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Segmented risk sharing in a continuous-time setting*

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Summary. The economy we study is comprised of a continuum of individuals. Each has a stochastic endowment that evolves continuously and independently of all other individuals' endowment processes. Individuals are risk averse and would therefore like to insure their endowment processes. The mutual independence of their endowment processes makes it feasible for them to obtain this insurance by pooling their endowments. We investigate whether such a scheme would survive as an equilibrium in a noncooperative setting.

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1 Introduction

The economy we study is comprised of a continuum of individuals. Each has a stochastic endowment that evolves continuously and independently of all other individuals' endowment processes. Individuals are risk averse and would therefore like to insure their endowment processes. The mutual independence of their endowment processes makes it feasible for them to obtain this insurance by pooling their endowments.

A contract implements this pooling. All individuals whose endowment processes lie within a finite segment pool their endowments and consume the average

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within that segment so that there is equal division within the segment.[1] In the event their endowment process crosses the boundary of the segment, they exit the contract. The continuous-time setting allows us to calculate conditional welfare of this arrangement using differential equation and stopping time methods that would be impractical in discrete-state and discrete-time settings.

In the first dynamic model we analyze there is a single such segment. Individuals who enter the contract enjoy a period of constant consumption, but eventually hit the boundary and exit. While away from the upper or lower boundary of the segment they consume their endowments autarkically, but eventually their endowments return to the segment and they again enter the contract. We then investigate whether such a contract would be free from defection, i.e., whether an individual whose endowment currently was within the contract bounds but exceeded the contractual consumption level would wish to exit the contract early and consume autarkically, gaining the persistent higher consumption of his endowment. Even if a permanent punishment is exacted in this case – in game-theoretic parlance, a grim punishment – the incentive to defect would still exist. In the absence of an external commitment mechanism, such a contract could not enforce itself by threatening such punishments, regardless of the size of the segment and magnitude of insurance. Since grim punishments are the maximum available in this setting, and since the feasibility of the contract depends on the averaging of the high as well as low endowments, this contract could not be sustained as an equilibrium. The fear of loss of the insurance provided by the contract just does not have enough punishment power.

One way to increase efficiency is to increase the completeness of the insurance. We do this by replicating the segment contract described above so that every level of endowment lies within the boundary of at least one segment. The contracts can be structured so that an individual who exits one segment joins the contract associated with an adjoining segment, and we consider the case of infinitely many such segments. Each individual's consumption process thus jumps at discrete times and is constant between jumps. In this setting an individual who defects from a segment can be punished with the denial of insurance not only from the segment he prematurely exits, but from adjacent segments as well, or even all of them in the grim punishment case. One of our central results is that this idea works when there are infinitely many segments, but that the grim punishment is unnecessarily harsh. A grim punishment equilibrium would require all individuals and segments to cooperate for all time in the punishment of any defector. We demonstrate that a finite domain of punishment, with limited cooperation between finite collections of segments, is sufficient to create punishments that deter defection. The segment equilibrium provides a lower value of insurance than the equivalent complete-market insurance, however: this reduced value is the cost of obeying the nondefection constraint.[2]

[1] We will use the terms risk-pooling group, segment contract, and contract interchangeably.

[2] A referee suggests the following interpretation. The major credit card companies deny credit to defaulters for six years (The companies coordinate, as with our segment construct). There are minor credit card companies that nevertheless offer credit to these defaulters, albeit for high interest rates.

Related Literature. Farrell and Scotchmer [10] analyzed a static model of group formation with the constraint that coalitions must divide their output equally among their members. They demonstrate that the groups that form are consecutive; i.e., individuals interact with others with similar characteristics. Our first model reproduces this result in a risk-sharing context in order to set the stage for the dynamic model. The durability of cooperative groups has previously been examined in Greenberg and Weber [12]; their focus is on general static cooperative game structures rather than the non-cooperative dynamic risk sharing that is our focus here.

Our noncooperative approach stems from the literature on dynamic insurance without commitment. Taub [24], Kehoe and Levine [16], Kocherlakota [19], and Alvarez and Jermann [2] characterize risk-sharing contracts when individuals can defect at any given time, and the punishment threat is reversion to autarky forever; the last two papers are also able to characterize the set of constrained efficient contracts in this setting, as well as the allocation dynamics that ensue. A recent paper by Kletzer and Wright [18] refines the equilibrium concept used in the aforementioned papers and characterizes contracts that are also renegotiation-proof; the punishments they construct yield a path of behavior that resembles what is actually observed when a sovereign country defaults on its debt. Krueger and Perri [20], and Attanasio and Rios-Rull [3] characterize constrained efficient insurance contracts in economies with a large number of agents; they focus on how changes in taxes or in the provision of public insurance against aggregate shocks affect the severity of punishment threats and therefore the amount of risk sharing that can be achieved using private contracts. Unlike these contributions, our model focuses on the sustainability of equilibria when risk pooling is conducted within groups that need not be perfectly coordinated, and where the identity of their members changes continuously. This difficulty forces us to focus on a narrower set of arrangements than those papers, i.e. contracts with equal division of the aggregate endowment of the group; we provide some motivation for this assumption in our dynamic framework. Krueger and Uhlig [21] also analyze a dynamic risk-sharing model with a large number of groups or 'villages' each containing a continuum of agents. In their framework, an individual can, at any time, defect from the contract signed with the 'village king' and instantly join another village in which she starts anew. Under the assumption that the groups behave in a competitive way, they show that the only equilibrium outcome is the autarkic allocation. Although their discrete-time model model differs in several ways from ours, two crucial differences are that we allow groups to coordinate their behavior in the punishment of defectors, and that it takes agents time to move across groups; these features permit us to sustain equilibria that dominate autarky.

The advantages of the continuous-time methods we use are evident in the literature on bandit and experimentation problems and their application to industrial organization theory, as in the recent papers of Bolton and Harris [6], Moscarini and Smith [23], Keller and Rady [17], and Bergemann and Valimaki [4], which also take advantage of boundary-crossing and stopping-time methods.

In the next section we construct a one-period model of risk sharing as an intuitive foundation for the dynamic model. We then set out the technical features of the dynamic model and analyze a benchmark case. The properties of a contract with a single segment are then analyzed. The subsequent sections examine the welfare and equilibrium properties of the model when there are infinitely many segments, first with full commitment, then when the assumption of full commitment is dropped.

2 A static model

We intuitively motivate the segmentation and equal division assumptions that we impose in the sequel by considering a one-period setting[3] in which there is a fundamental motivation to form groups in order to share risks. There is a discrete and finite collection $N = \{1, 2, \ldots, n\}$ of individuals whose endowments y_i are drawn from Gaussian distributions $N(m_i, \sigma^2)$ with publicly known parameters; moreover, the means are ordered so that $m_1 > m_2 > \ldots > m_n$. The agents are risk averse and have von Neumann-Morgenstern utility function $u(c) = -e^{-ac}$.[4] Because the variances are common across individuals, the endowment distributions are ordered by first order stochastic dominance, so that under autarky individuals with a low index are better off.[5]

Contracts are possible prior to the draw of endowments, and despite the ordering of preferences over autarkic endowments, risk sharing contracts are desirable. The one restriction that will be imposed on contracts is that they be of the equal division form, as in Farrell and Scotchmer [10]: all realized endowment within a contracting group must be split evenly among its members. While this restriction is ad hoc at this point, we provide a more compelling motivation in our dynamic model.

Using the distributional assumption, the ith agent receives expected utility under autarky as follows:

$$v^i_{\{i\}} = -e^{-am_i + \frac{\sigma^2 a^2}{2}} \tag{2.1}$$

If a coalition $G \in N$ decides to pool their stochastic endowments and share them equally, each agent $i \in G$ has expected utility

$$v^i_{\{G\}} = E\left[-e^{-a\frac{\sum_{i\in G} y_i}{|G|}}\right] = -e^{\left(-a\frac{\sum_{i\in G} m_i}{|G|} + \frac{a^2\sigma^2}{2|G|}\right)}$$

since $\frac{\sum_{i\in G} y_i}{|G|}$ is Gaussian $N(\frac{\sum_{i\in G} m_i}{|G|}, \frac{\sigma^2}{|G|})$.

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[3] In other words, there is a contracting period prior to the realization of endowment.

[4] These narrowly defined preferences are needed in the more complicated model to come in order to avoid wealth effects.

[5] Without constant variance, and without a common risk aversion parameter, it might not be possible to unanimously order the endowment distributions.

The size and composition of a coalition affect the welfare of its members in two ways: first, due to the common variance, independent of the composition of the group, the larger the group, the smaller the risk that each agent bears. Second, the expected consumption of each member is affected by the composition of the group; groups containing agents with lower expected endowments enjoy lower levels of expected consumption. These effects reveal a trade-off: large coalitions allow for a better diversification of risks but for more heterogeneity, which affects the expected consumption of their members. Agents with better expected endowments may be reluctant to share risks with agents that have low expected endowments, in which case we would expect the formation of risk pooling groups containing “similar” agents.

We define a *stable coalition structure* as a partition of the set of players into coalitions such that no new coalition could form and make its players better off. It is straightforward to demonstrate that stable coalition structures that form must be consecutive; i.e., if i and j are in G and k is another agent such that $i < k < j$, then k is also in G.

Proposition 2.1 *In any stable coalition structure coalitions are consecutive.*

Proof. Suppose there is a stable coalition such that there is a group G containing i and $i+2$ but not $i+1$, with $i+1 \in G'$. Then either $(G \backslash \{i+2\}) \cup \{i+1\}$ or $(G' \backslash \{i+1\}) \cup \{i\}$ forms a blocking coalition, which contradicts the stability of the coalition structure. □

It is clear that this result depends on the unanimity of the ordering of the endowment lotteries, and it is also dependent on the equal division constraint. The next result is a characterization of the stable coalition structure.

Proposition 2.2 *There is a generically unique stable coalition structure.*

Proof. Consider agent 1, and let $j^1 \geq 1$ be the j that maximizes $v^{\{1\}}_{\{1,\ldots,j\}}$. If $j^1 \geq n$ then by the previous proposition N is the unique stable coalition structure. Otherwise, define $G_1 = \{1, \ldots, j^1\}$ as the first element of the partition. From the remaining agents $\{j^1+1, \ldots, n\}$, let $j^2 \geq j^1+1$ solve $\max_j v^{\{j_1+1\}}_{\{j^1+1,\ldots,j^1+1+j\}}$. If $j^2 \geq n-(j^1+1)$, then $\{G_1, N \backslash G_1\}$ is the unique stable structure. Otherwise, repeat the same argument for the remaining agents. The process leads to a coalition structure $\{G_1, \ldots, G_k\}$. By the previous proposition, it is evident that no single individual or coalition would block the partition. Moreover, the only nonuniqueness problem that might arise is when a coalition is indifferent between accepting or not accepting an additional agent; obviously this is non-generic. □

The stable coalition structure is characterized by an acute form of segmentation by income: high-income agents share risks among themselves, and they do not mingle with the rest of the population; the same is true for other groups, although everybody wishes to belong to a higher-mean coalition. Again, the equal-division rule makes it optimal for the high income agents to separate themselves from the rest of the population, since a better diversification of risk cannot be achieved without introducing more heterogeneity.[6]

This “segmentation” arose endogenously, conditional on the equal-division rule. However, these contracts are *not* stable if ex post defection is permitted: individuals with high ex-post endowments will wish to renege and consume autarkically. We now investigate a setting in which a similar form of segmentation is imposed exogenously, but with a more compelling motivation for equal division, and explore ways in which contracts could be stable to defection.

3 The dynamic model: preferences, endowments and interactions

In the dynamic setting the endowment process for each individual is $y(t)$, a nongeometric, driftless diffusion described by

$$dy(t) = dz(t)$$

where $z(t)$ is a standard Brownian motion. Each individual is indexed at time t by his endowment process $y(t)$. In some initial period $t = 0$, there is a uniform distribution of individuals, and therefore of endowments, on the entire real line R. The measure ν of individuals in an interval A is simply uniform Lebesgue measure.[7]

Preferences. Each individual maximizes discounted expected utility, with exponential instantaneous return function:

$$-E\left[\int_0^\infty e^{-\rho s} e^{-ac(s)} ds\right] \qquad \rho > \frac{a^2}{2}$$

where ρ is the subjective rate of time preference, a is the coefficient of absolute risk aversion, and $c(t)$ is the consumption process, and the condition $\rho > a^2/2$ is an assumption of our model needed for the convergence of the integral. We restrict attention to these preferences in conjunction with the non-geometric and driftless diffusion processes because it is easy to construct contracts due to the absence of wealth effects.[8]

In autarky individuals consume their endowment: $c(t) = y(t)$. Under permanent autarky, it is straightforward to calculate discounted expected utility for an agent whose initial endowment is $y(0) = y$:

$$v_A(y) = -E\left[\int_0^\infty e^{-\rho s} e^{-a(y+z(s))} ds\right]$$

where the z process is assumed to start at zero. Interchanging the expectation and integration yields

[6] We note in passing that the induction argument works if there is a countable infinity of agents as long as $m_1 < \infty$.

[7] Since we encompass the entire real line, ν is not a probability measure. However, this will not present any difficulty since our main focus will be the analysis of finite segments.

[8] The absence of wealth effects means that the segments we construct can be of equal width. In the presence of wealth effects this would probably not hold.

$$= -e^{-ay}\int_0^\infty e^{-\rho s}e^{\frac{1}{2}a^2 s}ds = -e^{-ay}\frac{1}{\rho - \frac{1}{2}a^2} \qquad \rho > \frac{a^2}{2} \tag{3.1}$$

This solution will serve as a benchmark in the subsequent analysis.

A benchmark contract. Consider a complete-insurance contract. At some initial date $t = 0$, an individual with endowment process $y(t)$ and initial endowment $y(0)$ can sign a contract that keeps his consumption constant at that level indefinitely, so $c(t) = y(0)$. If his endowment process subsequently shrinks, he will continue to consume $c(0)$, with the contract providing transfers to cover the shortfall; if his endowment exceeds $y(0)$ he must hand endowment in excess of $y(0)$ to the contract. Although each situation is equally probable, risk aversion makes the contract desirable.

All individuals have similar preferences, and they will all agree to a contract that allows them to consume a constant amount equal to their initial endowment. If those initial endowments are distributed uniformly over the real line, then the contract can be internally self-financing because individuals whose endowment wanders below its initial level can be subsidized by the equal number of individuals whose endowments have symmetrically risen above their initial value.[9]

A participant in such a contract whose endowment has evolved to a very high value at time T might weigh the benefits of continuing to consume at the low level $y(0)$ or defecting from the contract and consuming autarkically henceforth, with initial value $y(T)$. The defector would lose the insurance benefit of the constant consumption, but would be able to consume the current high level of endowment. This high level would persist because of the properties of Brownian motion. Eventually endowment would decline, but this eventuality is discounted.

The value of defection can be calculated. The value of remaining in the contract is simply the value of constant consumption. With our assumptions about preferences, this is

$$-\frac{e^{-ay(0)}}{\rho}$$

The value of defecting at time T with endowment $y(T) = y^*$ is

$$-\frac{e^{-ay^*}}{\rho - \frac{a^2}{2}}$$

An individual will be indifferent between defecting and remaining in the contract if these two quantities are equal:

$$-\frac{e^{-ay^*}}{\rho - \frac{a^2}{2}} - \left(-\frac{e^{-ay(0)}}{\rho}\right) = 0 \tag{3.2}$$

This occurs if

$$y^* = y(0) - \frac{1}{a}\ln\left(1 - \frac{a^2}{2\rho}\right)$$

[9] We are using a law of large numbers assumption here which is discussed at length in Appendix C.

If $\rho < a^2/2$, the insurance value of the contract is infinite and defection will not occur; under our assumption that $\rho > a^2/2$, the punishment for defection is finite. The solution y^* is a finite distance from the initial value of endowment, and so will be attained with positive probability in finite time.

The comparison of the values of staying in the contract and defecting appears unsophisticated because it allows defection to occur at just a single point in time. The decision problem fails to account for the option value of defecting later, which affects the value of staying in the contract. An individual who has not yet defected but who realizes that he will defect once he reaches y^* will account for this in his calculation of discounted utility:

$$E\left[-\int_t^{T_{y^*}} e^{-\rho s}e^{-ay(0)}ds - e^{-\rho T_{y^*}}\frac{e^{-ay^*}}{\rho - \frac{a^2}{2}}\right]$$

where T_{y^*} is now the time until passage of endowment to the defection value y^* – it is a random process. It is apparent that this option value will not change the value of defection calculated above.[10]

A full-insurance contract of this kind is therefore not feasible: the existence of an endowment y^* that initiates defection means that high-endowment individuals will not participate, and it is therefore infeasible to support the consumption of low-endowment individuals.

Is there a contract, perhaps with less insurance value, which is feasible? Feasibility could be attained by imposing symmetry on the participants so that those with extremely low endowments who would have insurance provided by those with very high endowments are excluded from the benefits, since their counterparties have defected. Individuals would be insured inside a segment, $[y(0) - (y^* - y(0)), y^*]$. Those with endowment beyond the upper boundary of the segment would be excluded because of defection; those below the lower boundary would be excluded to maintain symmetry. Such a segment uses equal division in this dynamic setting just as the segments of our static model.

The equal division construct can be motivated as follows. It is desirable to maximize insurance ex ante while maintaining feasibility. Individuals enter the contract at $y(0)$. Their insurance value is maximized by providing them with constant consumption during their sojourn in the contract. It is feasible for this constant to be $y(0)$ due to the averaging of endowments across individuals whose endowments start from $y(0)$ and whose endowments have subsequently remained in the interval $[y(0) - (y^* - y(0)), y^*]$ – but this is entirely equivalent to equal division.

[10] Temporarily postponing the defection decision leads to a comparison of the temporary value of autarky with the temporary value of remaining in the contract: $E\left[-\int_t^{T_{y^*}} e^{-\rho s}e^{-ay(s)}ds\right] - E\left[-\int_t^{T_{y^*}} e^{-\rho s}e^{-ay(0)}ds\right] = 0$. This is a fixed point problem in y^*. If there were no solution then y^* would be infinite, thereby making T_{y^*} infinite as well, and the value comparison would become $E\left[-\int_t^{\infty} e^{-\rho s}e^{-ay(s)}ds\right] - E\left[-\int_t^{\infty} e^{-\rho s}e^{-ay(0)}ds\right] < 0$. This comparison has already been demonstrated to hold with equality in (3.2) however for initial value y^*, a contradiction.

We can immediately see that this contract will fail because of the incentive to defect. If individuals hitting the lower boundary are permanently excluded for the sake of symmetry, then the value of insurance is reduced – this makes the punishment of defectors at y^* weaker, and their value of defection consequently increases. A more subtle contract is needed, and we now begin its construction.

4 A single-segment contract

In this section we explore the idea of an equal-sharing contract defined by a single segment more formally. Because the preferences in our model do not exhibit wealth effects, we can restrict our focus to a single segment, denoted by S, with boundaries 0 and $2D$, where D is some fixed constant. An individual with endowment in the segment – $0 < y(t) < 2D$ – receives consumption D.

Outside of the segment S the individual receives nothing from that segment. We begin by calculating the discounted payoff from this contract. The value is

$$-E\left[\int_0^\infty e^{-\rho t} e^{-aD} \mathbf{1}_{\{y(t)\in S\}} dt | y(0) = y\right] \tag{4.1}$$

where $\mathbf{1}_A$ is the indicator function, taking value 1 for elements in A and 0 otherwise. Although the methods we will use below do not depend on the constancy of e^{-aD}, we will make use of it here by noting that we can reduce the problem of finding the discounted payoff to solving

$$h(y) = E\left[\int_0^\infty \mathbf{1}_{\{y(t)\in S\}} e^{-\rho t} dt | y(0) = y\right] \tag{4.2}$$

This integral conforms to the structure of a similar expression in Karatzas and Shreve ([15], p. 271), equation (4.14), with $k(y(t)) = 0$ and $f(y(t)) = \mathbf{1}_{\{y(t)\in S\}}$. We can therefore make use of their Theorem 4.4.9, which states a differential equation whose solution is (4.2):

$$(\rho + k(y))h(y) = \frac{1}{2}h''(y) + f(y) \tag{4.3}$$

The theorem notes that the integral is a Laplace transform and makes use of that fact in deriving occupation times; our interest is in the integral directly, as it is a discounted utility. What makes the theorem especially useful is that f and k only need be piecewise continuous, and in addition if they are piecewise continuous then the solution h will be piecewise-C^2 (twice differentiable with continuous derivative) which yields the appropriate boundary conditions.

In the absence of defection our model has the following differential equations:

$$\rho h = \frac{1}{2}h'' + 1, \qquad x \in S \tag{4.4}$$

$$\rho h = \frac{1}{2}h'', \qquad x \notin S; \tag{4.5}$$

The boundary conditions are established by the requirement that the level and first derivative of the solution must be continuous at the boundaries of S:

$$h(0-) = h(0+); \qquad h'(0-) = h'(0+)$$

$$h(2D-) = h(2D+); \qquad h'(2D-) = h'(2D+)$$

The general solutions of (4.3-4.4) are

$$h(y) = A_1 e^{\sqrt{2\rho}y} + A_2 e^{-\sqrt{2\rho}y} + \frac{1}{\rho}, \qquad y \in S$$

$$h(y) = B_1 e^{\sqrt{2\rho}y} + B_2 e^{-\sqrt{2\rho}y}, \qquad y \notin S$$

The boundary conditions need to be applied to these equations. We derive the solutions in the Appendix.

If defection does not occur, then consumption is constant only when endowment lies inside the segment boundary. Whenever endowment wanders outside the segment boundary, endowment must be consumed directly. If an individual defects when endowment is inside the segment boundary he will immediately and permanently revert to autarkic consumption. The following proposition states that this incentive does indeed exist, and the contract is therefore infeasible.

Proposition 4.1 *Let there be a single segment* $[0, 2D]$. *Then, for any* $\rho > a^2/2$, *there exists* $y^* \in (D, 2D]$ *such that the defection incentive is positive at* y^*.

The proof is in the Appendix. The proof alters the boundary conditions to take account of defection, including the option value of defection. When this is done, there is not enough punishment power in the threat of permanent exclusion from the single-segment contract. In order to prevent defection, it is necessary to increase the size of the carrot – insurance – that can be removed. We develop more notation and begin this construction in the next section.

5 Multi-segment contract

The previous section illustrated that a single segment with grim punishments would fail to deter defection and would therefore be infeasible. In this section we set out notation for more general contracts of this type.

Contract structure. All the individuals with current endowments in a finite interval (risk pooling group) $[y_{k-1}, y_{k+1}]$ cooperate in the sense that each turns over all his endowment to an agent of the group, which we will simply refer to as the *contract*. In return, he receives the average endowment of the group as consumption. Since the distribution is uniform, this contractual consumption is

$y_k = (y_{k+1} + y_{k-1})/2$.[11] We refer to the interval and its associated contract as the kth *segment*, S_k. Individuals within the segment in the initial instant t can give up their endowment at time $s > t$ in exchange for the same payoff y_k regardless of how their individual endowments evolve, because each individual's endowment evolves independently, and the average endowment remains at y_k according to the law of large numbers; this is true even if endowments evolve beyond the boundaries of the segment.

Each individual participates in the contract until the first time his endowment goes beyond the upper or lower bound of the segment. Denote this stopping time as T_k; it is actually the minimum $T_{y_{k-1}} \wedge T_{y_{k+1}}$. Thus individuals leak out of this contract as their endowments hit the boundary.[12] Nevertheless, they will still value membership because of the temporary insurance value. This type of contract will be our central focus.

Boundary behavior. Because Brownian motion has unbounded variation, the endowment path after the escape time T_k will cross the boundary infinitely often after T_k in any finite time interval.[13] It is nevertheless possible to characterize contracts of this type because they can be represented as occupation times within the segment, as we did in the previous section and in Appendix A. Consumption that fluctuates infinitely often between y_k and y_{k+2} as these crossings occur makes it difficult to define such a value function recursively, however.

We can define contracts recursively by using a *hysteresis* approach: an individual entering segment $S_{k+1} \equiv [y_k, y_{k+2}]$ does so at endowment y_{k+1}; he receives that endowment until his escape from S_{k+1}. If the escape is into the lower segment S_k, it is at y_k, the contractual consumption level of S_k. He will thus consume y_k until his escape from $S_k \equiv [y_{k-1}, y_{k+1}]$, and so on. Thus the segments S_k and S_{k+1} share the interval $[y_k, y_{k+1}]$, but an individual whose endowment process is in that overlapping range can belong either to S_k or to S_{k+1} depending on the recent history of his endowment.[14] In other words, the only difference with the segmentation derived in the static model of Section 2 is that the characteristics of the members of adjacent groups are not disjoint; this makes the analysis more tractable without losing any of the features of the problem under study.

Under the hysteresis assumption each individual commences his sojourn in the contract from the midpoint of the segment. His ex ante insurance value is maximized by a contract that provides the maximum feasible constant con-

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[11] In a nonuniform or geometric Brownian motion setting we would have $c_k = \int_{y_{k-1}}^{y_{k+1}} y\,d\nu(y)$ where k is an integer. The measure ν is presumed stationary and hence the integration is independent of t, but again this depends on a law of large numbers property which we are simply assuming here. That is, we are assuming the measure of an interval of endowments reflects the underlying Gaussian distribution of individual sample paths, and we then integrate with respect to the measure of initial endowments, and we assign measure zero to to sample paths that do not reflect the underlying distribution. The Appendix elaborates on this point.

[12] The continuous-time setting is helpful because the mathematical expression of this boundary-hitting behavior is a boundary condition.

[13] A reference is the stop-loss start-gain paradox, [7].

[14] The approach is standard in inventory models using the (S, s) method over a single interval; see for example [22].

sumption. Because of the symmetry of the subsequent evolution of endowment processes it is feasible for the constant to have the same value as the midpoint of the segment.

The hysteresis construction bears on the static model we discussed initially. In that model the equal sharing rule was somewhat ad hoc in that agents within a coalition have different conditional means m_i. In the segment with hysteresis, all segment members commence their membership at the midpoint of the segment, where equal sharing is identical with full insurance during the sojourn in the segment.

Strategies. A strategy for an agent y is a function $\sigma_y(t)$ that, at each t, maps the agent's information into an action in $A = \{0, 1\}$, where 1 =*pool*, and 0 =*defect*. Let $(\sigma_y)_{y \in R}$ be the strategy vector of all the players, and denote by $v(y, (\sigma_y)_{y\in R}, t)$ the expected discounted utility of agent y at t under this strategy vector. Letting ν denote Lebesgue measure, define

$$\Delta(S_k, t) = \nu(\{y : \sigma_y(t) = 1, y(t) = y \in S_k\})$$

that is, the measure of agents in segment S_k who pursue the pooling strategy. At each t, an individual knows the following about the history of the economy: the punishment regime put in place by the segments, the history of his endowment, and $\Delta(S_k, t')$ for all $t' < t$.

Each *segment* has a strategy as well. We denote this the *punishment regime*. Punishment regimes consist of the denial of insurance to an individual based on the history of his actions and endowments: $C[(y, A, \sigma_y)^t, E]$, where the superscript indicates a measurable history within a domain of punishment E. A domain of punishment is the boundaries (over time or endowment) during which punishment occurs. Since punishment requires observation, it is synonymous with the information needed to impose it.

Equilibrium. Our equilibrium concept is standard. A vector $(\sigma_y)_{y\in R}$ is a Subgame Perfect Equilibrium if, for (almost) all $y \in R$ and all t,

$$v(y, (\sigma_y)_{y\in R}, t) \geq v(y, (\hat{\sigma}_y, \sigma_{-y}), t)$$

for any alternative strategy $\hat{\sigma}_y$. That is, agent y at each t, finds it optimal to use σ_y given the strategies of the rest of the agents, which in turn determine the resources that each S_k will have and therefore y's consumption from t onwards.

Our goal is to study the restrictions on the parameters of the model (mainly the discount rate) that are sufficient for agents to pool their endowments at every t. In order to do that, we focus on the following simple class of trigger strategies:

$$\sigma_y(t) = \begin{cases} 1 & \text{if } t = 0 \text{ or if } \Delta(S_k, t') = y_{k+1} - y_{k-1} \text{ for all } k, t' < t \\ 0 & \text{otherwise} \end{cases}$$

In other words, an agent will pool his endowment as long as everyone pools; otherwise, he will cease to pool. Because these trigger strategies entail extreme punishments, they delineate the largest set of equilibria; in particular we wish to establish whether this set is nonempty. If a vector $(\sigma_y)_{y\in R}$ of these strategies conform to an SPE, the observable outcome would be pooling at every t.

In order to check whether this can be an SPE, we need to examine if agents are responding optimally after any possible history. To this end, we can partition histories in two groups: (i) histories where $\Delta(S_k, t') = y_{k+1} - y_{k-1}$ for all k and $t' < t$, and (ii) histories where this does not hold for some $t' < t$ and k.

If the history of the game at any t belongs to the second group ("off the equilibrium path") then, given that nobody is pooling, each agent is responding optimally by not pooling. In other words, agents are consuming autarkically, which is a Nash equilibrium of the subgame that starts at a history of the second group.

It remains to show whether they are acting optimally "on the equilibrium path"; i.e., at histories belonging to the first group where almost everybody is pooling. This requires calculating the expected discounted utility of an individual who always pools his endowment, which can be done using recursive methods. We undertake this task in the remainder of the section.

Convergence in the infinite extent model. We now contemplate an arbitrary number of segments. Because of our assumptions of exponential utility (that is, CARA) and Brownian endowment, we focus on segments that all have the same width $2D$: that is, $y_k - y_{k-1} = D$ for all k. There are infinitely many, so each individual's consumption process jumps at discrete times and is constant in between jumps.

Locally eliminating risk does not fully eliminate risk: even though constant consumption is feasible within coalitions because each has a continuum of individuals, risk is not eliminated because of the jumps in consumption that occur in transitions across segments, and this risk is a decreasing function of segment size, analogous with the decreasing risk of coalition size in the static discrete model.

Let there be an infinite collection of contract segments $\{S_k\}_{k=-\infty}^{\infty}$, defined by $S_k \equiv [y_{k-1}, y_{k+1}]$, and with the width of the segments defined by $y_{k+1} - y_{k-1} = 2D$. Thus each coalition is defined by a central endowment level y_k, and two boundary levels of endowment y_{k+1} and y_{k-1}. If an individual's endowment process $y(t)$ hits y_k, he then enters the kth contract segment and his consumption is constant and equal to y_k until he hits one of the boundaries y_{k+1} or y_{k-1}. At that time he enters either the $k+1$ or the $k-1$ contract segment respectively.

Define the following stopping times: $T_{y,y_{k+1}}$ is the time to go from y to y_{k+1}. Also, $T_{y,y_{k-1}} \wedge T_{y,y_{k+1}}$ is a double boundary escape time starting at $y \in S_k$. The following value function recursion then holds:

$$\begin{aligned} v(y, S_k) \quad &= \quad -e^{-ay_k} \frac{1 - E\left[e^{-\rho T_{y,y_{k-1}} \wedge T_{y,y_{k+1}}}\right]}{\rho} \\ &+ E\left[e^{-\rho T_{y,y_{k-1}} \wedge T_{y,y_{k+1}}} v\left(y\left(T_{y,y_{k-1}} \wedge T_{y,y_{k+1}}\right)\right)\right], \\ & y \in S_k, \qquad k = \ldots, -1, 0, 1, \ldots \end{aligned} \tag{5.1}$$

where the state variable S_k has been dropped for notational brevity on the right hand side. The instantaneous consumption rate before the first passage, y_k, is constant and unconnected to the initial or subsequent endowment $y(s)$. Also, the consumption level is known when the passage occurs. Therefore this formulation is separable in the stopping time and the instantaneous payoffs. This means the expectation of the $e^{-\rho T}$ terms can be independently calculated; conveniently, they are simply the Laplace transforms of the stopping times, for which explicit formulae can be developed. They are (Karatzas and Shreve [15], p. 100)

$$q(\rho, y - y_{k-1}, y_{k+1} - y) \equiv E\left[e^{-\rho T_{y,y_{k-1}} \wedge T_{y,y_{k+1}}}\right] = \frac{\cosh\left((y - D)\sqrt{2\rho}\right)}{\cosh\left(D\sqrt{2\rho}\right)} \tag{5.2}$$

where we are assuming $0 < y < 2D$.

The value recursion can now be solved at the segment boundaries using difference equation methods. Define $v_k = v(y_k, S_k)$ and

$$\lambda = e^{-\sqrt{2\rho}D} < 1.$$

After algebraic manipulation and appropriate application of boundary conditions (also detailed in the Appendix), the solution is

$$v_k = -\frac{1}{\rho}\frac{(1-\lambda)^2}{(1-\lambda e^{aD})(1-\lambda e^{-aD})}e^{-aDk} \tag{5.3}$$

which can be used to calculate $v(y, S_k)$.

We can now pose the following question: as the width D of the segments increases, does welfare for an individual centered within a segment asymptotically approach some level, and as the width approaches zero, does welfare approach that of the pure Brownian autarkic consumption case? The answer is Yes, showing that Brownian endowment is effectively approximated by a jump process:[15]

Proposition 5.1 *Let $y_k = kD$ be constant. Then full insurance is attained as the intervals grow large:*

(*i*) $$\lim_{D\to\infty} v_k = -\frac{1}{\rho}e^{-ay_k}$$

and as the intervals shrink to zero, all insurance value is lost:

(*ii*) $$\lim_{D\to 0} v_k = -\frac{1}{\rho - \frac{1}{2}a^2}e^{-ay_k}$$

which is autarkic utility.

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[15] The limiting jump process is in fact singular; see [9]. The average individual is an individual with an average initial endowment. However, since initial endowments range over the entire real line, the average is not well defined. The proposition is the weaker assertion that for an individual with initial endowment at the midpoint of a segment, full insurance is attained asymptotically as the width of the segments increases symmetrically about that midpoint.

Proof. Result (i) is straightforward. Result (ii) uses L'Hôpital's rule:

$$\lim_{D\to 0} v_k = -\frac{1}{2\rho}\left(\frac{\sqrt{2\rho}}{\sqrt{2\rho}-a} + \frac{\sqrt{2\rho}}{\sqrt{2\rho}+a}\right)e^{-ay_k}$$

After algebra this reduces to

$$-\frac{1}{\rho - \frac{1}{2}a^2}e^{-ay_k}$$

which is autarkic utility. □

The value of full insurance relative to autarky is thus

$$-\frac{1}{\rho}e^{-ay_k} + \frac{2}{2\rho - a^2}e^{-ay_k} = \frac{\frac{1}{2}a^2}{\rho(\rho - \frac{1}{2}a^2)}e^{-ay_k}$$

which is positive, increasing in a, and decreasing in ρ, as is intuitively sensible.

6 Defection and localized punishments

An individual in this setting whose endowment is near the upper boundary of his current segment, say S_k, might want to defect to autarky. In so doing he gets an immediate initial endowment that is near y_{k+1} which is significantly higher than the contractual consumption y_k. He takes on risk however, because consumption is now equal to endowment, at least until he hits y_k again or y_{k+1}. Is there some value of ρ low enough so that the defection temptation exceeds the fear of risk? For endowment close to the center of the current segment, y_k, there is no temptation for an individual to defect. But as the upper boundary is approached, the short run gains of consuming current endowment outweigh the long run loss of insurance. The reason is that since he is close to the upper boundary y_{k+1}, he will likely hit the boundary before the loss of insurance becomes compelling.

It is implicit that individuals defect alone, and that non-defectors cooperate in the punishment of excluding the defector from insurance within the local segment. This cooperation is costless because there is a continuum of agents. Of course the possibility of other equilibria exists, but we focus on this case of atomistic defection.

There are multiple possibilities for punishing a deviation. The easiest is *(i)* permanent defection to autarky, as might be imposed on a deviator as a grim trigger strategy, viewing the collection of contract segments $\{S_k\}_{-\infty}^{\infty}$ as a single contract. A second is *(ii)*, temporary deviation of an individual from the current contract S_k to autarky until a new segment, either S_{k-1} or S_{k+1}, is entered. After that time the agent can enter any contract. A third possibility, is intermediate between these: *(iii)* defectors are punished by the temporary exclusion from insurance, with reversion to the contract after traversing N segments. The temptation to defect arises when endowment is high, because that high endowment can be retained without paying the tax, and that benefit persists because Brownian motion is a martingale process. We find that arrangement *(ii)* therefore does not work, while *(i)*, the grim punishment, does work, unlike the single-segment case analyzed in the previous section. But grim punishments are not necessary: *(iii)*, temporary punishments can work, and we can characterize those situations. A limited folk theorem property holds, relating patience to the degree of efficiency.

Grim punishment. Suppose permanent autarky is imposed on defectors. As in Section 4, the temptation to defect can still occur because an individual at the top boundary of his segment (i.e., $2D$) could consume endowment in excess of his contractual consumption (i.e., D) for a finite interval and this might outweigh the loss of insurance. What is in essence the consequence of a folk theorem rules this out for low values of the discount rate:

Proposition 6.1 *Let defection from the kth segment be punished by imposing permanent autarky on the defector; all future insurance from all contract segments is barred. Then for values of ρ sufficiently close to $\frac{a^2}{2}$, defection is deterred.*

The proof is in the Appendix. The intuition for the result is clear if one views increasing the segment size as increasing the value of insurance; defection becomes more costly in lost insurance as the value of insurance rises. Increasing risk aversion works in the same direction, and minimizing the discount rate increases the cost of lost insurance against future downturns in endowment.

Proposition 6.1 shows that by using permanent punishments it is possible to deter defection and to support the equilibrium when there is an infinite number of segments, unlike the single-segment contract. However, this punishment construct has an informational attribute: in order to punish defectors, it is necessary to track them after they have left a segment. Thus, we cannot view the segments as acting autonomously without coordination of the punishment with other contract segments. Rather, there must be a great deal of information about defectors and there must be complete coordination across segments.

Localized punishments. We now analyze punishments that require only a local form of information-sharing among groups. Suppose that individuals may defect from a segment (say S_k) at any time, and the segment then withholds insurance from the individual for the duration of his stay in the segment. When he enters an adjacent segment, S_{k-1} or S_{k+1}, he resumes the contract in that segment. Moreover, if he ever re-enters the segment from which he defected, his prior defection is forgotten (bygones are bygones). Thus, the punishment is temporary and *mild*. Can this sustain the equilibrium? Because the punishment is temporary, it cannot.

Proposition 6.2 *Consider the coalition segment $[0, 2D]$ with constant consumption D under the contract. Let defection from the contract be punished by imposing temporary autarky on the defector until the random time $T_0 \wedge T_{2D}$. Then for any $\rho > \frac{a^2}{2}$ there is a defection threshold y^* such that $y^* \in (D, 2D]$.*

Proof. The proposition is a corollary of Proposition 4.1. At re-entry the defector has the option of continuing in the contract or defecting again; this option value must be non-negative relative to strictly remaining in the contract. Thus the temporary gain from defection documented in Proposition 4.1 is not diminished by the temporary punishment considered here. □

The import of the proposition is that within-segment punishments are too mild, and there are no parameter values that result in such punishments deterring defection. It was already established in the previous section that grim punishments can work, but as we pointed out, grim punishments require permanent tracking of defectors. Is there a set of punishments that requires less memory? We now turn to this question.

Extended localized punishments. We now analyze defection that results in temporary punishments that are more lasting than defection from just a single segment, but which eventually terminate. In particular, a defector from the kth segment must maintain autarkic consumption until he hits the entry point of the $k+N$th or $k-N$th segment.[16] The above analysis considered the case of $N = 1$. Because permanent autarky punishments can deter defection and single-segment defection fails to do so, we seek the intermediate cases in which defection is marginally prevented. If we find an N such that defection is always deterred, then we have a model of *local* coordination in which *finite* collections of segments cooperating in punishments of defectors is an equilibrium. While not full anonymity, this *partial anonymity* suffices for an equilibrium. It is partial anonymity in that a defection is forgotten once the boundary of the Nth additional segment has been crossed by the defector.[17]

As with the single-segment punishments, we assume permanent reversion to the contract value when the barriers are hit, so we need compare only the during-sojourn value of defection versus staying in the contract. This requires only the verification that the value of defection is never positive within a segment as in Section 4, and so only a comparison of the no-defection value with the defection value need be calculated. This value is relatively easy to compute for the autarkic payoff: the $N = 1$ method holds but with wider boundaries. The contract value is harder to calculate, however, because of the multiple steps that are followed until the Nth boundary. We follow the strategy of subtracting the continuation value at those boundaries.

As in our earlier analysis the absence of wealth effects permits us to focus on just one segment, $[0, 2D]$. We assume that $y \in (D, 2D)$. Thus the value of the contract (i.e. not defecting) until hitting is

$$\begin{aligned} v_C(y, N) = v(y) \quad &- \quad E\left[e^{-\rho T_{(N+1)D}} \mathbf{1}_{\{T_{(N+1)D} < T_{-(N-1)D}\}}\right] v((N+1)D) \\ &- \quad E\left[e^{-\rho T_{-(N-1)D}} \mathbf{1}_{\{T_{-(N-1)D} < T_{(N+1)D}\}}\right] v(-(N-1)D) \end{aligned}$$

On the other hand, we already have the temporary value of defecting and consuming autarkically until hitting:

[16] Because this occurs at a stopping time, the punishment regime is in the spirit of temporary punishments as in Abreu, Pearce and Stacchetti [1].

[17] Of course in equilibrium no defection occurs and no punishment is ever exacted.

$$v_A(y,N) = -\frac{e^{-ay}}{\rho - \frac{a^2}{2}} \left(1 - e^{a(y+(N-1)D)}\frac{\sinh(\sqrt{2\rho}((N+1)D-y))}{\sinh(\sqrt{2\rho}2ND)} - e^{-a((N+1)D-y)}\frac{\sinh(\sqrt{2\rho}(y+(N-1)D))}{\sinh(\sqrt{2\rho}2ND)}\right) \quad (6.1)$$

The main work is therefore in calculating $v_C(y,N)$. Continuing,

$$v_C(y,N) = v(y) - \frac{\sinh(\sqrt{2\rho}((N+1)D-y))}{\sinh(\sqrt{2\rho}2ND)}v(-(N-1)D) - \frac{\sinh(\sqrt{2\rho}(y+(N-1)D))}{\sinh(\sqrt{2\rho}2ND)}v((N+1)D) \quad (6.2)$$

Recalling that $\lambda \equiv e^{-\sqrt{2\rho}D}$, substitute the following values for $v(y)$, $v((N+1)D)$ and $v((1-N)D)$:

$$v(y) = -\frac{e^{-aD}}{\rho}\left(1 - \frac{\sinh(\sqrt{2\rho}(2D-y))}{\sinh(\sqrt{2\rho}2D)} - \frac{\sinh(\sqrt{2\rho}y)}{\sinh(\sqrt{2\rho}2D)}\right) + \frac{\sinh(\sqrt{2\rho}(2D-y))}{\sinh(\sqrt{2\rho}2D)}v(0) + \frac{\sinh(\sqrt{2\rho}y)}{\sinh(\sqrt{2\rho}2D)}v(2D) \quad (6.3)$$

$$v((N+1)D) = -\frac{1}{\rho}\frac{(1-\lambda)^2}{(1-\lambda e^{aD})(1-\lambda e^{-aD})}e^{-a(N+1)D} \quad (6.4a)$$

$$v((1-N)D) = -\frac{1}{\rho}\frac{(1-\lambda)^2}{(1-\lambda e^{aD})(1-\lambda e^{-aD})}e^{-a(1-N)D} \quad (6.4b)$$

We can use the continuity property of these formulas to demonstrate the result.

Proposition 6.3 *Let defection from a segment S_k be punished by imposing temporary autarky on the defector until the random time $T_{(k+1-N)D} \wedge T_{k+2ND}$, where N is a positive integer. Then for ρ sufficiently close to $\frac{a^2}{2}$ there exists $N^* < \infty$ such that for all $y \in S_k$ there is no incentive to defect from the local contract.*

The proof is in the Appendix. It uses the continuity of the value functions v_A and v_C in N and the transition of the defection value from positive in Proposition 6.2 ($N = 1$) to negative in Proposition 6.1 ($N = \infty$).

7 Conclusion

We have implemented a technical framework which expands on classical continuous-time inventory models, in which the interactions of many individuals can be analyzed. We can construct equilibria that have the following realistic properties: (i) Individuals are long-lived and their endowments fluctuate stochastically; (ii) Individuals share risk through pooling arrangements; (iii) Risk-pooling institutions are long-lived but live independently of the individuals who comprise them in the long run; (iv) Institutions must cooperate in potential punishments (which do not occur in equilibrium), so that isolated risk-sharing arrangements are not sustainable, but only finite collections of adjacent segments must cooperate;[18] (v) Individuals are anonymous in the long run, and complete simultaneous coordination across segments is not needed; (vi) Risk-sharing is incomplete.

Our results began with risk-sharing as the motivation for forming coalitions, and we found that a higher level of coalition formation was needed to make enforcement possible. We interpreted the minimum degree of segment cooperation needed for equilibrium as the minimum *information* needed, since enforcement of punishments rests on being able to observe defectors and their endowment processes.

In a longer working-paper version of this paper [8] we have developed a non-cooperative notion of segment size by considering when a segment equilibrium would also be robust against the dynamic formation of coalitions of segments, and show that there is a natural segment size that is finite. An economy in this setting would thus be expected to crystallize with an infinite number of segments of this size. Although our model is obviously abstract, it does put forward a clear and potentially empirically testable theory of the size of institutions indexed by risk aversion and impatience parameters. The concrete expression of this is a folk theorem result: as patience increases, so does the equilibrium segment size, and hence so does the value of insurance that can be attained in equilibrium.

We have not fully explored the role of markets in this setting. As we have already seen, we can easily envision a (complete) market in which individuals sell their endowments in return for a permanent fixed endowment equal to their initial endowment. Risk aversion will drive individuals to do this ex ante and it is feasible. As in most dynamic general equilibrium models, however, the market equilibrium will lead to states in which endowment processes are completely out of alignment with asset-determined consumption, with the resultant temptation to defect. Incorporating defection with the standard grim punishment approach requires ascribing a huge degree of commitment to institutions and presumes an extreme amount of information in order to track and punish defectors indefinitely.[19] Our model's temporary punishment concept and partial anonymity is a tentative beginning toward filling in these lacunae.

Appendix

A Non-hysteresis segment contract value functions

In this section we prove Proposition 4.1. As preparation we set out more general notation and develop solutions for a fundamental segment contract. In the general

[18] This resembles overlapping generations equilibria. There is interaction between adjacent generations but sustaining an efficient equilibrium requires infinitely many generations.

[19] As in the papers of Taub [24], Kehoe and Levine [16] and Alvarez and Jermann [2]. It is straightforward to construct another incomplete market structure that is more conventional-looking within our framework: a credit equilibrium. A portfolio of bonds is held and used as the medium of risk-sharing. Feasibility requirements determine the interest rate: $(\sqrt{1+2\rho a^2}-1)/a^2$. If defection results in grim punishments, defection cannot be deterred, whereas our segment equilibrium is sustainable in those settings.

case there are multiple contract segments of the sort in Section 4, each indexed by k. Each contract segment S_k is marked by boundaries y_k and y_{k+1}. An individual with endowment in the segment – $y_k < y(t) < y_{k+1}$ – receives $y_k + D$. The segments we consider here need not overlap, unlike the hysteresis model we consider in Section 5.

Outside of the segment S_k the individual receives nothing from that segment. We begin by calculating the discounted payoff from a single segment with this feature: that is, we suppress consumption outside the segment. The value is

$$\phi(k,y) = -E\left[\int_0^\infty e^{-\rho t} e^{-a(y_k+D)} \mathbf{1}_{\{y(t)\in S_k\}} dt|y\right] \tag{A.1}$$

Although the methods we will use below do not depend on the constancy of $e^{a(y_k+D)}$, we will make use of it here by noting that we can reduce the problem to solving

$$h(k,y) = E\left[\int_0^\infty \mathbf{1}_{\{y(t)\in S_k\}} e^{-\rho t} dt|y(0) = y\right] \tag{A.2}$$

which satisfies the following boundary value problem.

$$\rho h = \frac{1}{2}h'' + 1, \qquad x \in S_k; \tag{A.3}$$

$$\rho h = \frac{1}{2}h'', \qquad x \notin S_k; \tag{A.4}$$

The boundary conditions are established by the requirement that the level and first derivative of the solution must be continuous at the boundaries of S_k:

$$h(y_k-) = h(y_k+); \qquad h'(y_k-) = h'(y_k+)$$

$$h(y_{k+1}-) = h(y_{k+1}+); \qquad h'(y_{k+1}-) = h'(y_{k+1}+)$$

The general solutions of the DE's (A.3-4) are

$$h(y) = A_1 e^{\sqrt{2\rho}y} + A_2 e^{-\sqrt{2\rho}y} + \frac{1}{\rho}, \qquad y \in S_k$$

$$h(y) = B_1 e^{\sqrt{2\rho}y} + B_2 e^{-\sqrt{2\rho}y}, \qquad y \notin S_k$$

The boundary conditions need to be applied to these equations.

Stationarity and boundary conditions. Because h is the discounted occupation time for S_k, we can intuitively see that the value of the integral must shrink rather than grow as the initial value deviates from the boundaries of S_k. This requires that for $y > y_{k+1}$, $B_1 = 0$, and for $y < y_k$, $B_2 = 0$. The boundary condition equations are then

$$\begin{aligned}
A_1 e^{\sqrt{2\rho}y_k} + A_2 e^{-\sqrt{2\rho}y_k} + \frac{1}{\rho} &= B_1 e^{\sqrt{2\rho}y_k} \\
A_1 e^{\sqrt{2\rho}y_k} - A_2 e^{-\sqrt{2\rho}y_k} &= B_1 e^{\sqrt{2\rho}y_k} \\
A_1 e^{\sqrt{2\rho}y_{k+1}} + A_2 e^{-\sqrt{2\rho}y_{k+1}} + \frac{1}{\rho} &= B_2 e^{-\sqrt{2\rho}y_{k+1}} \\
A_1 e^{\sqrt{2\rho}y_{k+1}} - A_2 e^{-\sqrt{2\rho}y_{k+1}} &= -B_2 e^{-\sqrt{2\rho}y_{k+1}}
\end{aligned}$$

In matrix form,

$$\begin{pmatrix} e^{\sqrt{2\rho}y_k} & e^{-\sqrt{2\rho}y_k} \\ e^{\sqrt{2\rho}y_k} & -e^{-\sqrt{2\rho}y_k} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} B_1 e^{\sqrt{2\rho}y_k} \\ B_1 e^{\sqrt{2\rho}y_k} \end{pmatrix} - \begin{pmatrix} 1/\rho \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} e^{\sqrt{2\rho}y_{k+1}} & e^{-\sqrt{2\rho}y_{k+1}} \\ e^{\sqrt{2\rho}y_{k+1}} & -e^{-\sqrt{2\rho}y_{k+1}} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} B_2 e^{-\sqrt{2\rho}y_{k+1}} \\ -B_2 e^{-\sqrt{2\rho}y_{k+1}} \end{pmatrix} - \begin{pmatrix} 1/\rho \\ 0 \end{pmatrix}$$

The solutions are

$$A_1 = -\frac{1}{2\rho} e^{-\sqrt{2\rho}y_{k+1}}; \qquad A_2 = -\frac{1}{2\rho} e^{\sqrt{2\rho}y_k};$$

$$B_1 = \frac{1}{2\rho}\left(e^{-\sqrt{2\rho}y_k} - e^{-\sqrt{2\rho}y_{k+1}}\right); \qquad B_2 = \frac{1}{2\rho}\left(e^{\sqrt{2\rho}y_{k+1}} - e^{\sqrt{2\rho}y_k}\right)$$

As a check on the results, observe that $A_i > 0$, $B_i > 0$. These derivations can be expanded to include the many-segment case in a straightforward manner: the value is simply the sum of the values $\phi(k, y)$ for each segment.

It is immediate that the solution of $\phi(k, y)$ is $e^{-a(y_k+D)}h(y)$. If there are multiple segments, the contracts of this type are independent and their values can be added.

In Proposition 4.1, there is only a single segment, so this independence property is not needed, and the proof accounts for consumption that takes place outside the segment.

Proof of Proposition 4.1. The method of proof is to look for a solution of y^* from the smooth pasting approach – namely the solution of second-order ordinary differential equations and boundary conditions like the function ϕ above – and then demonstrate that $y^* < 2D$ for all $\rho > \frac{a^2}{2}$. Unlike the Feynman-Kac approach used above, we will retain the utility terms e^{-aD} and e^{-ay} throughout, and we will also account for the consumption that occurs outside the segment.

Inside the segment, if the result is true, then continuation will occur only in $[0, y^*)$. Below 0, endowment is consumed but the value function takes account of the future re-entry into the segment. This leads to the following solutions for the value function, similar to those in Section 4 of the text:

$$v_C(y) = -\frac{e^{-aD}}{\rho} + A_1 e^{-\sqrt{2\rho}y} + A_2 e^{\sqrt{2\rho}y}, \qquad y \in [0, y^*)$$

$$v_{A-}(y) = \frac{-e^{-ay}}{\rho - \frac{a^2}{2}} + B_2 e^{\sqrt{2\rho} y} \qquad y \in (-\infty, 0)$$

$$v_{A+}(y) = \frac{-e^{-ay}}{\rho - \frac{a^2}{2}} \qquad y \in [y^*, \infty)$$

The boundary conditions are

$$v_C(y^*) = v_{A+}(y^*) \qquad v'_C(y^*) = v'_{A+}(y^*)$$

$$v_C(0) = v_{A-}(0) \qquad v'_C(0) = v'_{A-}(0)$$

In matrix form these boundary conditions are

$$\begin{pmatrix} g(y^*) & g(y^*)^{-1} & 0 \\ -g(y^*) & g(y^*)^{-1} & 0 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} \frac{e^{-aD}}{\rho} - \frac{e^{-ay^*}}{\rho - \frac{a^2}{2}} \\ \frac{a}{\sqrt{2\rho}} \frac{e^{-ay^*}}{\rho - \frac{a^2}{2}} \\ \frac{e^{-aD}}{\rho} - \frac{1}{\rho - \frac{a^2}{2}} \\ \frac{a/\sqrt{2\rho}}{\rho - \frac{a^2}{2}} \end{pmatrix}$$

Solving three of the equations for the linear coefficients and substituting into the fourth yields the following nonlinear equation in y^*:

$$\left(-\frac{e^{-aD}}{\rho} + (1 + a/\sqrt{2\rho}) \frac{e^{-ay^*}}{\rho - \frac{a^2}{2}} \right) e^{\sqrt{2\rho} y^*} = \frac{1 + a/\sqrt{2\rho}}{\rho - \frac{a^2}{2}} - \frac{e^{-aD}}{\rho}$$

Algebraic simplification yields

$$f(y^*) \equiv \left(\frac{e^{-ay^*}}{1 - a/\sqrt{2\rho}} - e^{-aD} \right) e^{\sqrt{2\rho} y^*} = \frac{1}{1 - a/\sqrt{2\rho}} - e^{-aD} \equiv K \qquad \text{(A.5)}$$

The following properties are straightforward to verify: $K > 0$, $f(0) = K$, $f'(y^*) > 0$ if $y^* < D$, $f'(y^*) < 0$ if $y^* > D$, and $\lim_{y^* \to \infty} f(y^*) = -\infty$. Therefore (A.5) has a solution $y^* \in (0, \infty)$. In addition, for $\sqrt{2\rho} = a/(1 - e^{aD})$, $f(2D) < 0$, and so there is a $\rho < \infty$ such that $y^* \in [0, 2D)$. Equation (A.5) can be rearranged as follows:

$$1 > e^{-aD} \left(1 - \frac{a}{\sqrt{2\rho}} \right) = \frac{1 - e^{\left(\sqrt{2\rho} - a\right) 2D}}{1 - e^{\sqrt{2\rho} 2D}} > 1$$

demonstrating that no solution exists for $y^* = 2D$ for any value of $\rho > \frac{a^2}{2}$. Since a solution $y^* < 2D$ exists for $\rho < \infty$ then by the continuity of f and K in ρ there is a solution $y^* < 2D$ for all $\rho > \frac{a^2}{2}$. □

B Derivation of v_k and proofs of Propositions 6.1 and 6.3

The probability in (5.2) can be decomposed, and one way is as follows ([15], p. 100). Define the passage probability to the lower boundary as:

$$r(\rho, y - y_{k-1}, y_{k+1} - y) \equiv E\left[e^{-\rho T_{y,y_{k-1}}}\mathbf{1}_{\{T_{y_{k-1}} < T_{y_{k+1}}\}}|y\right] = \frac{\sinh\left((y_{k+1} - y)\sqrt{2\rho}\right)}{\sinh\left((y_{k+1} - y_{k-1})\sqrt{2\rho}\right)}$$

and

$$r(\rho, y_{k+1} - y, y - y_{k-1}) \equiv E\left[e^{-\rho T_{y,y_{k+1}}}\mathbf{1}_{\{T_{y_{k+1}} < T_{y_{k-1}}\}}|y\right] = \frac{\sinh\left((y - y_{k-1})\sqrt{2\rho}\right)}{\sinh\left((y_{k+1} - y_{k-1})\sqrt{2\rho}\right)}$$

The following relation also holds:

$$q(\rho, y - y_{k-1}, y_{k+1} - y) = r(\rho, y - y_{k-1}, y_{k+1} - y) + r(\rho, y_{k+1} - y, y - y_{k-1})$$

The value recursion with equal-width segments. The value recursion can be restated using these terms. For $y \in S_k$ we have:

$$\begin{aligned} v(y, S_k) = & - \frac{e^{-ay_k}}{\rho}\left(1 - q(\rho, y - y_{k-1}, y_{k+1} - y)\right) \\ & + r(\rho, y - y_{k-1}, y_{k+1} - y)v_{k-1} \\ & + r(\rho, y_{k+1} - y, y - y_{k-1})v_{k+1} \end{aligned}$$

In particular, the equation will hold at $y = y_k$, and this can be used to state a pure difference equation. Defining

$$v_k \equiv v(y_k, S_k),$$

we have

$$v_k = -\frac{e^{-ay_k}}{\rho}(1 - q(\rho, D, D)) + r(\rho, D, D)v_{k-1} + r(\rho, D, D)v_{k+1} \tag{B.1}$$

Because this equation is defined only on the coalition boundaries, notation can be simplified as follows:

$$w_k \equiv -\frac{1}{\rho}(1 - q)e^{-ay_k}$$

with explicit formulae for q and r,

$$q = q(\rho, D, D) = \frac{\cosh\left((D - D)\sqrt{2\rho}\right)}{\cosh\left(D\sqrt{2\rho}\right)} = \frac{1}{\cosh\left(D\sqrt{2\rho}\right)} = \frac{2}{e^{D\sqrt{2\rho}} + e^{-D\sqrt{2\rho}}}$$

and

$$r = \frac{\sinh\left(D\sqrt{2\rho}\right)}{\sinh\left(2D\sqrt{2\rho}\right)} = \frac{1}{2\sinh\left(D\sqrt{2\rho}\right)} = \frac{1}{e^{D\sqrt{2\rho}} + e^{-D\sqrt{2\rho}}} = \frac{q}{2}$$

Note that $0 < q < 1$; it is a probability.

Welfare calculation. Equation (B.1) can now be solved using difference-equation methods. Recapping (B.1) in lag-operator notation,

$$\frac{1}{1+\lambda^2}(1-\lambda L)(1-\lambda L^{-1})v_k = w_k \tag{B.2}$$

where

$$\lambda = e^{-\sqrt{2\rho}D} < 1.$$

In lag operator notation, the equation can be stated as:

$$(1 - rL - rL^{-1})v_k = w_k.$$

It is possible to analyze the solution using ordinary factorization methods. The crucial issue will be convergence; it must be demonstrated that the inversion of the left hand side dominates the growth of the w_k terms on the right hand side.

The growth of the w_k terms is easy to calculate. We have

$$w_k = -\frac{e^{-ay_k}}{\rho}(1 - q(\rho, D, D)) = -\frac{e^{-akD}}{\rho}\left(1 - \frac{2}{e^{\left(D\sqrt{2\rho}\right)} + e^{-\left(D\sqrt{2\rho}\right)}}\right)$$

Thus, the growth rate of w_k is $-aD$. This growth explodes in the negative-k region and is a potential cause of nonconvergence due to risk aversion; observe that risk neutrality ($a = 0$) implies zero growth in this term.

Factorization. The factorization is as follows. The characteristic equation is

$$\lambda_0(1-\lambda z)(1-\lambda z^{-1}) = 1 - rz - rz^{-1}$$

inducing a pair of equations:

$$\lambda_0\lambda = r; \qquad \lambda_0 = \frac{1}{1+\lambda^2}$$

The factors are

$$\lambda = \frac{1}{2r} \pm \left(\frac{1}{4r^2} - 1\right)^{1/2};$$

After some algebraic manipulation, it can be shown that

$$\lambda = e^{\pm\sqrt{2\rho}D}$$

The location of these roots relative to the unit circle can be analyzed. We are interested in whether

$$\lambda e^{-\rho D}$$

is fractional; if it is, we can construct a convergent solution. Note first that since $0 < q < 1$,

$$\frac{1}{r} > 2.$$

Thus, the radical term is positive and we have distinct roots.

Next, the magnitude of λ and $1/\lambda$ must be assessed relative to e^{aD}. The roots are such that we will have a solution of the form

$$\frac{A_1}{1-\lambda L}e^{-aDk} - \frac{A_2L^{-1}}{1-\lambda L^{-1}}e^{-aDk}$$

Using the partial fractions solutions yields

$$\frac{-\lambda^2}{1-\lambda^2}\frac{1}{1-\lambda L}e^{-aDk} - \frac{1}{1-\lambda^2}\frac{L^{-1}}{1-\lambda L^{-1}}e^{-aDk}$$

The difficulty here is that the root is repeated, so both forward and backward convergence must follow. In order to obtain convergence, it must be the case that both of the following conditions hold:

$$\left|\sum_{k=0}^{\infty}\lambda^k e^{-aDk}\right| < \infty; \qquad \left|\sum_{k=1}^{\infty}\lambda^k e^{+aDk}\right| < \infty;$$

This in turn means that the following conditions must hold:

$$\lambda e^{-aD} < 1; \qquad \lambda e^{aD} < 1.$$

Substituting for λ, it is obvious that the smaller root must be chosen:

$$\lambda = e^{-\sqrt{2\rho}D} < 1.$$

We immediately have

$$\lambda e^{-aD} = e^{-aD-\sqrt{2\rho}D} < 1.$$

However, it is also necessary to demonstrate

$$e^{(a-\sqrt{2\rho})D} < 1.$$

This is the component that measures declines of endowment, and thus represents risk. It is already established that under autarky the integrability condition $\rho > \frac{1}{2}a^2$ must hold, and that condition holds here as well. Thus, the existence of the contracts does not (for this form of utility) alter the convergence condition. This seems to be because it is still possible to reach the tails of the process with positive probability; it happens by jumps under the contract.

Derivation of (5.3). Recall (B.2); inverting the left hand side and using partial fractions yields

$$\begin{aligned}\frac{1+\lambda^2}{(1-\lambda L)(1-\lambda L^{-1})} &= -\lambda^{-1}L\frac{1+\lambda^2}{(1-\lambda L)(1-\lambda^{-1}L)}\\ &= -(1+\lambda^2)\lambda^{-1}L\frac{1}{1-\lambda^2}\left(\frac{-\lambda^2}{1-\lambda L}+\frac{1}{1-\lambda^{-1}L}\right)\end{aligned}$$

$$= \frac{1+\lambda^2}{1-\lambda^2}\left(\frac{\lambda L}{1-\lambda L} + \frac{1}{1-\lambda L^{-1}}\right)$$

Thus, the solution is

$$v_k = \frac{1+\lambda^2}{1-\lambda^2}\left(\frac{\lambda L}{1-\lambda L} + \frac{1}{1-\lambda L^{-1}}\right) w_k$$

$$= -\frac{1}{\rho}\left(1 - \frac{1}{\cosh(\sqrt{2\rho}D)}\right)\frac{1+\lambda^2}{1-\lambda^2}\left(\frac{\lambda L}{1-\lambda L} + \frac{1}{1-\lambda L^{-1}}\right) e^{-aDk} + C_1\lambda^k + C_2\lambda^{-k}$$

We then have

$$-\frac{1}{\rho}\left(1 - \frac{1}{\cosh(\sqrt{2\rho}D)}\right)\frac{1+\lambda^2}{1-\lambda^2}\left(\frac{\lambda e^{-aD(k-1)}}{1-\lambda e^{aD}} + \frac{e^{-aDk}}{1-\lambda e^{-aD}}\right) + C_1\lambda^k + C_2\lambda^{-k}$$

The remaining task is to apply the boundary conditions. A natural boundary condition is to consider $k = 0$: in that case,

$$v_0 = -\frac{1}{\rho}\left(1 - \frac{1}{\cosh(\sqrt{2\rho}D)}\right)\frac{1+\lambda^2}{1-\lambda^2}\left(\frac{\lambda L}{1-\lambda L} + \frac{1}{1-\lambda L^{-1}}\right) e^{-aD(0)} + C_1 + C_2$$

which makes sense if $C_1 = C_2 = 0$.

Assuming that $C_1 = C_2 = 0$, observe that

$$\cosh(\sqrt{2\rho}D) = \frac{\lambda + \lambda^{-1}}{2}$$

so,

$$v_k = -\frac{1}{\rho}\frac{(1-\lambda)^2}{1+\lambda^2}\frac{1+\lambda^2}{1-\lambda^2}\left(\frac{\lambda e^{-aD(k-1)}}{1-\lambda e^{aD}} + \frac{e^{-aDk}}{1-\lambda e^{-aD}}\right)$$

Cancelling,

$$v_k = -\frac{1}{\rho}\frac{1-\lambda}{1+\lambda}\left(\frac{\lambda e^{aD}}{1-\lambda e^{aD}} + \frac{1}{1-\lambda e^{-aD}}\right) e^{-aDk}$$

Further algebra yields

$$-\frac{1}{\rho}\frac{1-\lambda}{1+\lambda}\left(\frac{1-\lambda^2}{(1-\lambda e^{aD})(1-\lambda e^{-aD})}\right) e^{-aDk}$$

$$= -\frac{1}{\rho}\frac{(1-\lambda)^2}{(1-\lambda e^{aD})(1-\lambda e^{-aD})} e^{-aDk}$$

which is (5.3).

Proof of Proposition 6.1. Without loss of generality we consider the segment $S_1 = [0, 2D]$. The payoff from permanent defection for an individual with endowment $y(t) = y \in S_1$ is

$$v_A(y) = -\frac{e^{-ay}}{\rho - \frac{a^2}{2}} \tag{B.3}$$

The payoff from cooperation with no future defection is

$$v_C(y) = -\frac{e^{-aD}}{\rho}\left(1 - \frac{\sinh(\sqrt{2\rho}(2D-y))}{\sinh(\sqrt{2\rho}2D)} - \frac{\sinh(\sqrt{2\rho}y)}{\sinh(\sqrt{2\rho}2D)}\right.$$

$$\left. + \frac{\sinh(\sqrt{2\rho}(2D-y))}{\sinh(\sqrt{2\rho}2D)}v_C(0) + \frac{\sinh(\sqrt{2\rho}y)}{\sinh(\sqrt{2\rho}2D)}v_C(2D)\right) \quad \text{(B.4)}$$

with the continuation values

$$v_C(0) = -\frac{1}{\rho}\frac{(1-\lambda)^2}{(1-\lambda e^{aD})(1-\lambda e^{-aD})}$$

$$v_C(2D) = -\frac{1}{\rho}\frac{(1-\lambda)^2}{(1-\lambda e^{aD})(1-\lambda e^{-aD})}e^{-a2D}$$

We remind the reader that at the point of defection $v_C = v_A$ despite the fact that the option value of future defection is not explicitly taken into account, because the marginal value of defection must be zero at the optimum. Therefore it is appropriate to demonstrate that there is no solution to $v_C = v_A$ in the interior of S_k for low values of the discount rate, which is equivalent to

$$\lim_{\rho \to \frac{a^2}{2}} \frac{v_C(y)}{v_A(y)} < 1, \qquad y \in S_1$$

since $v_A < 0$, $v_C < 0$. The ratio has two components:

$$\frac{v_C(y)}{v_A(y)} = \alpha(y,\rho) + \beta(y,\rho)$$

with

$$\alpha(y,\rho) = \frac{\rho - \frac{a^2}{2}}{\rho}e^{a(y-D)}\left(1 - \frac{\sinh(\sqrt{2\rho}(2D-y))}{\sinh(\sqrt{2\rho}2D)} - \frac{\sinh(\sqrt{2\rho}y)}{\sinh(\sqrt{2\rho}2D)}\right)$$

$$\beta(y,\rho) = \left(\frac{\sinh(\sqrt{2\rho}(2D-y))}{\sinh(\sqrt{2\rho}2D)}e^{ay} + \frac{\sinh(\sqrt{2\rho}y)}{\sinh(\sqrt{2\rho}2D)}e^{a(y-2D)}\right)$$
$$\frac{1}{\rho}\frac{(1-\lambda)^2}{(1-\lambda e^{aD})(1-\lambda e^{-aD})}$$

It is immediate that

$$\lim_{\rho \to \frac{a^2}{2}} \alpha(y,\rho) = 0$$

Also, by an application of L'Hôpital's rule,

$$\lim_{\rho \to \frac{a^2}{2}} \beta(y,\rho) = \frac{1+e^{-aD}}{1-e^{-aD}}\frac{2}{aD}\left[\frac{1}{e^{a(2D-y)}}\frac{e^{aD}\sinh(a(2D-y)) + \sinh(ay)}{\sinh(a2D)}\right]$$

After some tedious algebra the term in square brackets can be demonstrated to be equal to one. Therefore

$$\lim_{\rho \to \frac{a^2}{2}} \frac{v_C(y)}{v_A(y)} = \frac{1+e^{-aD}}{1-e^{-aD}}\frac{2}{aD}$$

The appropriate inequality can be rearranged as

$$g(aD) = \frac{aD}{2} + \frac{aDe^{-aD}}{2} - 1 + e^{-aD} > 0$$

Since $g(0) = 0$ and $g'(x) > 0$ for $x > 0$, it is apparent that the inequality is true for $a > 0$ and $D > 0$.

We note that the condition $\rho \to a^2/2$ is necessary as well as sufficient. We have

$$\lim_{\rho\to\infty} \beta(y, \rho) = 0 \qquad \lim_{\rho\to\infty} \alpha(y, \rho) = e^{a(y-D)} > 1$$

and hence

$$\lim_{\rho\to\infty} \frac{v_C(y)}{v_A(y)} > 1$$

if $y \in (D, 2D]$. □

Proof of Proposition 6.3. Using the definitions (6.1) and (6.2) we demonstrate that the difference $v_C(y, N) - v_A(y, N)$ is continuous and has a zero for some $N < \infty$. Proposition 6.2 establishes that

$$v_C(y, 1) - v_A(y, 1) < 0$$

We now demonstrate that

$$\lim_{N\to\infty} \lim_{\rho\to\frac{a^2}{2}} v_C(y, N) - v_A(y, N) > 0$$

First use the expressions (6.1-6.2) and the terms (6.4a - 6.4b) to define the terms involving N in the definitions of $v_A(y, N)$ and $v_C(y, N)$:

$$\begin{aligned} &\lim_{N\to\infty} e^{-a(1-N)D} \frac{\sinh(\sqrt{2\rho}((N+1)D - y)}{\sinh(\sqrt{2\rho}2ND)} \\ = \quad &\lim_{N\to\infty} e^{-aD} \frac{e^{ND(a+\sqrt{2\rho}}e^{\sqrt{2\rho}(D-y)} - e^{ND(a-\sqrt{2\rho})}e^{-\sqrt{2\rho}(D-y)}}{e^{\sqrt{2\rho}2ND} - e^{-\sqrt{2\rho}2ND}} = 0 \end{aligned}$$

and

$$\begin{aligned} &\lim_{N\to\infty} e^{-a(N+1)D} \frac{\sinh(\sqrt{2\rho}((N-1)D + y)}{\sinh(\sqrt{2\rho}2ND)} \\ = \quad &\lim_{N\to\infty} e^{-aD} \frac{e^{ND(a+\sqrt{2\rho}}e^{-\sqrt{2\rho}(D-y)} - e^{ND(a+\sqrt{2\rho})}e^{\sqrt{2\rho}(D-y)}}{e^{\sqrt{2\rho}2ND} - e^{-\sqrt{2\rho}2ND}} = 0 \end{aligned}$$

Therefore

$$\lim_{N\to\infty} \lim_{\rho\to\frac{a^2}{2}} v_C(y, N) = \lim_{\rho\to\frac{a^2}{2}} v_C(y)$$

and

$$\lim_{N\to\infty} \lim_{\rho\to\frac{a^2}{2}} v(A(y, N) = \lim_{\rho\to\frac{a^2}{2}} v_A(y)$$

where $v_C(y)$ and $v_A(y)$ are defined in $B.3 - 4$. By Proposition 6.1

$$\lim_{\rho \to \frac{a^2}{2}} v_C(y) - v_A(y) > 0$$

and hence

$$\lim_{N \to \infty} \lim_{\rho \to \frac{a^2}{2}} v_C(y, N) - v_A(y, N) > 0.$$

Consider N as a continuous variable. It is easy to demonstrate that $v_A(y, N)$ and $v_C(y, N)$ are continuous in N since the denominator of the above expressions is necessarily positive. Thus $v_C(y, N) - v_A(y, N)$ is negative for $N = 1$, becomes strictly positive as N approaches infinity, and is continuous in N. Therefore there is an integer N^* such that $v_C(y, N^*) - v_A(y, N^*) > 0$. □

C The continuum assumption

As Judd [14] and Feldman and Gilles [11] point out, there are some issues associated with the continuum-of-agents assumption that must be addressed. We now apply their results to our construction. We have assumed that the distribution of realized endowment processes across agents reflects the underlying Gaussian distribution of each unrealized path, and that these Gaussian distributions evolve in an even fashion, resulting in a uniform or Lebesgue measure of endowments in any interval across agents. We have also implicitly assumed measurability of sets of agents within a segment.

These assumptions are only a slight variation on the analysis of [14]. The difference stems from the fact that each draw of a random variable from the index set produces a random variable $\omega(t)$, in Judd's notation, but the distribution is also a function of t, namely its mean is determined by t. The continuum of draws from the index set does not (under the iid assumption) replicate a single distribution, but rather produces a continuum of distributions that must be integrated to produce the appropriate measure. Each of these distributions is Gaussian, but the integrated measure is uniform, that is, Lebesgue. In other words, we assume that there is a continuum of individuals at each starting value of endowment; we view this collection of agents at each starting value as belonging to Judd's index set. Then we aggregate across these index sets and compute the product measure.

We now express the construction more formally: Let there be family of random variables X^a_μ such that for each (μ, a) the distribution F^a_μ is the Wiener distribution, such that for any realized path $\omega(a)$, the time-t distribution of $y(\omega(a)|t)$, given $y(0) = \mu$, is Gaussian $N(\mu, t)$.[20] Let $\mu \in R$ and let there be a probability space $(R, \mathscr{B}, \ell)$, where ℓ is Lebesgue measure. Also, let $a \in I_\mu$ and let there be

[20] As pointed out in standard developments of the Kolmogorov extension, the continuous sample paths on the time domain do not form a measurable set under the Kolmogorov construct. (For example see ([25], p. 161), ([15], p. 53)) Therefore an alternative model of Brownian motion is developed in order that the Brownian paths are measurable. Moreover, the Brownian construct has no other (discontinuous) paths. Our construction uses this notion, and the Gaussian property is preserved for the values of the sample paths at time t; our F-distribution is not on the terminal values but on the continuous sample paths $C[0, t_1]$.

a probability space $(I_\mu, \mathscr{B}, \ell)$ with $\ell(I_\mu) = 1$ for each μ. The interpretation is that I_μ is an index set of individuals indexed by a whose endowments are initially μ; since they evolve according to Brownian motion, their future endowments are $N(\mu, t)$. The index set at μ has measure 1. The remaining issue is to agglomerate these index sets.

For fixed μ, the family $\{X_\mu^a : a \in I_\mu\}$ can be defined on $(\Omega_\mu, \overline{\mathscr{F}}_\mu, \overline{P}_\mu)$ where $\overline{P}_\mu$ is the extension that makes $\{a \in I_\mu : X_\mu^a(t) \le x\}$ Borel measurable and such that the Glivenko-Cantelli theorem holds: Judd [14] demonstrates that the extension exists.[21] Observe that distinct μ have distinct extensions. Let $(\Omega, \mathscr{F}, P)$ be the uncountable product of these spaces (Halmos [13], p. 158, remark 2).

The probability measure of a segment is the measure over the segment of all the initial endowments that can lead to that segment. Define the measure of the segment $[a, b]$ by

$$\nu([a,b]) = \int_{-\infty}^{\infty} \int_a^b \frac{1}{\sqrt{2\pi t_1}} e^{\frac{(y-x)^2}{t_1}} dy dx$$

That is, we integrate over all the initial endowment points x that can reach the segment $[a, b]$, and compute the probability of y in each point of the segment.

Proposition .1 *ν is Lebesgue measure.*

Proof. Interchange the order of integration:

$$\nu([a,b]) = \int_a^b \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t_1}} e^{\frac{(y-x)^2}{t_1}} dx dy$$

The roles of x and y are now reversed, and the inner integral is 1. The integral is then

$$\nu([a,b]) = \int_a^b dy = b - a$$

□

In the above construction we could have used Proposition 2 of Feldman and Gilles [11], which abandons the independence assumption, since our agents would remain negligible in strategic power in the absence of independence.

This idea is formalized in the concept of Wiener measure ([15], p. 71) on the measure space $(C[0,\infty), \mathscr{B}(C[0,\infty)))$, where $C([0,\infty))$ is the space of continuous functions on the nonnegative real line, such that the coordinate mapping process $W_t(\omega)$, where ω is a sample path $\omega(t)$, is a one-dimensional Brownian motion ([15], Th. 2.4.20, p. 70). Sampling from this distribution at an instant t yields the Gaussian measure used to describe $y(t)$ in the discussion here.

[21] The Glivenko-Cantelli theorem states that the empirical distribution of an F-distributed sequence of random variables, $F_n(x,\omega) = \frac{1}{n}\sum_{k=1}^n \mathbf{1}_{(-\infty,x]}(X_k(\omega))$, tends to the underlying distribution with probability 1; the law of large numbers is a special case.

References

1. Abreu, D., Pearce, D., Stacchetti, E.: Optimal cartel equilibria with imperfect monitoring. Journal of Economic Theory **39**, 251–269 (1986)
2. Alvarez, F., Jermann, U.: Asset pricing when risk sharing is limited by default. Working paper, University of Chicago (1998)
3. Attanasio, O., Rios-Rull, V.: Consumption smoothing in island economies: Can public insurance reduce welfare? Mimeo, Dept. of Economics, University of Pennsylvania (1999)
4. Bergemann, D., Valimaki, J.: Experimentation in markets. Working paper, Yale University (1999)
5. Billingsley, P.: Probability and measure. New York: Wiley 1979
6. Bolton, P., Harris, Ch.: Strategic experimentation. Econometrica **67** (2), 349–374 (1999)
7. Carr, P., Jarrow, R.: The stop-loss start-gain paradox and option valuation: A new decomposition into intrinsic and time value. Review of Financial Studies **3** (3), 469–492 (1990)
8. Chade, H., Taub, B.: Segmented risk sharing in a continuous-time setting. Working paper, ASU (2001)
9. Chung, K. L.: A course in probability theory. Second edition. New York: Academic Press 1974
10. Farrell, J., Scotchmer, S.: Partnerships. Quarterly Journal of Economics **103**, 279–297 (1988)
11. Feldman, M., Gilles, C.: An expository note on individual risk without aggregate uncertainty. Journal of Economic Theory **35**, 26–32 (1985)
12. Greenberg, J., Weber, Shl.: Stable coalition structures in consecutive games. In: Binmore, K., Kirman, A., Tani, P. (eds.) Frontiers of game theory Cambridge, MA: MIT Press (1993)
13. Halmos, P.: Measure theory. Berlin Heidelberg New York: Springer 1974
14. Judd, K.: The law of large numbers with a continuum of iid random variables. Journal of Economic Theory **35**, 19–25 (1985)
15. Karatzas, I., Shreve, S.: Brownian motion and stochastic calculus. Berlin Heidelber New York: Springer 1991
16. Kehoe, T., Levine, D.: Debt-constrained asset markets. Review of Economic Studies **60**, 865–888 (1993)
17. Keller, G., Rady, S.: Optimal experimentation in a changing environment. Review of Economic Studies **66** (3), 475–507 (1999)
18. Kletzer, K., Wright, B.: Sovereign debt as intertemporal barter. American Economic Review **90**, 621–639 (2000)
19. Kocherlakota, N.: Implications of efficient risk-sharing without commitment. Review of Economic Studies **63** (4), 595–610 (1996)
20. Krueger, D., Perri, F.: Risk sharing: private insurance markets or redistributive taxes? Federal Reserve Bank of Mineapolis, Research Dept. Staff Report 262 (1999)
21. Krueger, D., Uhlig, H.: Competitive risk-sharing contracts with one-sided commitment. Working paper, Stanford University (2000)
22. Miller, M., Orr, D.: A model of the demand for money by firms. Quarterly Journal of Economics **80**, 413–435 (1966)
23. Moscarini, G., Smith, L.: Wald revisited: The optimal level of experimentation. Mimeo, Department of Economics, University of Michigan (1999)
24. Taub, B.: Dynamic consistency of insurance contracts under enforcement by exclusion. Journal of Economic Dynamics and Control **13**, 93–112 (1989)
25. Taylor, S. J.: Introduction to probability and measure. Cambridge: Cambridge University Press 1966