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An optimal auction with identity-dependent externalities

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We analyze the problem of a seller of multiple identical units of a good who faces a set of buyers with unit demands, private information, and identity-dependent externalities. We derive the seller's optimal mechanism and characterize its main properties. We show that the probability that a buyer obtains a unit is an increasing function of the externalities he generates and enjoys. Also, the seller's allocation of the units of the good need not be ex post *efficient. As an illustration, we apply the model to the problem faced by a developer of a shopping mall who wants to allocate and price its retail space among anchor and non-anchor stores. We show that a commonly used sequential mechanism is not optimal unless externalities are large enough.*

1. Introduction

■ This article solves an optimal multi-unit auction design problem with private information and identity-dependent externalities. The model we study is best understood as a model of space allocation in a mall (although we shall see below that it subsumes other applications as well). Consider a developer of a new shopping mall who wants to sell its retail space to a set of stores. An important constraint in this allocation problem is that each store has private information about its profit function, such as the cost of production or the demand for its product. Another key feature of this problem is the existence of interstore externalities: the identity of the stores located in the mall determines its customer traffic, which in turn affects the stores' volume of sales. Thus, a store's willingness to pay for retail space depends on the identity of the other stores that locate in the mall. If the developer wants to maximize her profits, what is the optimal selling procedure?

We develop a simple model that captures the most salient aspects of the problem. We consider a seller who has two identical units of a good and faces a set of potential buyers with unit demands. A buyer's valuation for the good depends on his privately known type and on an externality parameter that depends upon the identity of the buyer who obtains the other unit.

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In this setting, we characterize the revenue-maximizing mechanism for the seller. We find that the seller should allocate the units of the good to the pair of buyers that generates the largest sum of virtual surpluses, weighted by the external effects they enjoy. The probability that a buyer obtains a unit of the good is increasing in both the externality he imposes on other buyers and the one that he enjoys. More importantly, the allocation that ensues need not be *ex post* efficient for the following reasons: (i) as in the case without externalities, the seller sometimes keeps one or both units of the good; (ii) because the presence of external effects makes buyers asymmetric, those who receive the good need not have the largest sum of valuations; and (iii) when externalities are negative, the seller may sell a unit when it is *ex post* efficient for her to keep it.

We also characterize an optimal payment rule that internalizes the externalities generated and enjoyed by a buyer, and makes the optimal mechanism a dominant strategy one. In particular, we show that a buyer's payment is a decreasing function of the externalities he generates.

As an application, we elaborate on the shopping mall problem. A standard procedure in practice is to sign the anchor stores first (i.e., department stores), which are the main externality generators, and then approach the remaining interested stores. The empirical evidence also suggests that anchors receive large discounts that are increasing in the externalities they generate.[1] In turn, the non-anchor stores that enjoy these externalities pay a premium that is increasing in their magnitude. We characterize the properties of this sequential selling procedure, and show that it is not an optimal one for the seller unless externalities are large enough.

In order to simplify the exposition, we focus on the case of two units, positive externalities, and buyers' payoff functions that are multiplicatively separable in types and externality parameters. We later show that all the results extend to the case of N units, negative externalities, and also to a more general class of complementarities in the buyers' payoff functions.

□ **Related literature.** To the best of our knowledge, this is the first article to analyze an optimal multi-unit auction problem with private information and identity-dependent externalities.

The closest related papers are Jehiel, Moldovanu, and Stacchetti (1996), Das Varma (2002), Figueroa and Skreta (2008), and Brocas (2007). Jehiel, Moldovanu, and Stacchetti (1996) and Das Varma (2002) both analyze auctions with externalities. Jehiel, Moldovanu, and Stacchetti (1996) characterize the optimal mechanism and allow the external effects to be private information, albeit buyers are *ex ante* identical in their model. Das Varma (2002) studies open ascending-bid auctions with commonly known identity-dependent externalities, and shows that when they are nonreciprocal the open auction yields a higher expected revenue than a sealed-bid auction. Unlike our article, these references deal with single-unit auctions and focus on the case in which the winner imposes a negative externality on the losers through their reservation utility.[2] Both Figueroa and Skreta (2008) and Brocas (2007) analyze optimal auctions with externalities, with the emphasis placed on the role played by (privately known) outside options in the optimal allocation of a single unit of a good.[3]

Finally, the article also relates to Segal (1999), who analyzes contracting situations with externalities under complete information. Our model is a contracting problem with externalities, but unlike Segal, we study the effects that private information has on the optimal contract.

The rest of the article proceeds as follows. Section 2 presents the model and some preliminary results. Section 3 contains the derivation of the optimal mechanism and its main properties. Section 4 applies the model to the shopping mall problem. Section 5 presents several extensions of the analysis. Section 6 concludes. The Appendix contains proofs omitted from the text.

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[1] See Pashigian and Gould (1998) and Gould, Pashigian, and Prendergast (2005) for a detailed empirical analysis of the size and impact of inter store externalities on the allocation and pricing of retail space in shopping malls.

[2] Jehiel and Moldovanu (2001) analyze a general model that can accommodate multi-unit auctions with identity-dependent externalities. Their focus, however, is on efficiency rather than revenue maximization. We are grateful to Benny Moldovanu for pointing this reference out to us.

[3] An earlier version of Figueroa and Skreta (2008) allows for multiple units and more general externalities, but they focus on the aforementioned outside option effect.

2. The model

■ There are $I + 1$ risk-neutral agents: a seller, whom we call agent 0, and $I \geq 2$ potential buyers, numbered $1, 2, \ldots, I$. The seller owns two identical units of an indivisible good, and buyers have unit demands (i.e., each one demands at most one unit of the good).[4]

□ **Valuations and external effects.** Without loss of generality (henceforth wlog), we assume that the seller derives no value from the two units. The valuation of buyer i, $i = 1, 2, \ldots, I$, for a unit of the good depends on two factors. First, it depends on a parameter (type) θ_i that is private information and is distributed on $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i]$, $0 \leq \underline{\theta}_i < \overline{\theta}_i$, with positive and atomless density $\phi_i(\cdot)$ and cumulative distribution function $\phi_i(\cdot)$. Let $J_i(\theta_i) = \theta_i - \frac{1-\Phi_i(\theta_i)}{\phi_i(\theta_i)}$; we assume that $J_i(\cdot)$ is a strictly increasing function. Moreover, buyers' types are independently distributed. Second, a buyer's valuation depends also on who obtains the other unit of the good. We model this feature by introducing a matrix of external effects $\{\alpha_{ij}\}_{1\leq i\leq I, 0\leq j\leq I}$, which is assumed to be common knowledge among the agents.[5]

To simplify the presentation of the main results, we assume that externalities are positive and that a buyer's type and the externality parameter interact multiplicatively in his payoff function.[6] Formally, if buyers i and j each obtains a unit of the good, and i pays the seller $-t_i$, then buyer i's payoff is $\alpha_{ij}\,\theta_i + t_i$; moreover, $1 \leq \alpha_{ij} \leq \overline{\alpha} < \infty$, $\alpha_{ii} = \alpha_{i0} = 1$. For simplicity, the reservation utility of a buyer is assumed to be equal to zero. Notice that i does not derive value from a second unit of the good ($\alpha_{ii} = 1$), and he does not enjoy a positive externality if the seller keeps it ($\alpha_{i0} = 1$).

□ **The seller's problem.** The goal of the seller is to design a mechanism that maximizes her expected revenue, taking into account that (i) buyers have private information, (ii) ownership entails external effects, and (iii) participation is voluntary.

By the revelation principle, we can restrict the search for the optimal selling scheme to direct revelation mechanisms (DRM) that are incentive compatible and individually rational. In the present case, because the two units are identical and buyers have unit demands, we can describe a DRM as follows. Let $\Lambda = \{[i, j] \mid i, j = 0, 1, \ldots, I\}$ be the set consisting of the $\frac{(I+2)(I+1)}{2}$ unordered pairs $[i,j]$ (the notation is borrowed from Shiryaev, 1996); that is, (i, j) is the same pair as (j, i). Let $y = (y_{[i,j]})_{0\leq i,j\leq I}$ be a probability distribution over Λ; that is, $y_{[i,j]}$ is interpreted as the probability that i obtains one unit and the other one goes to j. A DRM is a pair of functions $(y(\theta), t(\theta))$, consisting of an allocation rule $y(\cdot)$ and a payment rule $t(\cdot)$, with $y(\theta) = (y_{[i,j]}(\theta))_{0\leq i,j\leq I}$, $t(\theta) = (t_0(\theta), \ldots, t_I(\theta))$, $t_0(\cdot) = -\sum_{i=1}^{I} t_i(\cdot)$, $1 - y_{[0,0]}(\cdot) = \sum_{[i,j]\neq[0,0]} y_{[i,j]}(\cdot)$, and $\theta = (\theta_1, \ldots, \theta_I)$. For example, if θ is the reported vector of types, then $y_{[i,j]}(\theta)$ is the probability that i and j obtain the two units of the good, and $t_i(\theta)$, $i = 1, \ldots, I$, is the amount of money transferred to buyer i.

Define $v_i(\theta_i, \theta_{-i}) = \sum_{j=0}^{I} \alpha_{ij} y_{[i,j]}(\theta_i, \theta_{-i})$. Then the seller's problem can be written as follows:

$$\max_{(y_{[i,j]}(\cdot))_{[i,j]\neq[0,0]},(t_i(\cdot))_{1\leq i\leq I}} E_\theta\left[-\sum_{i=1}^{I} t_i(\theta)\right] \tag{1}$$

$$\text{subject to} \quad U_i(\theta_i) \geq \theta_i \overline{v}_i(\hat{\theta}_i) + \overline{t}_i(\hat{\theta}_i) \quad \forall(i, \theta_i, \hat{\theta}_i) \tag{2}$$

$$U_i(\theta_i) \geq 0 \quad \forall(i, \theta_i) \tag{3}$$

[4]The extension to more than two units is immediate, albeit notationally cumbersome. See Section 5 for details.

[5]A more complete model would allow for the external effects to be private information. Notice, however, that in many applications of the model, such as the shopping mall case, this common knowledge assumption is plausible.

[6]In Section 5, we extend the results to the case of negative externalities, and also allow for a more general interaction between a buyer's type and the externality parameter.

$$y_{[i,j]}(\theta) \geq 0 \quad \forall([i,j],\theta) \tag{4}$$

$$1 - \sum_{[i,j]\neq[0,0]} y_{[i,j]}(\theta) \geq 0 \quad \forall\theta, \tag{5}$$

where $\overline{v}_i(\hat{\theta}_i) = E_{\theta_{-i}}[v_i(\hat{\theta}_i, \theta_{-i})] = E_{\theta_{-i}}[\sum_{j=0}^{I} \alpha_{ij} y_{[i,j]}(\hat{\theta}_i, \theta_{-i})]$ is the expected external effect buyer i enjoys if he reports $\hat{\theta}_i$; $\overline{t}_i(\hat{\theta}_i) = E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ is i's expected transfer if he reports $\hat{\theta}_i$; and $U_i(\theta_i) = \theta_i \overline{v}_i(\theta_i) + \overline{t}_i(\theta_i)$ is buyer i's expected utility if his type is θ_i and he reports it truthfully.

□ **Simplification of the problem.** Using standard arguments, we can simplify the seller's optimal mechanism design problem as follows:

- The incentive compatibility constraints (2) are tantamount to the following conditions:

Lemma 1 (Myerson). A DRM is incentive compatible if and only if for $i = 1, 2, \ldots, I$

(i) $\overline{v}_i(\cdot)$ is increasing[7]; and
(ii) $U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(s)\, ds, \forall\theta_i \in \Theta_i$.

  - Lemma 1 reveals that (3) holds if and only if $U_i(\underline{\theta}_i) \geq 0$. For $i = 1, 2, \ldots I$.
  - Because $-\overline{t}_i(\theta_i) = \theta_i \overline{v}_i(\theta_i) - U_i(\theta_i)$, we can use condition (ii) in Lemma 1 and rewrite the objective function as follows:

$$\begin{aligned}
E_\theta\left[-\sum_{i=1}^{I} t_i(\theta)\right] &= \sum_{i=1}^{I} E_{\theta_i}\left[\theta_i \overline{v}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \overline{v}_i(s)\, ds\right] - \sum_{i=1}^{I} U_i(\underline{\theta}_i) \\
&= \sum_{i=1}^{I} E_{\theta_i}\left[\left(\theta_i - \frac{1-\Phi_i(\theta_i)}{\phi_i(\theta_i)}\right)\overline{v}_i(\theta_i)\right] - \sum_{i=1}^{I} U_i(\underline{\theta}_i) \\
&= \sum_{i=1}^{I} E_\theta\left[\left(\theta_i - \frac{1-\Phi_i(\theta_i)}{\phi_i(\theta_i)}\right)\left(\sum_{j=0}^{I} \alpha_{ij} y_{[i,j]}(\theta)\right)\right] - \sum_{i=1}^{I} U_i(\underline{\theta}_i) \\
&= \sum_{i=1}^{I} E_\theta\left[J_i(\theta_i)\left(\sum_{j=0}^{I} \alpha_{ij} y_{[i,j]}(\theta)\right)\right] - \sum_{i=1}^{I} U_i(\underline{\theta}_i),
\end{aligned} \tag{6}$$

  where the second line follows by integration by parts, and the last line follows from the definition of $\overline{v}_i(\cdot)$ and $J_i(\cdot)$.
  - It is clear from (6) that $U_i(\underline{\theta}_i) = 0$ at the optimum for $i = 1, 2, \ldots, I$.

Summarizing, the seller's problem becomes

$$\max_{(y_{[i,j]}(\cdot))_{[i,j]\neq[0,0]}} \sum_{i=1}^{I} E_\theta\left[J_i(\theta_i)\left(\sum_{j=0}^{I} \alpha_{ij} y_{[i,j]}(\theta)\right)\right] \tag{7}$$

subject to (4) and (5) and condition (i) in Lemma 1.

□ **Some examples.** We envision the externalities among buyers as emerging from a downstream interaction among the agents who acquire the units of the goods, without imposing any noticeable effect on the losers. This interaction is modelled in "reduced form" as a multiplicative term that is identity dependent. Notice also that the externalities imposed and enjoyed by the buyers who obtain the units of the good are determined only by their identity; the prices paid

[7]Throughout the article, increasing and decreasing are used in the weak sense.

for those units play no role in the buyers' interaction after the auction. Despite these restrictions, our model encompasses several substantive economic applications, as the following examples illustrate.

Allocation of retail space in shopping malls. Let the seller be a developer who owns two structures, and let the buyers be stores each wishing to acquire one of them. Assume that each store i faces a separate linear demand for its product, and let the slope be private information and the intercept depend upon the identity of the other store that acquires a structure. For instance, suppose $P_i = \sqrt{\alpha_{ij}} - b_i Q_i$ and let the cost of production be zero. Then, store i's valuation for one unit is its profit function (after choosing the optimal Q_i) given by $\alpha_{ij}\frac{1}{4b_i} = \alpha_{ij}\theta_i$, where $\theta_i = \frac{1}{4b_i}$. It is clear in this case that the stores that end up locating in the mall benefit from the traffic flow generated by the presence and identity of the other store, which is the main source of interaction in this example. We will revisit this application in Section 4.[8]

Optimal choice of tenants by a landlord. The seller is the owner of two apartments or rooms, and the buyers are potential tenants or roommates, who are heterogeneous along two dimensions, one observable (the externality they generate) and one that is private information (a preference parameter for an apartment or a room). If their utility function is quasilinear and multiplicatively separable in the two characteristics, then our model subsumes this one-sided matching problem. In this case, the externalities can be thought of as emerging from complementarities in tasks and effort allocation within the household, similarity in tastes, habits, and so on, which are identity dependent.

Filling multiple positions in a firm. Let the seller be a firm trying to fill two positions in its R& D department. Each applicant i is privately informed about his disutility of effort, which also depends on who will be the candidate j filling the remaining position. For instance, i's payoff could be given by $t_i - \alpha_{ij}\,\theta_i$. Here the identity of the other candidate makes the working environment more pleasant for i; alternatively, one could model the externality as an identity-dependent improvement in the productivity of i. In either case, our model subsumes this contracting problem.

3. Main results

■ Consider the "relaxed problem" of maximizing (7) subject to (4) and (5) only.

Rewrite the seller's objective function as follows:

$$E_\theta\left[\sum_{i=1}^{I}\sum_{j=0}^{I} J_i(\theta_i)\alpha_{ij}y_{[i,j]}(\theta)\right].$$

By inspection, the seller's relaxed problem is equivalent to solving, for each $\theta = (\theta_1, \dots, \theta_I)$,

$$\max_{(y_{[i,j]}(\theta))_{[i,j]\neq[0,0]}} \sum_{i=1}^{I}\sum_{j=0}^{I} J_i(\theta_i)\alpha_{ij}y_{[i,j]}(\theta) \tag{8}$$

subject to (4) and (5).

Straightforward algebra shows that (recall $\alpha_{i0} = \alpha_{ii} = 1$)

$$\begin{aligned}\sum_{i=1}^{I}\sum_{j=0}^{I} J_i(\theta_i)\alpha_{ij}y_{[i,j]}(\theta) &= \sum_{i=1}^{I} J_i(\theta_i)\left(y_{[i,0]}(\theta) + y_{[i,i]}(\theta)\right) + \sum_{i=1}^{I-1}\sum_{j=i+1}^{I} S_{[i,j]}(\theta_i,\theta_j)y_{[i,j]}(\theta)\\ &= \sum_{i=1}^{I} J_i(\theta_i)\left(y_{[i,0]}(\theta) + y_{[i,i]}(\theta)\right) + \sum_{1\le i<j\le I} S_{[i,j]}(\theta_i,\theta_j)y_{[i,j]}(\theta), \end{aligned} \tag{9}$$

where, to shorten the notation, we have defined $S_{[i,j]}(\theta_i, \theta_j) = \alpha_{ij}\,J_i(\theta_i) + \alpha_{ji}\,J_j(\theta_j)$.

[8] Note that we could allow stores to make multiple decisions (investment, advertising, etc.) after the auction of the two structures takes place, so long as the final profits are multiplicatively separable in the externality parameter.

□ The optimal mechanism

Solution to the relaxed problem. It is evident from (9) that the following allocation rule $y^*(\cdot)$ solves problem (8): for every pair $[i, j]$, $1 \le i < j \le I$, set

$$y^*_{[i,j]}(\theta) = \begin{cases} 1 & \text{if } S_{[i,j]}(\theta_i, \theta_j) > \max\{0, \max_l J_l(\theta_l), \max_{[l,k],l\neq k,l,k\geq 1} S_{[l,k]}(\theta_l, \theta_k)\} \\ 0 & \text{otherwise;} \end{cases}$$

for every pair $[i, 0]$, $i = 1, \ldots, I$, set

$$y^*_{[i,0]}(\theta) = \begin{cases} 1 & \text{if } J_i(\theta_i) > \max\{0, \max_{l,l\neq i} J_l(\theta_l), \max_{[l,k],l\neq k,l,k\geq 1} S_{[l,k]}(\theta_l, \theta_k)\} \\ 0 & \text{otherwise;} \end{cases}$$

and for every pair $[i, i]$, $i = 1, \ldots, I$, set $y_{[i,i]}(\theta) = 0$.[9]

In words, the seller ranks all pairs of buyers according to their weighted sum of virtual surpluses, where the weights are given by the external effects they impose on each other. The best pair obtains the units of the goods if their weighted sum is nonnegative and greater than the virtual surplus of every single buyer. Otherwise, the seller allocates one unit to the buyer who has the largest nonnegative virtual surplus. If no buyer or pair of buyers satisfies the aforementioned condition, then the seller keeps the two units of the good.

Monotonicity of $\overline{v}_i(\cdot)$. A necessary condition for $y^*(\cdot)$ to be part of an optimal mechanism is that it satisfy condition (i) of Lemma 1, namely, that $\overline{v}_i(\cdot)$ be increasing in θ_i for all $i = 1, 2, \ldots, N$. It is not obvious that $y^*(\cdot)$ satisfies this condition, for $y^*_{[i,j]}(\cdot)$ need not be increasing in θ_i (unlike the case with a single unit or with multiple units and no externalities).

Example 1 (Non-monotonicity of $y^*(\cdot)$). There are three bidders with valuations distributed uniformly on [0, 1], so $J_i(\theta_i) = 2\theta_i - 1$. Let $\theta_1 = 0.7$, $\theta_2 = 0.8$, $\theta_3 = 0.6$, $\alpha_{21} = \alpha_{31} = 1$, $\alpha_{12} = 1.5$, $\alpha_{13} = 2$, and $\alpha_{23} = \alpha_{32} = 1.3$. Exhaustive checking reveals that $y^*_{[1,2]}(\theta_1, \theta_{-1}) = 1$. Consider $\theta_1' = 0.95 > 0.7$. Now $S_{[1,3]}(\theta'_1, \theta_{-1})$ is the largest weighted sum of virtual surpluses, and thus $y^*_{[1,2]}(\theta_1', \theta_{-1}) = 0$ and $y^*_{[1,3]}(\theta_1'\ \theta_{-1}) = 1$, thereby proving that $y^*_{[1,2]}(\cdot, \theta_{-1})$ is not increasing in θ_1.

This example notwithstanding, the allocation rule $y^*(\cdot)$ implies that $\overline{v}_i(\cdot)$ is increasing in θ_i for all $i = 1, 2, \ldots, N$.

Lemma 2 (Monotonicity). The allocation rule $y^*(\cdot)$ satisfies condition (i) of Lemma 1.

Proof. We need to prove that condition (i) in Lemma 1 is satisfied, namely that $\overline{v}_i(\cdot)$ is an increasing function. Note that it suffices to prove that $v_i(\cdot, \theta_{-i}) = \sum_{j=0}^{I} \alpha_{ij} y^*_{[i,j]}(\cdot, \theta_{-i})$ is increasing. Take $\theta_i' > \theta_i''$; we will show that $v_i(\theta_i', \theta_{-i}) \geq v_i(\theta_i'', \theta_{-i})$. The result follows trivially when $v_i(\theta_i'', \theta_{-i}) = 0$, because $v_i(\cdot, \theta_{-i})$ is a nonnegative function. So suppose $v_i(\theta_i'', \theta_{-i}) > 0$. The allocation rule $y^*(\cdot)$ and the definition of $v_i(\cdot, \theta_{-i})$ reveal that we can, wlog, set $v_i(\theta_i'', \theta_{-i}) = \alpha_{il}$ for some l.

Note that it must be the case that $v_i(\theta_i', \theta_{-i}) > 0$, for $\alpha_{il}\ J_i(\theta_i') + \alpha_{li}\ J_l(\theta_l) > \alpha_{il}\ J_i(\theta_i'') + \alpha_{li}\ J_l(\theta_l)$ implies that, if the solution to the relaxed problem allocates a unit to i when he reports θ_i'', it must do so when he reports θ_i' (recall θ_{-i} is kept fixed). Thus $v_i(\theta_i', \theta_{-i}) = \alpha_{ik}$ for some k.

To complete the proof, we show that $\alpha_{ik} \geq \alpha_{il}$. Because $y^*_{[i,l]}(\theta_i'', \theta_{-i}) = 1$ and $y^*_{[i,k]}(\theta_i', \theta_{-i}) = 1$,

$$\alpha_{il} J_i(\theta_i'') + \alpha_{li} J_l(\theta_l) \ \geq \alpha_{ik} J_i(\theta_i'') + \alpha_{ki} J_k(\theta_k)$$

$$\alpha_{ik} J_i(\theta_i') + \alpha_{ki} J_k(\theta_k) \geq \alpha_{il} J_i(\theta_i') + \alpha_{li} J_l(\theta_l).$$

These inequalities yield $(\alpha_{il} - \alpha_{ik})(J_i(\theta_i'') - J_i(\theta_i')) \geq 0$. But because $J_i(\cdot)$ is strictly increasing, $J_i(\theta_i'') < J_i(\theta_i')$ and hence $\alpha_{ik} \geq \alpha_{il}$, thereby completing the proof.

[9]Setting $y_{[i,i]}(\theta) = 0$ is wlog as the seller is indifferent between giving a second unit to buyer i and keeping it herself. Note also that we ignore ties, as they occur with zero probability.

The intuition of the proof is as follows. Suppose that, given his type, buyer i receives a unit of the good along with buyer j. If buyer i's type increases, then his willingness to pay for a unit increases regardless of the identity of the other buyer. That is, the weighted sum of virtual surpluses of buyer i and any $k \neq i$ increases. Hence, the probability that he obtains a unit of the good must go up when his type is higher. But because the external effect enjoyed by i when he is paired with k need not be the same for all k, the ranking of all pairs containing i may change as buyer i's type increases. Thus (in the relaxed problem), the seller may find it optimal to pair him with $k \neq j$.[10]

An optimal mechanism. It follows from Lemma 2 that $y^*(\cdot)$, along with any payment rule $t(\cdot)$ that satisfies $-\bar{t}_i(\theta_i) = \theta_i \bar{v}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \bar{v}_i(s)\,ds$ for all θ_i and $i = 1, 2, \ldots, N$ (see condition (ii) in Lemma 1), will constitute an optimal mechanism for the seller. In particular, this holds if we set, for every i and every $\theta = (\theta_i, \theta_{-i})$,

$$\begin{aligned}
-t_i^*(\theta_i, \theta_{-i}) &= \theta_i v_i(\theta_i, \theta_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} v_i(s, \theta_{-i})\,ds \\
&= \theta_i \sum_{j=0}^{I} \alpha_{ij} y^*_{[i,j]}(\theta_i, \theta_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} \left(\sum_{j=0}^{I} \alpha_{ij} y^*_{[i,j]}(s, \theta_{-i}) \right) ds. \qquad (10)
\end{aligned}$$

Summarizing, we have shown the following result:

Theorem 1. The mechanism $(y^*(\cdot), t^*(\cdot))$ is an optimal selling procedure for the seller.

The no-externality case. It is easy to show that the allocation rule $y^*(\cdot)$ reduces to the one derived in Maskin and Riley (1989) when there are no external effects, that is, when $\alpha_{ij} = 1$ for all $i = 1, 2, \ldots, I$, $j = 0, 1, \ldots, I$. In this case, the optimal allocation rule $y^*(\cdot)$ is such that, for every pair $[i, j]$, $1 \leq i < j \leq I$, $y^*_{[i,j]}(\theta) = 1$ if and only if (i) $J_i(\theta_i) \geq 0$, (ii) $J_j(\theta_j) \geq 0$, (iii) $J_i(\theta_i) \geq J_k(\theta_k)$ for $k \neq j$, and (iv) $J_j(\theta_j) \geq J_k(\theta_k)$ for $k \neq i$. In other words, the seller simply allocates the units of the good to the buyers with the largest virtual surpluses. It follows that if types are identically distributed, then a standard auction with a reserve price is an optimal mechanism.

Interpretation of the payment rule. It is illuminating to manipulate the payment rule (10) further. Recall that $v_i(\cdot, \theta_{-i}) = \sum_{j=0}^{I} \alpha_{ij} y^*_{[i,j]}(\cdot, \theta_{-i})$ is an increasing function. Indeed, given the shape of $y^*_{[i,j]}(\cdot)$, $v_i(\cdot, \theta_{-i})$ is actually a step function that, wlog, can be assumed to be right-continuous. Let $\theta_i^k(\theta_{-i})$, $k = 1, 2, \ldots, n$, with $\theta_i^1(\theta_{-i}) < \theta_i^2(\theta_{-i}) < \cdots < \theta_i^n(\theta_{-i})$, be the points at which the function jumps as θ_i increases. Notice that $\theta_i^1(\theta_{-i})$ is the smallest type that i could report and still obtain a unit of the good. It is evident from (10) that if $\theta_i < \theta_i^1(\theta_{-i})$ then, $-t_i^*(\theta_i, \theta_{-i}) = 0$. Suppose $\theta_i \geq \theta_i^1(\theta_{-i})$, and let j^k, $k = 1, 2, \ldots, n$, be the identity of the buyer who, along with i, will obtain a unit of the good when buyer i's type reaches the point $\theta_i^k(\theta_{-i})$. Notice that $\alpha_{ij^1} < \alpha_{ij^2} < \cdots < \alpha_{ij^n}$ by the monotonicity of $v_i(\cdot, \theta_{-i})$. Obviously, if buyer i reports θ_i, then he obtains a unit of the good and j^n obtains the other unit. Then (setting $\theta_i^{n+1}(\theta_{-i}) = \theta_i$),

$$\int_{\underline{\theta}_i}^{\theta_i} v_i(s, \theta_{-i})\,ds = \sum_{p=1}^{n} \alpha_{ij^p} \left(\theta_i^{p+1}(\theta_{-i}) - \theta_i^p(\theta_{-i}) \right),$$

and therefore,

$$\begin{aligned}
-t_i^*(\theta_i, \theta_{-i}) &= \theta_i \alpha_{ij^n} - \sum_{p=1}^{n} \alpha_{ij^p} \left(\theta_i^{p+1}(\theta_{-i}) - \theta_i^p(\theta_{-i}) \right) \\
&= \theta_i^1(\theta_{-i}) \alpha_{ij^1} + \sum_{p=1}^{n-1} (\alpha_{ij^{p+1}} - \alpha_{ij^p}) \theta_i^{p+1}(\theta_{-i}).
\end{aligned}$$

[10] Notice that this also explains the intuition underlying Example 1.

FIGURE 1

PAYMENT RULE

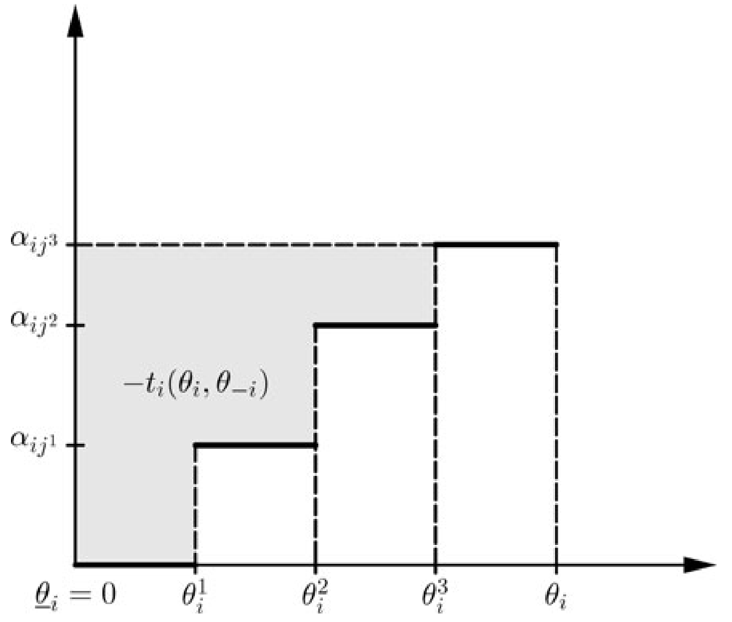

Figure 1 provides an illustration of the payment rule in which, given (θ_i, θ_{-i}), i obtains a unit of the good along with j^3, and there are two other buyers, j^1 and j^2, with whom i could be paired.

In summary, the optimal payment rule (10) is equivalent to

$$-t_i^*(\theta_i, \theta_{-i}) = \begin{cases} \theta_i^1(\theta_{-i})\alpha_{ij^1} + \sum_{p=1}^{n-1}(\alpha_{ij^{p+1}} - \alpha_{ij^p})\theta_i^{p+1}(\theta_{-i}) & \text{if } \theta_i \geq \theta_i^1(\theta_{-i}) \\ 0 & \text{if } \theta_i < \theta_i^1(\theta_{-i}). \end{cases} \tag{11}$$

The interpretation of (11) is as follows. Suppose that, given (θ_i, θ_{-i}), i obtains a unit of the good and j^n receives the other unit. Then the amount that i pays is the sum of two terms. The first term is $\alpha_{ij^1}\theta_i^1(\theta_{-i})$, which is the value the good would have had to him had he submitted the lowest winning report given θ_{-i}, namely, $\theta_i^1(\theta_{-i})$. The second term is the sum of the increments in utility derived from the higher external benefits that i would have enjoyed, had he submitted the lowest winning reports that would have paired him with, respectively, j^2, j^3, and so on. For instance, the smallest winning report above $\theta_i^1(\theta_{-i})$ that pairs i with j^2 is $\theta_i^2(\theta_{-i})$, and the incremental utility i enjoys is $(\alpha_{ij^2} - \alpha_{ij^1})\theta_i^2(\theta_{-i})$, which is internalized by the optimal payment rule (11).[11]

□ **Properties of the optimal mechanism.** The optimal mechanism derived above exhibits the following important properties:

Allocation and external effects. The probability that a buyer obtains a unit of the good is increasing in both the externality he enjoys and the one he generates. To see this, let $y_i^*(\theta) = \sum_{j=0}^{I} y_{[i,j]}^*(\theta)$ be the probability that buyer i obtains a unit of the good, given the vector of announced types θ. Careful inspection of the optimal allocation rule $y^*(\cdot)$ reveals that $y_i^*(\theta)$ is larger the bigger are α_{ij} or α_{ji} for any $j \neq i$.

[11] This payment rule is similar to the one in Levin (1997), who analyzes an optimal multi-unit auction problem with synergies among the objects instead of externalities among the buyers. He shows that a buyer's payment internalizes the complementarities among the goods in an incremental way that resembles (11).

Notice also that, unlike the case with a single unit or with multiple units but without externalities, a buyer could obtain a unit of the good despite having $J_i(\theta_i) < 0$, so long as he is paired with another buyer who enjoys a large external effect from being with *i*. For a numerical illustration, consider Example 1 but with $\theta_1 = .95$, $\theta_2 = 0.6$, and $\theta_3 = 0.4$. Then it is optimal to allocate the units of the good to buyers 1 and 3, even though $J_3(0.4) = -0.2$.

Payment and external effects. The amount of money paid by a buyer who obtains a unit of the good is a decreasing function of the externality he imposes on the buyer who obtains the other unit. To prove this assertion, consider *i*'s payment when he obtains a unit of the good along with j^n (see (11)). We will show that $-t_i(\theta)$ decreases in $\alpha_{j^n i}$. Recall that $\theta_i^n(\theta_{-i})$ is the threshold type of buyer *i* that pairs him with j^n instead of with j^{n-1}. Formally, $\theta_i^n(\theta_{-i})$ solves $S_{[i,j^n]}(\theta_i^n(\theta_{-i}), \theta_{j^n}) = S_{[i,j^{n-1}]}(\theta_i^n(\theta_{-i}), \theta_{j^{n-1}})$, which yields

$$\theta_i^n(\theta_{-i}) = J_i^{-1}\left(\frac{\alpha_{j^{n-1}i}J_{j^{n-1}}(\theta_{j^{n-1}}) - \alpha_{j^n i}J_{j^n}(\theta_{j^n})}{\alpha_{ij^n} - \alpha_{ij^{n-1}}}\right).$$

Now, $\alpha_{ij^n} > \alpha_{ij^{n-1}}$ and $J_i(\cdot)$ increasing imply that $\theta_i^n(\theta_{-i})$ is a decreasing function of $\alpha_{j^n i}$. And because $-t_i(\theta)$ increases in $\theta_i^n(\theta_{-i})$, it follows that $-t_i(\theta)$ decreases in $\alpha_{j^n i}$, thus proving the result.

Dominant strategy implementation. It is easy to verify that, under $(y^*(\cdot), t^*(\cdot))$, it is a dominant strategy for buyers to report their true types. For if a buyer's report were higher than his type, then he could be paired with a buyer who generates a larger externality, but any increment in his utility is paid to the seller (see (11)). Thus, he cannot gain by overstating his type. And if a buyer's report were lower than his type, then he could either cease to obtain a unit of the good, which clearly reduces his payoff, or be paired with a buyer who generates a lower externality, in which case his net payoff goes down as well.

As an illustration, consider Figure 1 and suppose buyer *i* reports $\hat{\theta}_i > \theta_i$. Two things can happen. If $\hat{\theta}_i < \theta_i^4$, then *i* will still be paired with j^3 and his payment remains unaltered. If $\hat{\theta}_i \geq \theta_i^4$, then *i* will now be paired with a buyer who generates a larger externality, say buyer j^k. But $(\alpha_{ij^k} - \alpha_{ij^3})\theta_i$ becomes part of buyer *i*'s payment. Hence, his payoff is the same as when he reports his true type θ_i. Similarly, he cannot gain by reporting $\hat{\theta}_i < \theta_i$. For instance, if $\theta_i^2 \leq \hat{\theta}_i < \theta_i^3$, then *i* will be paired with j^2, and his payoff is reduced by the amount $(\alpha_{ij^3} - \alpha_{ij^2})(\theta_i - \theta_i^3)$.

Inefficient allocation of the units. The allocation rule $y^*(\cdot)$ need not yield an *ex post* efficient allocation of the units of the good. When types are common knowledge, the seller's optimal mechanism is straightforward. To wit, she should allocate the units of the good to the pair $[i, j]$ with the largest $\alpha_{ij}\theta_i + \alpha_{ji}\theta_j$, and then charge $\alpha_{ij}\theta_i$ to buyer *i* and $\alpha_{ji}\theta_j$ to *j*, thereby extracting all the surplus from the two buyers. Obviously, the allocation that ensues is *ex post* efficient. Moreover, because $\underline{\theta}_i \geq 0$ for all *i* and $\alpha_{ij} \geq 1$, the seller should always sell the two units of the good to different buyers. Under the optimal allocation rule $y^*(\cdot)$, however, the seller may keep one or both units, and even when she sells both units, she need not allocate them to the pair $[i, j]$ with the largest $\alpha_{ij}\theta_i + \alpha_{ji}\theta_j$, as the following example illustrates.

Example 2 (Inefficient allocation). There are three bidders with valuations distributed uniformly on $[\underline{\theta}, \overline{\theta}]$, with $2\underline{\theta} > \overline{\theta}$. Thus, $J_i(\theta_i) = 2\theta_i - \overline{\theta} > 0$ for all θ_i and *i*, and the seller always sells both units. Let $\alpha_{21} = \alpha_{31} = \alpha > 1$, $\alpha_{12} = \alpha_{13} = \alpha_{23} = \alpha_{32} = 1$; that is, only buyer 1 generates externalities. Consider $\theta = (\theta_1, \theta_2, \theta_3)$, with $\theta_2 > \theta_3$. Notice that in this case it is never optimal for the seller to allocate the goods to [1, 3]. It is easy to verify that the two units will be sold to [1, 2] if and only if $(\theta_1 + \alpha\theta_2) - (\theta_2 + \theta_3) \geq \frac{\overline{\theta}}{2}(\alpha - 1) > 0$. Thus, it could happen that the seller allocates the two units to [2, 3] even though $(\theta_1 + \alpha\theta_2) > (\theta_2 + \theta_3)$; that is, the allocation is inefficient. For a numerical illustration, set $\underline{\theta} = 1.05$, $\overline{\theta} = 2$, $\alpha = 1.1$, $\theta_1 = 1.07$, $\theta_2 = 1.2$, and $\theta_3 = 1.1$.

The intuition underlying the inefficiency illustrated in Example 2 is as follows. The existence of identity-dependent externalities introduces an asymmetry in the model, for they make buyers *ex ante* heterogeneous even with identically distributed types. It is well known that, without externalities, the presence of asymmetric bidders can lead to the type of inefficiency portrayed in Example 2 (e.g., see Myerson, 1981). The analysis above suitably generalizes this result to the case with externalities. Another important implication of this asymmetry is that it makes it extremely difficult to find an indirect mechanism that implements the optimal auction $(y^*(\cdot), t^*(\cdot))$.

4. Application: shopping malls and interstore externalities

■ As an illustration, we apply the model to the shopping mall example described in Section 2. Our intention is not to provide a complete analysis of this problem, which is quite complex in practice. Instead, our objectives are (i) to show that our model can shed light on the empirical evidence on the allocation and pricing of shopping mall space, and (ii) to show that a sequential selling procedure commonly used in practice need not be an optimal mechanism for the seller.

□ **Evidence on interstore externalities.** Pashigian and Gould (1998) and Gould, Pashigian, and Prendergast (2005) provide a detailed empirical analysis of the importance of interstore externalities in the pricing and composition of retail space in shopping malls. Indeed, a common procedure used by developers is to first sign at a low price (rent) per square foot the so-called anchor stores (i.e., department stores), which are the ones that generate the largest mall traffic that benefits all the stores. Then they offer the remaining space to non-anchor stores (Pashigian and Gould, 1998). A large fraction of anchor stores pay no rent or a trivial amount, and among the stores that pay rent, the average price per square foot paid by non-anchor stores is about seven times higher than that paid by anchor stores (Gould, Pashigian, and Prendergast, 2005). Thus, the evidence shows that the probability of signing a store is an increasing function of the externalities it generates, whereas the price paid decreases (increases) in the externalities generated (enjoyed).

□ **A simple model with an anchor store.** To shed light on the main features of this problem, let us consider the case of three potential stores in which only store 1 generates externalities, that is, store 1 is an anchor store. Formally, let $\alpha_{21} = \alpha_{31} = \alpha > 1$, $\alpha_{12} = \alpha_{13} = \alpha_{23} = \alpha_{32} = 1$. For simplicity, assume that θ_i is distributed on $[\underline{\theta}, \overline{\theta}]$ with density $\phi(\cdot)$ for all i and $\underline{\theta}\phi(\underline{\theta}) > 1$, thereby ensuring that $J(\underline{\theta}) > 0$ and hence $J(\theta_i) > 0$ for all θ_i, $i = 1, 2, 3$.

□ **A sequential procedure.** Suppose the seller uses the following sequential procedure. In the first stage, she makes a take-it-or-leave-it offer to store 1. If store 1 accepts, it obtains one structure and then the seller uses a first-price auction to allocate the remaining one between stores 2 and 3 in the second stage. If store 1 rejects, then the seller sells one structure to store 2 and the other to store 3 at a price $\underline{\theta}$ per unit in the second stage. Notice that the seller uses an optimal mechanism in each possible case in which she deals with stores 2 and 3.[12]

Consider the second stage. If there are two structures left, stores 2 and 3 accept the offer and the seller's revenue is equal to $2\underline{\theta}$. If there is only one structure left, then the store with the highest type between 2 and 3 obtains it. It is straightforward to verify that the seller's expected revenue in this case is $\alpha E[\min\{\theta_2, \theta_3\}] = 2\alpha \int_{\underline{\theta}}^{\overline{\theta}} s(1 - \Phi(s))\phi(s)\, ds$.

Let us now turn to the first stage. The seller's problem is

$$\max_{p \geq \underline{\theta}} (1 - \Phi(p)) \left(p + 2\alpha \int_{\underline{\theta}}^{\overline{\theta}} s(1 - \Phi(s))\phi(s)\, ds \right) + \Phi(p) 2\underline{\theta},$$

where p is the take-it-or-leave-it offer tendered to store 1.

[12] If $J(\underline{\theta}) < 0$, we only need to introduce a "reserve price" θ_r in the second stage.

The solution p^* to this problem is the following: if $\alpha \geq \alpha^S$, then $p^* = \underline{\theta}$, whereas if $\alpha < \alpha^S$, then p^* is the unique solution to

$$J(p^*) = 2\underline{\theta} - 2\alpha \int_{\underline{\theta}}^{\overline{\theta}} s(1 - \Phi(s))\phi(s)\, ds. \tag{12}$$

The threshold value of the external effect, α^S, is given by

$$\alpha^S = \frac{\underline{\theta}\phi(\underline{\theta}) + 1}{2\phi(\underline{\theta}) \int_{\underline{\theta}}^{\overline{\theta}} s(1 - \Phi(s))\phi(s)\, ds}.$$

The properties of the sequential mechanism can be summarized as follows. Differentiation of (12) reveals that the offer the seller makes to store 1 is decreasing in the size of the external effect it generates. And if store 1 accepts the offer, then the price paid for the remaining unit is an increasing function of the externality enjoyed by the store that acquires it. Also, when the size of the externality is relatively small, there is a positive probability that store 1 will not obtain a structure. But if the externality that store 1 generates is sufficiently large, then the seller ensures that the anchor store receive a unit for sure, and her expected revenue is $\underline{\theta} + 2\alpha \int_{\underline{\theta}}^{\overline{\theta}} s(1 - \Phi(s))\phi(s)\, ds$.

□ **Comparison with the optimal mechanism.** The properties of the sequential mechanism are broadly consistent with the empirical evidence, and they are also closely related to those of the optimal mechanism discussed in Section 3. A natural question to ask is whether the sequential mechanism is indeed optimal for the seller. We will compare it with the optimal mechanism and show that the answer is negative. The comparison relies on the following result:

Lemma 3. Store 1 obtains a structure with probability one in the optimal mechanism if and only if $\alpha \geq \alpha^O = 2 - \frac{\underline{\theta}\phi(\underline{\theta}) - 1}{\overline{\theta}\phi(\underline{\theta})}$.

Proof. We need to find the smallest value of α such that, for every $\theta = (\theta_1, \theta_2, \theta_3)$, either $J(\theta_1) + \alpha\, J(\theta_2) \geq J(\theta_2) + J(\theta_3)$ or $J(\theta_1) + \alpha\, J(\theta_3) \geq J(\theta_2) + J(\theta_3)$. Equivalently, the following condition on α must hold for every $\theta = (\theta_1, \theta_2, \theta_3)$:

$$\alpha \geq 1 + \min\left\{ \frac{J(\theta_3) - J(\theta_1)}{J(\theta_2)}, \frac{J(\theta_2) - J(\theta_1)}{J(\theta_3)} \right\}. \tag{13}$$

Suppose $\theta_2 \geq \theta_3$. Then the right side of (13) becomes $1 + \frac{J(\theta_3) - J(\theta_1)}{J(\theta_2)}$. The largest value of this expression occurs when $\theta_1 = \underline{\theta}$, $\theta_2 = \theta_3$, and $\theta_3 = \overline{\theta}$, which is equal to $2 - \frac{\underline{\theta}\phi(\underline{\theta}) - 1}{\overline{\theta}\phi(\underline{\theta})}$. If $\alpha \geq 2 - \frac{\underline{\theta}\phi(\underline{\theta}) - 1}{\overline{\theta}\phi(\underline{\theta})}$, then (13) holds for all $\theta = (\theta_1, \theta_2, \theta_3)$ with $\theta_2 \geq \theta_3$. And if (13) holds for all $\theta = (\theta_1, \theta_2, \theta_3)$ with $\theta_2 \geq \theta_3$, then it holds for $\theta = (\underline{\theta}, \overline{\theta}, \overline{\theta})$. The case with $\theta_3 \geq \theta_2$ is analogous. We have thus shown that store 1 obtains a structure with probability one if and only if $\alpha \geq \alpha^O = 2 - \frac{\underline{\theta}\phi(\underline{\theta}) - 1}{\overline{\theta}\phi(\underline{\theta})}$.

It is straightforward to calculate the payments when $\alpha \geq \alpha^O$. Because store 1 always receives one structure, its smallest winning report is $\underline{\theta}$, and this is its payment. Store 2 obtains the other structure if and only if $\theta_2 \geq \theta_3$; the smallest report that allows it to receive the remaining unit is θ_3, and its payment is then $\alpha\, \theta_3$. Thus, the seller's expected revenue is $\underline{\theta} + \alpha E[\min\{\theta_2, \theta_3\}]$.

Notice that the allocation and payments under the optimal mechanism when $\alpha \geq \alpha^O$ are the same as in the sequential mechanism when $\alpha \geq \alpha^S$. Thus, the sequential mechanism is optimal if $\alpha \geq \max\, \{\alpha^O, \alpha^S\}$. This is intuitive, for if the externality generated by the anchor store is sufficiently large, then it is optimal for the seller to ensure that this store locates in the mall. This is accomplished by selling a structure to the anchor store at the lowest price, and then profit from the sale of the other structure, whose price increases due to the presence of the anchor store.

More interestingly, we now show that when externalities are not large enough, the sequential mechanism does not maximize the seller's expected revenue.

Proposition 1. The threshold α^O is strictly bigger than α^S. Thus, whenever $\alpha^S \leq \alpha \leq \alpha^O$, store 1 obtains a structure in the sequential mechanism more often than in the optimal mechanism.

Proof. Simple algebra reveals that $\alpha^O - \alpha^S > 0$ if and only if

$$(\phi(\underline{\theta})(2\overline{\theta} - \underline{\theta}) + 1)2 \int_{\underline{\theta}}^{\overline{\theta}} s(1 - \Phi(s))\phi(s)\, ds - \underline{\theta}\overline{\theta}\phi(\underline{\theta}) - \overline{\theta} > 0. \tag{14}$$

But $2\int_{\underline{\theta}}^{\overline{\theta}} s(1 - \Phi(s))\phi(s)\, ds = E[\min\{\theta_2, \theta_3\}] > \underline{\theta}$. Hence, the left side of (14) is greater than $(\underline{\theta}\phi(\underline{\theta}) - 1)(\overline{\theta} - \underline{\theta}) > 0$, thereby proving that $\alpha^O > \alpha^S$.

The analysis reveals that a common procedure used in practice is not an optimal mechanism for the seller unless the externality generated by the anchor store is large enough to ensure that it obtains a structure with probability one. When the external effects generated by the anchor store are not sufficiently strong, the seller can improve her expected revenue over that of the sequential mechanism by increasing the probability of selling the two structures to stores 2 and 3. In this way, she raises the amount that store 1 pays whenever it obtains a structure.

5. Extensions

■ Thus far, we have focused on the case of two units of the good, positive externalities, and multiplicative interaction between a buyer's type and the externality he enjoys. As we emphasized in Section 2, we made these assumptions with the sole purpose of simplifying the presentation of the main results. We now show that none of the insights depend on them.

□ **Negative externalities.** The analysis of the optimal mechanism does not depend on the assumption that externalities are positive. To wit, nowhere in Section 3 have we used the fact that $\alpha_{ij} \geq 1$ for all $i \neq j \neq 0$. Thus, all the results in that section also hold when $\alpha_{ij} \in (0, 1]$. The only exception is that, with negative externalities, *ex post* efficiency sometimes requires the seller to sell only one unit of the good. This gives rise to an interesting property, which has no counterpart in the single- or multi-unit cases without externalities: under the optimal mechanism, the seller may sell the second unit of the good in cases in which it is *ex post* efficient for her to keep it. The following example illustrates this phenomenon.

Example 3 (Seller sells when she shouldn't). There are three bidders with valuations distributed uniformly on $[0, \overline{\theta}_i]$, so $J_i(\theta_i) = 2\theta_i - \overline{\theta}_i$ for all i. Let $\overline{\theta}_1 = 10$, $\overline{\theta}_2 = \overline{\theta}_3 = 5$, $\alpha_{12} = \alpha_{13} = 0.3$, and $\alpha_{21} = \alpha_{31} = \alpha_{23} = \alpha_{32} = 1$. That is, buyers 2 and 3 impose a negative externality on buyer 1. Consider types $\theta_1 = 6$, $\theta_2 = 3.3$, and $\theta_3 = 1$. Under complete information, the seller should sell one unit to buyer 1 for a payment of 6, and keep the other unit. To see this, note that the best the seller can do if she sells both units is to allocate them to buyers 1 and 2, whose combined surplus is $0.3\theta_1 + \theta_2 = 5.1 < 6$. Under incomplete information, however, $J_1(6) = 2$, $J_2(3.3) = 1.6$, and $J_3(1) = -3$ imply that it is optimal for the seller to sell both units to buyers 1 and 2, whose combined virtual surplus is $0.3\, J_1(6) + J_2(3.3) = 2.2 > 2$. In other words, the seller sells the second unit of the good when the (*ex post*) efficient allocation instructs her to keep it.[13]

□ ***N* units of the good.** For simplicity, we have assumed that the seller owns only two units of the good. Barring notational complexity, the main insights readily extend to the N-unit case. We now outline the modifications needed in order to accommodate this case. Let $I \geq N$, and define Λ as the set of all unordered N-tuples λ in $\{0, 1, \ldots, I\}^N$. A DRM is a pair $(y(\theta), t(\theta))$, where $y(\theta) = (y_\lambda(\theta))_{\lambda \in \Lambda}$ and $t(\theta) = (t_0(\theta), \ldots, t_I(\theta))$. For each i, the externality parameters are defined as $0 \leq \alpha_{i\mathbf{j}} \leq \overline{\alpha}$ for every $\mathbf{j}$, where $\mathbf{j}$ is an unordered $N - 1$-tuple such that $\lambda = [i, \mathbf{j}\,] \in \Lambda$. If $\mathbf{j}$

[13] This inefficiency also arises in the single-unit case in which the winner imposes an externality on the losers (see Jehiel, Moldovanu, and Stacchetti, 1996; Brocas, 2007; Figueroa and Skreta, 2008). The example extends this result to the multi-unit case with externalities among those who obtain the units of the good.

consists of all zeros or all i's, then $\alpha_{i\mathbf{j}} = 1$, and the same holds for $i = 0$ and any $\mathbf{j}$. All the sums with respect to j in Sections 2 and 3 should now be replaced by sums over all $\mathbf{j}$ such that $[i, \mathbf{j}] \in \Lambda$. With these modifications, all the results extend to the case in which the seller owns N units of the good.

□ **Supermodular interaction of types and externalities.** We have assumed that a buyer's payoff is multiplicatively separable in his type and the externality he enjoys. The following example illustrates the usefulness of allowing for more general complementarities between them.

Example 4 (*Shopping mall*). Suppose that if store i locates in the shopping mall, then it faces a linear demand $P_i = \theta_i + \alpha_{ij} - Q_i$ for its product, where j denotes the identity of the neighboring store. For simplicity, let the cost of production be zero. It is easy to verify that i's profit function is equal to $\frac{(\alpha_{ij}+\theta_i)^2}{4}$. Notice that profits are increasing in α_{ij} and θ_i, (strictly) convex in θ_i, and (strictly) supermodular in (α_{ij}, θ_i).[14]

We will show that the results extend to buyers' payoff functions of the form $u_i(\alpha_{ij}, \theta_i) + t_i$, with $u_i(\cdot, \cdot)$ nonnegative, $u_i(\alpha_{ij}, \theta_i) > u_i(\alpha_{ik}, \theta_i)$ if $\alpha_{ij} > \alpha_{ik}$, $\frac{\partial u_i}{\partial \theta_i} > 0$, $\frac{\partial^2 u_i}{\partial \theta_i^2} \geq 0$, and $u_i(\cdot, \cdot)$ supermodular in (α_{ij}, θ_i).[15] Note that $\alpha_{ij}\,\theta_i$ and $\alpha_{ij} + \theta_i$ satisfy all these properties.

For each buyer i, let

$$V_i(\alpha_{ij}, \theta_i) = u_i(\alpha_{ij}, \theta_i) - \frac{(1 - \Phi_i(\theta_i))}{\phi_i(\theta_i)} \frac{\partial u_i(\alpha_{ij}, \theta_i)}{\partial \theta_i}. \tag{15}$$

We assume that $V_i(\alpha_{ij}, \theta_i)$ is strictly increasing in θ_i and supermodular in (α_{ij}, θ_i). In the multiplicatively separable case, $V_i(\alpha_{ij}, \theta_i) = \alpha_{ij}\, J_i(\theta_i)$, which clearly satisfies all these properties.

The Appendix contains the details of the analysis of this extension. Under the assumptions made, an optimal mechanism $(y^*(\cdot), t^*(\cdot))$ for the seller is given by the following allocation and payment rules: for every pair $[i, j]$, $1 \leq i < j \leq I$, set

$$y^*_{[i,j]}(\theta) = \begin{cases} 1 & \text{if } W_{[i,j]}(\theta_i, \theta_j) > \max\{0, \max_l V_l(\alpha_{l0}, \theta_l), \max_{[l,k]} W_{[l,k],l\neq k,l,k\geq 1}(\theta_l, \theta_k)\} \\ 0 & \text{otherwise,} \end{cases}$$

where $W_{[i,j]}(\theta_i, \theta_j) = V_i(\alpha_{ij}, \theta_i) + V_j(\alpha_{ji}, \theta_j)$; for every pair $[i, 0]$, $i = 1, \ldots, I$, set

$$y^*_{[i,0]}(\theta) = \begin{cases} 1 & \text{if } J_i(\theta_i) > \max\{0, \max_{l\neq i} V_l(\alpha_{l0}, \theta_l), \max_{[l,k]} W_{[l,k],l\neq k,l,k\geq 1}(\theta_l, \theta_k)\} \\ 0 & \text{otherwise;} \end{cases}$$

and for every pair $[i, i]$, $i = 1, \ldots, I$, set $y_{[i,i]}(\theta) = 0$. Finally, set

$$-t^*_i(\theta_i, \theta_{-i}) = \sum_{j=0}^{I} u_i(\alpha_{ij}, \theta_i) y^*_{[i,j]}(\theta_i, \theta_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} \left(\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij}, s)}{\partial \theta_i} y^*_{[i,j]}(s, \theta_{-i}) \right) ds.$$

Intuitively, the allocation and payment rules are suitable generalizations of those in the multiplicatively separable case. It is evident by inspection that the intuition of the allocation rule is the same as before, and a bit of work reveals that the same is true for the payment rule.

6. Concluding remarks

■ This article studies the optimal multi-unit auction design problem of a seller in the presence of buyers' private information and identity-dependent externalities. We show that it is optimal for the seller to allocate the units of the good to the set of buyers that generates the largest weighted

[14] A function $f(x, y)$ is supermodular if, given (x_1, y_1) and (x_2, y_2), $f(x_1 \vee x_2, y_1 \vee y_2) + f(x_1 \wedge x_2, y_1 \wedge y_2) \geq f(x_1, y_1) + f(x_2, y_2)$. If $f(x, y)$ is $\mathcal{C}^2$, this is equivalent to $\frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0$.

[15] An alternative way of stating some of these properties is as follows. Define the order $\succeq_i$ by $j \succeq_i k$ if and only if $\alpha_{ij} \geq \alpha_{ik}$ ($\succ_i$ is defined analogously). Then $u_i(\alpha_{ij}, \theta_i) > u_i(\alpha_{ik}, \theta_i)$ if $\alpha_{ij} > \alpha_{ik}$ is equivalent to $u_i(\alpha_{ij}, \theta_i)$ being increasing in j in the order $\succeq_i$; and $u_i(\cdot, \cdot)$ supermodular in (α_{ij}, θ_i) is equivalent to $u_i(\cdot, \cdot)$ supermodular in (j, θ_i).

sum of virtual surpluses. This rule often leads to an *ex post* inefficient allocation as the seller (i) may keep one or more units when it is *ex post* efficient to sell them, (ii) may sell to the set of buyers that does not maximize the joint valuation for the units, or (iii) may sell a unit when it is *ex post* efficient to keep it. We derive an optimal payment rule that illustrates how the seller can structure payments in such a way that buyers who obtain the goods pay according to the external benefits they enjoy and generate. As an application, we analyze the selling problem faced by a shopping center's developer, and we flesh out the main properties of a sequential mechanism commonly used in practice. It turns out that the sequential procedure is not an optimal selling mechanism unless interstore externalities are sufficiently large.

We have conducted the analysis under the assumption that the size of the identity-dependent externalities was common knowledge among the agents involved. Albeit this is a plausible assumption in many settings such as the shopping mall application that motivated this article, it would be desirable to relax it and solve the model allowing for two dimensions of private information. Given the well-known difficulties associated with mechanism design problems with multidimensional types, this extension is not only interesting but also apt to be nontrivial.

Appendix

■ In the general case with buyers' utility functions $u_i(\alpha_{ij}, \theta_i) + t_i$, the seller's problem is

$$\max_{(y_{[i,j]}(\cdot))_{[i,j]\neq[0,0]},(t_i(\cdot))_{1\leq i\leq I}} E_\theta\left[-\sum_{i=1}^{I} t_i(\theta)\right] \tag{A1}$$

$$\text{subject to}\quad U_i(\theta_i) \geq E_{\theta_{-i}}\left[\sum_{j=0}^{I} u_i(\alpha_{ij}, \theta_i) y_{[i,j]}(\hat{\theta}_i, \theta_{-i})\right] + \bar{t}_i(\hat{\theta}_i) \quad \forall(i, \theta_i, \hat{\theta}_i) \tag{A2}$$

$$U_i(\theta_i) \geq 0 \quad \forall(i, \theta_i) \tag{A3}$$

$$1 \geq y_{[i,j]}(\theta) \geq 0 \quad \forall([i,j], \theta) \tag{A4}$$

$$1 - \sum_{[i,j]\neq[0,0]} y_{[i,j]}(\theta) \geq 0 \quad \forall\theta \tag{A5}$$

The following conditions are necessary and sufficient for incentive compatibility:

(i) $E_{\theta_{-i}}[\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},\cdot)}{\partial\theta_i} y_{[i,j]}(\cdot, \theta_{-i})]$ is increasing; and
(ii) $U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} E_{\theta_{-i}}[\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},s)}{\partial\theta_i} y_{[i,j]}(s, \theta_{-i})]\, ds, \forall\theta_i \in \Theta_i$.

To prove necessity, let $(y(\cdot), t(\cdot))$ be incentive compatible. Then (ii) follows from an application of the envelope theorem. Regarding (i), let $\hat{\theta}_i > \theta_i$ wlog; then

$$U_i(\theta_i) \geq U_i(\hat{\theta}_i) + E_{\theta_{-i}}\left[\sum_{j=0}^{I}(u_i(\alpha_{ij}, \theta_i) - u_i(\alpha_{ij}, \hat{\theta}_i)) y_{[i,j]}(\hat{\theta}_i, \theta_{-i})\right]$$

$$U_i(\hat{\theta}_i) \geq U_i(\theta_i) + E_{\theta_{-i}}\left[\sum_{j=0}^{I}(u_i(\alpha_{ij}, \hat{\theta}_i) - u_i(\alpha_{ij}, \theta_i)) y_{[i,j]}(\theta_i, \theta_{-i})\right].$$

These inequalities yield

$$E_{\theta_{-i}}\left[\sum_{j=0}^{I}(u_i(\alpha_{ij}, \hat{\theta}_i) - u_i(\alpha_{ij}, \theta_i)) y_{[i,j]}(\hat{\theta}_i, \theta_{-i})\right] \geq E_{\theta_{-i}}\left[\sum_{j=0}^{I}(u_i(\alpha_{ij}, \hat{\theta}_i) - u_i(\alpha_{ij}, \theta_i)) y_{[i,j]}(\theta_i, \theta_{-i})\right],$$

which is equivalent to

$$E_{\theta_{-i}}\left[\sum_{j=0}^{I} u_i(\alpha_{ij},\hat{\theta}_i)y_{[i,j]}(\hat{\theta}_i,\theta_{-i})\right] + E_{\theta_{-i}}\left[\sum_{j=0}^{I} u_i(\alpha_{ij},\theta_i)y_{[i,j]}(\theta_i,\theta_{-i})\right]$$

$$\geq E_{\theta_{-i}}\left[\sum_{j=0}^{I} u_i(\alpha_{ij},\hat{\theta}_i)y_{[i,j]}(\theta_i,\theta_{-i})\right] + E_{\theta_{-i}}\left[\sum_{j=0}^{I} u_i(\alpha_{ij},\theta_i)y_{[i,j]}(\hat{\theta}_i,\theta_{-i})\right],$$

thereby showing that $E_{\theta_{-i}}[\sum_{j=0}^{I} u_i(\alpha_{ij},\theta_i)y_{[i,j]}(\hat{\theta}_i,\theta_{-i})]$ is supermodular in $(\theta_i,\hat{\theta}_i)$ or, equivalently,

$$E_{\theta_{-i}}\left[\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},\theta_i)}{\partial\theta_i} y_{[i,j]}(\cdot,\theta_{-i})\right] \tag{A6}$$

is increasing. Because $\frac{\partial^2 u_i}{\partial\theta_i^2} \geq 0$, (A6) is increasing in θ_i as well. Take $\theta_i'' > \theta_i'$; then

$$E_{\theta_{-i}}\left[\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},\theta_i'')}{\partial\theta_i} y_{[i,j]}(\theta_i'',\theta_{-i})\right] \geq E_{\theta_{-i}}\left[\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},\theta_i')}{\partial\theta_i} y_{[i,j]}(\theta_i'',\theta_{-i})\right]$$

$$\geq E_{\theta_{-i}}\left[\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},\theta_i')}{\partial\theta_i} y_{[i,j]}(\theta_i',\theta_{-i})\right],$$

where the first inequality follows by $\frac{\partial^2 u_i}{\partial\theta_i^2} \geq 0$ and the second by supermodularity.

To prove sufficiency, suppose that (i) and (ii) hold, and let $\hat{\theta}_i > \theta_i$. Then,

$$U_i(\hat{\theta}_i) - U_i(\theta_i) = \int_{\theta_i}^{\hat{\theta}_i} E_{\theta_{-i}}\left[\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},s)}{\partial\theta_i} y_{[i,j]}(s,\theta_{-i})\right] ds$$

$$\geq \int_{\theta_i}^{\hat{\theta}_i} E_{\theta_{-i}}\left[\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},s)}{\partial\theta_i} y_{[i,j]}(\theta_i,\theta_{-i})\right] ds, \tag{A7}$$

where the inequality follows from (A6). Similarly,

$$E_{\theta_{-i}}\left[\sum_{j=0}^{I} u_i(\alpha_{ij},\hat{\theta}_i)y_{[i,j]}(\theta_i,\theta_{-i})\right] + \bar{t}_i(\theta_i) - U_i(\theta_i) = E_{\theta_{-i}}\left[\sum_{j=0}^{I}(u_i(\alpha_{ij},\hat{\theta}_i) - u_i(\alpha_{ij},\theta_i))y_{[i,j]}(\theta_i,\theta_{-i})\right]$$

$$= \int_{\theta_i}^{\hat{\theta}_i} E_{\theta_{-i}}\left[\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},s)}{\partial\theta_i} y_{[i,j]}(\theta_i,\theta_{-i})\right] ds. \tag{A8}$$

Expressions (A7) and (A8) yield

$$U_i(\hat{\theta}_i) \geq E_{\theta_{-i}}\left[\sum_{j=0}^{I} u_i(\alpha_{ij},\hat{\theta}_i)y_{[i,j]}(\theta_i,\theta_{-i})\right] + \bar{t}_i(\theta_i),$$

which completes the proof of sufficiency.

Let $V_i(\alpha_{ij},\theta_i) = u_i(\alpha_{ij},\theta_i) - \frac{(1-\Phi_i(\theta_i))}{\phi_i(\theta_i)}\frac{\partial u_i(\alpha_{ij},\theta_i)}{\partial\theta_i}$; the seller's problem can be written as follows:

$$\max_{(y_{[i,j]}(\cdot))_{[i,j]\neq[0,0]}} \sum_{i=1}^{I} E_{\theta}\left[\sum_{j=0}^{I} V_i(\alpha_{ij},\theta_i)y_{[i,j]}(\theta)\right] \tag{A9}$$

subject to (A.4) and (A.5) and condition (i).

Consider the relaxed problem in which condition (i) is ignored. It is immediate to show that the solution to this problem is given by the allocation rule $y^*(\cdot)$. If this allocation rule satisfied (i) then, along with the payment rule $t^*(\cdot)$ (which can be derived following the same steps that led to (10)), they would constitute an optimal mechanism for the seller.

We now show that $y^*(\cdot)$ satisfies (i); it suffices to show that $\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},\cdot)}{\partial\theta_i} y^*_{[i,j]}(\cdot,\theta_{-i})$ is increasing in θ_i. Take $\theta_i' > \theta_i''$ and suppose $y^*_{[i,l]}(\theta_i'',\theta_{-i}) = 1$ (the other case is trivial); that is, $\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},\theta_i'')}{\partial\theta_i} y^*_{[i,j]}(\theta_i'',\theta_{-i}) = \frac{\partial u_i(\alpha_{il},\theta_i'')}{\partial\theta_i}$ for some l. As in Lemma 2, it is easy to show that $\sum_{j=0}^{I} \frac{\partial u_i(\alpha_{ij},\theta_i')}{\partial\theta_i} y^*_{[i,j]}(\theta_i',\theta_{-i}) > 0$; wlog, suppose that this sum is equal to $\frac{\partial u_i(\alpha_{ik},\theta_i')}{\partial\theta_i}$ for some k.

To complete the proof, we need to show that $\frac{\partial u_i(\alpha_{ik},\theta_i')}{\partial\theta_i} \geq \frac{\partial u_i(\alpha_{il},\theta_i'')}{\partial\theta_i}$. By supermodularity and convexity, it suffices to show that $\alpha_{ik} \geq \alpha_{il}$. Because $y^*_{[i,l]}(\theta_i'',\theta_{-i}) = 1$ and $y^*_{[i,k]}(\theta_i',\theta_{-i}) = 1$,

$$V_i(\alpha_{il},\theta_i'') + V_l(\alpha_{li},\theta_l) \geq V_i(\alpha_{ik},\theta_i'') + V_k(\alpha_{ki},\theta_k)$$

$$V_i(\alpha_{ik},\theta_i') + V_k(\alpha_{ki},\theta_k) \geq V_i(\alpha_{il},\theta_i') + V_l(\alpha_{li},\theta_l).$$

These inequalities yield

$$V_i(\alpha_{ik}, \theta_i'') + V_i(\alpha_{il}, \theta_i') \leq V_i(\alpha_{ik}, \theta_i') + V_i(\alpha_{il}, \theta_i''). \quad \text{(A10)}$$

If $\alpha_{ik} < \alpha_{il}$, then (A10) would violate the supermodularity of $V_i(\alpha_{ij}, \theta_i)$. Thus, $\alpha_{ik} \geq \alpha_{il}$.

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