2</td>
<td>0.0662</td>
<td>0.0221</td>
<td>0.9858</td>
</tr>
<tr>
<td>Comp 3</td>
<td>0.0427</td>
<td>0.0142</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Figure 6: Eigenvalues Later Period
C.1.3 NLSY 79/97

Our analysis requires a measure of ability, $\theta$, and its distribution, $Q$, for both time periods. To this end, we use AFQT test scores from the National Longitudinal Surveys (NLSY79 and NLSY97) to obtain an empirical ability distribution for our early and later period.

The NLSY79 and NLSY97 are longitudinal projects that follow a sample of American’s of a specific cohort over time. Respondents in the NLSY79 are born between 1957-1964 and were 14-22 years old when first interviewed in 1979. In the NLSY97, respondents are born between 1980-1984 and were 12-17 years old when first interviewed in 1997. The AFQT test scores of the NLSY79 and NLSY97 are from the Armed Services Vocational Aptitude Battery (ASVAB), which is the main screening test for applicants to the U.S. military. The AFQT is equal to the sum of the arithmetic reasoning, word knowledge and paragraph comprehension component plus one half times the score from the numerical comprehension component of the ASVAB (Altonji, Bharadwaj, and Lange (2009)). The test was administered in the second half of 1979 for the NLSY79 and in the second half of 1997 (and the first months of 1998) for the NLSY97.

However, the AFQT test scores from these two surveys are not directly comparable for two reasons. First, the test format has changed from paper & pencil in 1979 to computer administered in 1997. Second, the distribution of the test taking age differs significantly across the two samples, with the 1997 sample being much younger. We rely on Altonji, Bharadwaj, and Lange (2009) who account for these two issues and construct test scores that are comparable across the NLSY79 and NLSY97. They rely on two percentile mappings: The first one transforms the computer administered test scores of the NLSY97 into paper & pencil scores. The authors make use of a mapping provided by Segall (1997), which is based on a study of a sample of randomly assigned test takers (to either computer or paper & pencil). The second one addresses the problem that individuals are observed at different ages when taking the test. They exploit the fact that, in both samples, a large group of respondents is 16 years old when taking the test. They thus map all test scores within cohorts into the age 16-distribution based on within age ranking of test scores.

In line with the model’s $\theta$ we normalize the test scores to lie between 0 and 1, and weight each observation with the weights provided by Altonji, Bharadwaj, and Lange (2009) to make them representative, resulting in normalized test scores that are comparable across both cohorts. The test scores of the NLSY79 are used for constructing the ability distribution $Q$ in the early period of our analysis, while the test scores of the NLSY97 are used for $Q$ in the later period.

---

56 NLSY: https://www.nlsinfo.org/content/getting-started
57 In the NLSY79, data is available from round 1 (1979 survey year) to round 27 (2016 survey year), meaning this cohort has been interviewed 27 times to date. The NLSY97 cohort has been surveyed 18 times to date, with the first interview in 1997 and the 18th interview in 2017.
58 Altonji, Bharadwaj, and Lange (2009) use custom weights to achieve representative samples, with the year 1979 (1997) for the year during which the ASVAB was administered.
59 See Altonji, Bharadwaj, and Lange (2009) for a more detailed discussion.
C.2 Summary Statistics

This section provides some summary statistics of the data. Figure 7 shows the cdf’s of monthly wages for both time periods, showing an increase in inequality (IR shift). Table 6 provides basic summary statistics along with measures of inequality and the college share (fraction of individuals with at least a college degree). Summary statistics of the distribution of test scores in both periods are in Table 7. The occupational productivity distribution, discussed in Section C.1.2, is summarized in Table 8.

Table 6: Summary Statistics and Inequality Measures

<table>
<thead>
<tr>
<th>Year</th>
<th>N</th>
<th>Mean</th>
<th>St. Dev.</th>
<th>90/10</th>
<th>90/50</th>
<th>75/25</th>
<th>Skill Premium</th>
<th>College Share</th>
</tr>
</thead>
<tbody>
<tr>
<td>1979-81</td>
<td>82,913</td>
<td>3,112</td>
<td>1,702</td>
<td>3.87</td>
<td>1.86</td>
<td>2.03</td>
<td>1.38</td>
<td>0.26</td>
</tr>
<tr>
<td>2015-17</td>
<td>80,171</td>
<td>3,118</td>
<td>2,008</td>
<td>4.99</td>
<td>2.25</td>
<td>2.29</td>
<td>1.66</td>
<td>0.39</td>
</tr>
</tbody>
</table>

Note: This graph shows the cdf’s of monthly wages for both periods. The early period includes the years 1979-81 and the later period the years 2015-17. Wages are adjusted for inflation using the consumer price index in 1999.

Figure 7: CDF’s of Monthly Wages for Both Periods

Table 7: Summary Statistics $Q$ Distribution

<table>
<thead>
<tr>
<th>Year</th>
<th>N</th>
<th>Mean</th>
<th>St. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1979-81</td>
<td>314,828</td>
<td>0.60</td>
<td>0.23</td>
</tr>
<tr>
<td>2015-17</td>
<td>153,825</td>
<td>0.63</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Table 8: Summary Statistics $G$ Distribution

<table>
<thead>
<tr>
<th>Year</th>
<th>N</th>
<th>Mean</th>
<th>St. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1979-81</td>
<td>82,913</td>
<td>0.54</td>
<td>0.24</td>
</tr>
<tr>
<td>2015-17</td>
<td>80,171</td>
<td>0.55</td>
<td>0.28</td>
</tr>
</tbody>
</table>
C.3 Estimation

C.3.1 Parameters

We estimate the following seven parameters:

- Cost function parameter $\lambda$
- Skill (risk) distribution parameter $b$
- Standard deviation $\tau$ of shock distribution $L$
- Production function parameters
  - TFP $q$,
  - elasticity of output with respect to $x$, $\gamma_1$,
  - elasticity of output with respect to $y$, $\gamma_2$,
  - Constant $K$

For the ability distribution $Q$, we input the empirical distribution of test scores described in Section C.1.3 into the model. For the productivity distribution $G$, we input the empirical distribution of occupations’ cognitive task requirement described in Section C.1.2 into the model. We assume a risk aversion parameter of the CRRA utility, $\sigma = 1.1$, which is within a standard range, as the meta analysis of Chetty (2006) shows (see his Table 1 for a range of CRRA coefficients for Macroeconomic and Trend Evidence).

C.3.2 Identification

Having identified distributions $(G,Q)$ directly from their empirical counterparts, we need to identify the following remaining parameters/objects ($H_0, H_1, f, L, c$). We seek (semi)-parametric identification, using some of the functional form assumptions used in the calibration of the model. But we note that most of the following arguments hold beyond our specific functional forms.

Production Function. Parameter $K$ is identified from minimum income $w(0) = K$ (constant of integration in the wage function). In turn, we follow arguments on the estimation of hedonic models to show identification of the production function. In principle, one can do this non-parametrically, but we explain the approach in the context of our parametric production function. In particular, we follow Ekeland, Heckman, and Nesheim (2004), Section IV.D, but also make use of their discussion of the identification strategy proposed by Rosen (1974) and criticized by Brown and Rosen (1982). The identification is based on the firm’s FOC and exploits the non-linearity of our matching model, which is an important source of identification just as in Ekeland, Heckman, and Nesheim (2004). The firm’s FOC is given by:
\[ w_x(x) = f_x(x, \mu(x)) \]  

and the marginal wage income is \( w_x(x)\eta(\alpha) \). We can identify the production function parameters in two steps:

1. Estimate the marginal wage income \( w_x\eta \) as the derivative of the kernel regression of wage income (observed) on \( x \) (e.g. proxied by workers’ education). Denote this estimate by \( \hat{w}_x\eta \). We can then treat this derivative of wage income as observable.

2. Estimate a version of FOC (31) after taking the income shock into account and applying a log transformation:

\[
\log(\hat{w}_x(x)\eta(\alpha)) = \log(f_x(x, \mu(x))\eta(\alpha))
\]

where we assume the functional forms \( f(x, y) = qx_1y_2 + K \) and \( \eta(\alpha) = e^{\alpha} \) (see Section 6.1), and where we treat \( x \) and the matched occupation \( y \) as observable. We obtain:

\[
\log(\hat{w}_x(x)\eta(\alpha)) = \log(q) + \log(\gamma_1) + (\gamma_1 - 1) \log(x) + \gamma_2 \log(y) + \alpha
\]

Note that this functional form of \( f \) circumvents the identification problem of Rosen (1974), discussed in Brown and Rosen (1982) and Ekeland, Heckman, and Nesheim (2004), since the slope of the wage gradient in \( x \) is not equal to the slope of the marginal product in \( x \). Regression (32) identifies \((q, \gamma_1, \gamma_2)\).

While we chose to lay out this two-step approach since it is particularly transparent, it is not the only way to achieve identification. More generally, one can apply the identification argument along the lines of Section IV.B in Ekeland, Heckman, and Nesheim (2004).

**Shock Distribution.** We need to identify the variance \( \tau^2 \) of the income shock distribution \( L \). From the distribution of the residual in (32), which is the un-forecastable component of the marginal product of income and also of income, we can identify \( Var[\alpha] = \tau^2 \).

**Skill Distribution.** Given our normalization of \( H_0 \) (uniform), we need to identify \( H_1 \), which amounts to identifying \( b \). This parameter will be identified from the skill premium, defined as the ratio of wage income of educated workers and non-educated workers, where we denote wage

\text{Note that income shock } \alpha \text{ is realized after investment and matching takes place and thus does not impact } x \text{ or } y.
income of agent $x$ with shock realization $\alpha$ by $i(x, \alpha) = w(x)\eta(\alpha)$:

$$
\frac{\mathbb{E}[i|x \sim H_1]}{\mathbb{E}[i|x \sim H_0]} = \frac{\int i(x, \alpha)dH_1(x)dL(\alpha)}{\int i(x, \alpha)dH_0(x)dL(\alpha)} = \frac{\int e^{\alpha}\eta(\alpha)\int w(x)dH_1(x)}{\int e^{\alpha}\eta(\alpha)\int w(x)dH_0(x)} = \frac{\int w(x)dH_1(x)}{\int w(x)dH_0(x)}
$$

(33)

Note that the LHS is observed in the data (a number). Moreover, matching function $\mu$ is observed and $f$ and $K$ were identified above, meaning we can compute the wage function in numerator and denominator on the RHS. This yields one equation in one unknown, $b$, where only the numerator of the RHS depends on $b$ in an increasing way (since $b$ increases $H_1$ in the FOSD sense and since $w$ is increasing in $x$). So we are looking for the $b$ that rationalizes the LHS, given the estimated $f$ and the observed $\mu$. Since the LHS is a fixed number larger than one and the RHS starts at one (for $b = 1$) and increases in $b$, there is a unique $b$ for which (33) holds. Thus, the skill premium identifies $b$.

Cost Function. At this stage we know $(f, \eta, H_0, H_1, L, u, \theta^*, \beta)$ either because we fixed them ($u$, $H_0$, $\beta$) or because we identified them $(f, \eta, H_1, L)$ or because we observe them directly in the data ($\theta^*$). We can thus construct the LHS of the indifference condition of the marginal ability type,

$$
\frac{1}{1-\beta} (U_1(\theta^*) - U_0(\theta^*)) = c(\theta^*)
$$

$$
\frac{1}{1-\beta} \left( \int \int u(w(x, \alpha, \theta^*)dH_1(x))dL(\alpha) - \int \int u(w(x, \alpha, \theta^*))dH_0(x)dL(\alpha) \right) = c(\theta^*)
$$

and on the RHS we know everything except $\lambda$. For a given $\theta^*$, the LHS is constant in $\lambda$ (and larger than zero), while the RHS is increasing in $\lambda$ (starting at zero and increasing), giving a unique solution for $\lambda$, which rationalizes the observed share of uneducated workers $\theta^*$.

C.3.3 Moments

The estimation is based on the CPS sample described in Section C.1.1. Table 9 shows the 10 moments that we use to pin down the parameters (inspired by identification argument above), as well as a brief description of how they are constructed.

C.3.4 Simulated Method of Moments

There are seven parameters of the model, $\Lambda \equiv (\lambda, b, \tau, q, \gamma_1, \gamma_2, K)$, that are not directly identified from the data or pre-set. They are disciplined by the moments described above, which are chosen based on our formal identification argument discussed in Section C.3.2. To estimate those
parameters, we apply the method of simulated moments (McFadden (1989)). For any vector $\Lambda$ of structural parameters the model produces the 10 moments outlined above, $mom_{sim}(\Lambda)$, that will also be computed in the data, $mom_{data}$. We then use a global search algorithm (refined by a local search thereafter) to find the values of parameters that minimize the distance between simulated and observed moments.

Formally, the vector $\hat{\Lambda}$ solves

$$\hat{\Lambda} = \arg \min_{\Lambda} \ [mom_{sim}(\Lambda) - mom_{data}]^T V [mom_{sim}(\Lambda) - mom_{data}]$$

(34)

with weighting matrix $V$. We perform a ‘m out of n’ bootstrap on this set of moments to calculate the weighting matrix $V$, defined as the inverse of the diagonal of the covariance matrix of the bootstrapped moments.

Table 9: Description of Moments

<table>
<thead>
<tr>
<th>Moment</th>
<th>Construction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.) Share of college graduates</td>
<td>$\frac{# \text{ people with college degree}}{# \text{ people with college degree} + # \text{ people without college degree}}$</td>
</tr>
<tr>
<td>2.) College premium</td>
<td>$\frac{\text{mean wage if college degree}}{\text{mean wage if no college degree}}$</td>
</tr>
<tr>
<td>3).-7.) Percentiles</td>
<td>percentiles 5, 25, 50, 75, 95 of the wage distribution</td>
</tr>
<tr>
<td>8.) Average wage</td>
<td>mean of the wage distribution</td>
</tr>
<tr>
<td>9.) Standard deviation</td>
<td>standard deviation of the wage distribution</td>
</tr>
<tr>
<td>10.) Ratio of within to between wage variance</td>
<td>Ratio of within-y and between-y variance of wage. To compute this ratio, first run linear regression of wage on occupation fixed effects. The between variance is the variance of the resulting predicted values, and the within variance is the variance of the residuals of this regression.</td>
</tr>
</tbody>
</table>
C.4 Results with Endogenous Investment

Table 10: Pre-Set Parameters

<table>
<thead>
<tr>
<th>Value</th>
<th>Parameter Description</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Monthly Discount Rate</td>
<td>Standard</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Risk Aversion</td>
<td>In Standard Range for CRRA</td>
</tr>
<tr>
<td>$\bar{\alpha}$</td>
<td>Mean of Shock</td>
<td>Normalization</td>
</tr>
</tbody>
</table>

Table 11: Model Fit Early Period

<table>
<thead>
<tr>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 - Q(\theta^*)$</td>
<td>0.26 0.27</td>
</tr>
<tr>
<td>$E[w</td>
<td>\theta \geq \theta^*]$</td>
</tr>
<tr>
<td>$E[w</td>
<td>\theta &lt; \theta^*]$</td>
</tr>
<tr>
<td>$w_5$</td>
<td>1916.19 1912.50</td>
</tr>
<tr>
<td>$w_{25}$</td>
<td>2766.73 2792.25</td>
</tr>
<tr>
<td>$w_{75}$</td>
<td>3920.87 3882.54</td>
</tr>
<tr>
<td>$w_{95}$</td>
<td>6453.80 6387.50</td>
</tr>
<tr>
<td>$E[w]$</td>
<td>3113.00 3111.91</td>
</tr>
<tr>
<td>$\sqrt{Var[w]}$</td>
<td>1694.73 1701.16</td>
</tr>
<tr>
<td>$\frac{Var[w</td>
<td>within]}{Var[w]}$</td>
</tr>
</tbody>
</table>

Table 12: Model Fit Later Period

<table>
<thead>
<tr>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 - Q(\theta^*)$</td>
<td>0.40 0.40</td>
</tr>
<tr>
<td>$E[w</td>
<td>\theta \geq \theta^*]$</td>
</tr>
<tr>
<td>$E[w</td>
<td>\theta &lt; \theta^*]$</td>
</tr>
<tr>
<td>$w_5$</td>
<td>1731.41 1735.00</td>
</tr>
<tr>
<td>$w_{25}$</td>
<td>2651.32 2602.50</td>
</tr>
<tr>
<td>$w_{75}$</td>
<td>3981.42 3983.67</td>
</tr>
<tr>
<td>$w_{95}$</td>
<td>7113.37 7171.33</td>
</tr>
<tr>
<td>$E[w]$</td>
<td>3137.79 3117.66</td>
</tr>
<tr>
<td>$\sqrt{Var[w]}$</td>
<td>2001.84 2008.16</td>
</tr>
<tr>
<td>$\frac{Var[w</td>
<td>within]}{Var[w]}$</td>
</tr>
</tbody>
</table>

Figure 8: Model Fit Early Period (left) and Later Period (right)

Note for Figure 8: The figure plots model moments against data moments (45 degree line indicates a perfect fit). All moments are plotted on their original scale except wage moments. We divided the wage moments by 100 to fit them into the same graph along with the other moments that have a smaller scale.
C.5 Results with Exogenous Investment

C.5.1 Statistical Decomposition

Our objective of the statistical decomposition is to isolate changes in education and skill returns on wage inequality changes. To this end, we construct counterfactual wage distributions relying on the CPS data and impose the same sample restrictions described in Section C.1.1 with two minor differences. First, in contrast to Section C.1.1 where individuals are classified as either college educated or not, individuals now are assigned to one of five mutually exclusive groups according to the highest level of education completed: no or unclear high school degree (< 12
years of schooling), high school graduates (12 years of schooling), some college (<4 years of college), Bachelor’s degree (4 years of college) and more than Bachelor’s degree (> 4 years of college). Second, we round wages up to tens, as the wage values in our data are very detailed (including several decimal places). Without rounding we would not have enough observations per education group and wage level to construct counterfactual wage distributions. Similar to the main empirical analysis, we consider an early period 1979-1981 and a later period 2015-2017.

We construct two counterfactual wage distributions: the first one using a counterfactual education distribution and the second one using a counterfactual distribution of returns to skill (or wages). More formally, suppose the wage distribution in period $t$ is denoted by $N_t$, the conditional wage distributions (conditional on education) are denoted by $N_{t,e}(w|educ = e)$ and the probability distribution of education is $p_t(educ = e)$, for educ levels $e \in \{1, ..., 5\}$ explained above. Then, the actual (observed) wage distribution in period $t$ can be expressed as a mixture of the conditional wage distributions:

$$N_t(w) = p_t(educ = 1)N_{t,1}(w|educ = 1) + p_t(educ = 2)N_{t,2}(w|educ = 2) + ... + p_t(educ = 5)N_{t,5}(w|educ = 5)$$

We can run our two counterfactuals as follows:

1. Keeping education fixed in $t-1$ (‘counterfactual education distribution’):

$$N_{t}^{counter-e}(w) = p_{t-1}(educ = 1)N_{t-1,1}(w|educ = 1) + p_{t-1}(educ = 2)N_{t-1,2}(w|educ = 2) + ... + p_{t-1}(educ = 5)N_{t-1,5}(w|educ = 5)$$

2. Keeping wages (conditional on education) fixed in $t-1$ (‘counterfactual returns to skill’):

$$N_{t}^{counter-w}(w) = p_t(educ = 1)N_{t-1,1}(w|educ = 1) + p_t(educ = 2)N_{t-1,2}(w|educ = 2) + ... + p_t(educ = 5)N_{t-1,5}(w|educ = 5)$$

For the first counterfactual, we keep the pdf of education fixed in the early period 1979-1981 and implement the true conditional wage distribution in the later period 2015-2017. In turn, for the second one we keep the conditional wage distribution fixed in the early period and implement the true pdf of education in the later period.
C.5.2 Model Fit and Decomposition with Exogenous Investment

Table 15: Estimated Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate 1980</th>
<th>Estimate 2015</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>8.71</td>
<td>34.67</td>
<td>Skill Risk</td>
</tr>
<tr>
<td>$\tau$</td>
<td>0.49</td>
<td>0.59</td>
<td>Standard Deviation of Shock</td>
</tr>
<tr>
<td>$q$</td>
<td>5857.49</td>
<td>10328.45</td>
<td>TFP</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>0.32</td>
<td>0.36</td>
<td>Elasticity of $f$ in $x$</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>0.28</td>
<td>0.71</td>
<td>Elasticity of $f$ in $y$</td>
</tr>
<tr>
<td>$K$</td>
<td>473.47</td>
<td>719.17</td>
<td>Constant in $f$</td>
</tr>
</tbody>
</table>

Table 16: Model Fit Early Period

<table>
<thead>
<tr>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[w</td>
<td>\theta \geq \theta^*]$</td>
</tr>
<tr>
<td>$E[w</td>
<td>\theta &lt; \theta^*]$</td>
</tr>
<tr>
<td>$w_5$</td>
<td>1044.47</td>
</tr>
<tr>
<td>$w_{25}$</td>
<td>1916.17</td>
</tr>
<tr>
<td>$w_{50}$</td>
<td>2765.38</td>
</tr>
<tr>
<td>$w_{75}$</td>
<td>3919.89</td>
</tr>
<tr>
<td>$w_{95}$</td>
<td>6454.63</td>
</tr>
<tr>
<td>$E[w]$</td>
<td>3113.54</td>
</tr>
<tr>
<td>$\sqrt{Var[w]}$</td>
<td>1696.06</td>
</tr>
<tr>
<td>$\sqrt{Var[w]_{\text{within}}}$</td>
<td>2.42</td>
</tr>
<tr>
<td>$\sqrt{Var[w]_{\text{between}}}$</td>
<td>2.46</td>
</tr>
</tbody>
</table>

Table 17: Model Fit Later Period

<table>
<thead>
<tr>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[w</td>
<td>\theta \geq \theta^*]$</td>
</tr>
<tr>
<td>$E[w</td>
<td>\theta &lt; \theta^*]$</td>
</tr>
<tr>
<td>$w_5$</td>
<td>964.94</td>
</tr>
<tr>
<td>$w_{25}$</td>
<td>1732.02</td>
</tr>
<tr>
<td>$w_{50}$</td>
<td>2647.47</td>
</tr>
<tr>
<td>$w_{75}$</td>
<td>3978.69</td>
</tr>
<tr>
<td>$w_{95}$</td>
<td>7118.84</td>
</tr>
<tr>
<td>$E[w]$</td>
<td>3132.18</td>
</tr>
<tr>
<td>$\sqrt{Var[w]}$</td>
<td>2001.94</td>
</tr>
<tr>
<td>$\sqrt{Var[w]_{\text{within}}}$</td>
<td>2.25</td>
</tr>
<tr>
<td>$\sqrt{Var[w]_{\text{between}}}$</td>
<td>2.31</td>
</tr>
</tbody>
</table>

Table 18: Decomposition: Skill Premium

<table>
<thead>
<tr>
<th>Skill Premium</th>
<th>%-Change rel. to 1980</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980</td>
<td>1.38</td>
</tr>
<tr>
<td>2015</td>
<td>1.65</td>
</tr>
<tr>
<td>$G$</td>
<td>1.39</td>
</tr>
<tr>
<td>$H_1$</td>
<td>1.44</td>
</tr>
<tr>
<td>$Q$</td>
<td>1.38</td>
</tr>
<tr>
<td>$L$</td>
<td>1.39</td>
</tr>
<tr>
<td>$f$</td>
<td>1.53</td>
</tr>
</tbody>
</table>

Table 19: Decomposition: Wage Inequality

<table>
<thead>
<tr>
<th>$w_{75}/w_{25}$</th>
<th>%-Change rel. to 1980</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980</td>
<td>2.05</td>
</tr>
<tr>
<td>2015</td>
<td>2.30</td>
</tr>
<tr>
<td>$G$</td>
<td>2.05</td>
</tr>
<tr>
<td>$H_1$</td>
<td>2.06</td>
</tr>
<tr>
<td>$Q$</td>
<td>2.05</td>
</tr>
<tr>
<td>$L$</td>
<td>2.25</td>
</tr>
<tr>
<td>$f$</td>
<td>2.18</td>
</tr>
<tr>
<td>$G, H_1, f$ (Skill Premium)</td>
<td>2.27</td>
</tr>
<tr>
<td>$1 - Q(\theta^*)$ (Education)</td>
<td>2.01</td>
</tr>
</tbody>
</table>