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Journal of Economic Theory 129 (2006) 81 – 113

JOURNAL OF Economic Theory
www.elsevier.com/locate/jet

Matching with noise and the acceptance curse

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Received 18 October 2002; final version received 20 January 2005
Available online 10 May 2005

Abstract

This paper explores matching with both search and information frictions. Specifically, everyone observes only a noisy signal of the true type of any potential mate. In this context, matching decisions must incorporate not only information about a partner's attribute conveyed by the noisy signal, but also—as in the winner's curse in auction theory—information about a partner's type contained in his or her acceptance decision.

We show that there exists an equilibrium exhibiting a stochastic positive assorting of types, generalizing [Becker, J. Polit. Economy 81 (1973) 813–846]. In equilibrium, selection is adverse: being accepted reduces an agent's estimate of a potential partner's type, a phenomenon that we call the acceptance curse effect.

JEL classification: D82; D83

Keywords: Matching; Search; Adverse selection; Bayesian games

1. Introduction

The formation of partnerships in marriage and labor markets takes place under substantial uncertainty as to the true attributes of the parties involved, as information on attributes is often private. Thus, a match formation decision is usually based on noisy signals that potential partners observe about each other's types. The informational content of those signals is not, however, the only information that rational individuals should consider when

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0022-0531/$ - see front matter
doi:10.1016/j.jet.2005.01.008

they decide to form matches. For a motivational parallel, observe that bidders in a common value auction must condition their bids not only on their private information, but also on the 'winner's curse'—namely, the information revealed by the event that one is the highest bidder. By the same token, in the matching setting, partners must also condition decisions on the information revealed by a partner's acceptance decision, for this conveys information about the partner's hidden attributes. Unlike in the auction setting, the presence of a dynamic search option value substantially deepens the analysis of the problem.

The purpose of this paper is to analyze the implications of imperfect information about types in a dynamic environment with search frictions, heterogeneous agents, and nontransferable utility. Since the nontransferable utility assumption precludes most labor market applications, we will cast the model in terms of a marriage market. There are two populations of agents, men and women, who have diverse characteristics that are private information. Single agents meet randomly and pairwise in each period over an infinite horizon. After privately observing a noisy signal of the type of their potential mates, they decide whether to marry or not. The stochastic structure of the noisy signals is such that the higher an agent's type, the higher a potential partner's estimate of his or her type. A married agent enjoys a per-period utility equal to the type of his or her spouse. Couples that get married exit the market and are replaced by new singles with the same types as those of the departing pair.

In this environment, we obtain the following results. Intuitively, an agent's optimal strategy is to accept a potential partner if the expected discounted payoff from marriage exceeds the option value of remaining single. When everyone else uses this stationary strategy, each agent's optimal response is stationary as well, and has the 'reservation-signal property'; i.e., he or she only accepts a partner if the signal observed exceeds a threshold—the agent's reservation-signal—which is determined by equating the expected payoff from marriage with the option value of remaining single.

The analysis reveals that agents' optimal strategies take into account not only the information conveyed by the signal observed, but also the informational content of being accepted by a partner. The former has an unambiguous impact on the expected payoff from marriage, for observing a higher signal always increases an agent's estimate about a potential partner's type; i.e., higher signals are 'better news' about a partner's type than lower ones. The latter, however, affects the expected payoff from marriage in a more subtle way that depends on an agent's type and on how the reservation-signals of an agent's potential partners change with their types. In particular, if they become more selective as their types increase—i.e., the reservation-signal is an increasing function of their types—then being accepted always confers 'bad news' about a potential partner's type. Thus, the expected payoff of forming a match with a partner shrinks once the informational content of being accepted by this partner is taken into account. We call this phenomenon the *acceptance curse* effect.

In this setting, we show that there exists an equilibrium in which reservation-signals are increasing in agents' types. [1] The method of proof, which is of independent interest, can be succinctly described as follows. The reservation-signal property of optimal strategies allows us to show that the model can be reinterpreted as a Bayesian game with two players (representing the populations of men and women), a continuum of types (men and women's types), and a continuum of actions (each player chooses a reservation-signal from

[1] Throughout, we use the words increasing and decreasing in the weak sense.

an interval). Then, we prove that this static game has an equilibrium in pure strategies that are increasing in types, thereby showing the existence of an equilibrium with this property in the original model. To the best of our knowledge, this is the first paper to exploit the tools developed for *static* Bayesian games in the analysis of a *dynamic* equilibrium matching problem, a connection that may prove useful in other dynamic matching models as well.

An equilibrium in which reservation-signals are increasing in types exhibits a stochastic form of positive assortative mating, thereby generalizing [4]. The intuition relies on the interplay of two opposing forces. On the one hand, we show that, as the type of an agent increases, the acceptance curse becomes less severe, and hence the expected value of accepting a match is increasing in an agent's type. If the option value of being single remained constant, the acceptance curse effect would induce agents to become *less* selective in their acceptance decision as their types increased. On the other hand, since agents of higher types are accepted more often and suffer less from the acceptance curse, the option value of searching for a partner is increasing in an agent's type, thereby inducing agents to become *more* selective as their types increase. The net result of the interplay of these two forces is that, if the reservation-signals of one side of the market are increasing in types, the other side will also respond with a similar strategy. Therefore, the equilibrium exhibits positive assortative mating by signals, and since higher types are more likely to generate better signals, there is also positive sorting by types, albeit stochastically.

We also shed light on whether an equilibrium in which reservation-signals are not increasing in types could exist. To see how, at least in principle, such an equilibrium could emerge, suppose that women's reservation-signals are decreasing in their types and consider a male's decision problem. In this case, being accepted by a woman confers good news about her type, and it is better news the lower his type is. Hence, the expected payoff of forming a match with a given woman is a decreasing function of a man's type. Regarding the option value of remaining single, there are two opposite effects. On the one hand, the higher a man's type, the more likely he is to be accepted by women, and this increases his option value of being single. On the other hand, the expected payoff from being accepted by a given woman is a decreasing function of his type, and this tends to decrease his option value of being single. If the second effect is strong enough, it can engender reservation-signals that are also decreasing in men's types. We are able to show that matching equilibria in decreasing strategies do not exist.

Finally, we discuss several potential extensions of the model and the main complications that ensue in each case.

To be sure, the paper is closely related to the literature on two-sided search models with heterogeneous agents. Papers by McNamara and Collins [13], Smith [21], Burdett and Coles [6], Shimer and Smith [20], Bloch and Ryder [5], Eeckhout [9], Morgan [16], and Chade [8] also analyze environments with costly search and heterogeneity. All these authors address the question of whether positive assortative matching emerges in equilibrium, and their models proceed under alternative assumptions regarding replenishment of the pool of singles, existence of complementarities in utility functions, presence of explicit search costs, and possibility to transfer utility between partners (see the survey by Burdett and Coles [7] for more references). All the references, however, assume that types are perfectly observable when a meeting takes place, which precludes the adverse selection effect explored in this paper.[2] In fact, without noise, our model reduces to that in [6] without entry flows and to that in [21] without exogenous match dissolutions, and it exhibits—under automatic replenishment of agents—a unique nontrivial equilibrium, which is constructed by first analyzing the decision problems of agents with the highest types, and then continuing iteratively with the remaining types.

The paper is also related to [1], which analyzes a dynamic matching model with transferable utility, no search frictions, and unknown types that are publicly learned over time as partners produce output. The authors of [1] find that the dual role of output as current payoff and information leads to a robust failure of positive assortative mating. Unlike this model, we assume that types are private information and explore the effects of the aforementioned acceptance curse.

As will become clear below, we make extensive use of tools of comparative statics under uncertainty developed in [3], as well as the analysis of the existence of equilibrium in pure strategies that are increasing in types in Bayesian games with a continuum of types and actions, contained in [2]. Indeed, our proof of existence amounts to checking that the conditions of one of the theorems in [2] hold in the two-player Bayesian game we analyze.

The paper proceeds as follows. Section 2 describes the model. Section 3 contains the main results of the paper. Section 4 contains an illustrative example. Section 5 discusses several extensions of the model. Section 6 concludes. Some of the proofs are collected in the Appendix.

2. The model

The main characteristics of the model are summarized below.

Time: Time is discrete and is divided into periods of length equal to one.

Agents: There are two populations, one containing a continuum of males and the other a continuum of females. The measure of agents in each population is normalized to one. Each female is characterized by a type $x \in [\underline{x}, \overline{x}]$, $0 < \underline{x} < \overline{x}$, distributed according to a continuous and positive density function $f(x)$. Similarly, each male is characterized by a type $y \in [\underline{y}, \overline{y}]$, $0 < \underline{y} < \overline{y}$, distributed according to a continuous and positive density function $g(y)$.

Information structures: Agents do not observe the actual types of potential partners. Instead, each agent observes a realization of a random variable that provides imperfect information about their potential partner's true type. Formally, when a man with type y meets a woman with type x, he only observes a realization of a random variable θ, which takes values on $[\underline{\theta}, \overline{\theta}]$ with conditional density $m(\theta \mid x)$ and cumulative distribution function (henceforth c.d.f.) $M(\theta \mid x)$. Similarly, a woman only observes a realization of ω, distributed on $[\underline{\omega}, \overline{\omega}]$ according to $n(\omega \mid y)$, with c.d.f. $N(\omega \mid y)$. To avoid technical problems, we assume that $m(\theta \mid x)$ and $n(\omega \mid y)$ are continuous, positive everywhere, and that they

[2] The authors of [18] analyze a model with imperfect information about types, and, unlike in our paper, they allow for match dissolutions when revelation of the agents' types occurs after being matched for one period. They, however, restrict attention to strategies that *only* take into account the informational content of the signal. Thus their work does not consider the issues analyzed in this paper.

have constant supports, equal to $[\underline{\theta}, \overline{\theta}]$ for all $x \in [\underline{x}, \overline{x}]$, and to $[\underline{\omega}, \overline{\omega}]$ for all $y \in [\underline{y}, \overline{y}]$. Since in several places we need to differentiate under an integral sign, we also assume that $M(\theta \mid x)$ is differentiable with respect to x, $N(\omega \mid y)$ is differentiable with respect to y, and the derivatives are uniformly bounded. Finally, the families $\{m(\theta \mid x) : x \in [\underline{x}, \overline{x}]\}$ and $\{n(\omega \mid y) : y \in [\underline{y}, \overline{y}]\}$ satisfy the strict Monotone Likelihood Ratio Property (MLRP) or, equivalently, they are strictly log-supermodular (log-spm). [3]

Meeting technology: In every period, agents from opposite populations randomly meet in pairs.

Marriage game: When a pair meets, the man privately observes a realization of θ, and the woman privately observes a realization of ω. Then, they simultaneously announce *Accept* or *Reject*. Marriage takes place if and only if both announce *Accept*, in which case they leave the market. In any other case, they go back to the pool of singles and continue searching.

Replenishment: To keep the distribution of types invariant over time, we assume that when a pair exits the market, it is replaced by agents with the same types ('clones') as departing ones.

Match payoffs: A single agent's per-period utility is equal to zero. In order to focus on the effects of imperfect information about types, we assume that if a woman with type x marries a man with type y, her per-period utility is y and his is x. Agents discount the future at the common factor $0 < \beta < 1$.

Strategies: A stationary strategy for a woman with type x is a measurable mapping $\sigma_x : [\underline{\omega}, \overline{\omega}] \longrightarrow \{0, 1\}$, where $0 =$ *Reject* and $1 =$ *Accept*. Similarly, a stationary strategy for a man whose type is y is a measurable mapping $\sigma_y : [\underline{\theta}, \overline{\theta}] \longrightarrow \{0, 1\}$. [4] A profile of stationary strategies for women and men is denoted by $(\sigma_{\mathbf{x}}, \sigma_{\mathbf{y}})$, where $\sigma_{\mathbf{x}} = (\sigma_x)_{x \in [\underline{x}, \overline{x}]}$ and $\sigma_{\mathbf{y}} = (\sigma_y)_{y \in [\underline{y}, \overline{y}]}$.

Equilibrium: Let $\Phi(x, \sigma_x, \sigma_{\mathbf{y}})$ be the expected discounted utility for a woman with type x when her strategy is σ_x and the strategy of men is $\sigma_{\mathbf{y}}$; define $\Psi(y, \sigma_y, \sigma_{\mathbf{x}})$ analogously. [5] A *matching equilibrium* is a stationary strategy profile $(\sigma^*_{\mathbf{x}}, \sigma^*_{\mathbf{y}})$ such that, $\forall x \in [\underline{x}, \overline{x}]$,

$$\Phi(x, \sigma^*_x, \sigma^*_{\mathbf{y}}) \geqslant \Phi(x, \sigma'_x, \sigma^*_{\mathbf{y}})$$

for any alternative strategy σ'_x (not necessarily stationary); and $\forall y \in [\underline{y}, \overline{y}]$,

$$\Psi(y, \sigma^*_y, \sigma^*_{\mathbf{x}}) \geqslant \Psi(y, \sigma'_y, \sigma^*_{\mathbf{x}})$$

for any alternative strategy (stationary or not) σ'_y. (For simplicity, we omit the obvious 'almost everywhere' qualifier.)

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[3] A function $h : X \to \mathbb{R}$ defined on a lattice is log-spm if for all x and x', $h(x \vee x')h(x \wedge x') \geqslant h(x)h(x')$, where $x \vee x' = \max\{x, x'\}$ and $x \wedge x' = \min\{x, x'\}$. See [3,10,14].

[4] Throughout this paper, measurability is with respect to the appropriate Borel σ-algebra. All sets and functions used in the paper satisfy required measurability properties. To avoid repetition, we will henceforth omit the qualifier 'measurable.'

[5] It is easy to see that an agent is not directly affected by the strategies of agents belonging to the same population; he or she need only consider the strategies of members of the other population.

3. Main results

3.1. Optimal strategies

Consider the decision problem that a man of type y faces when women use a stationary strategy profile $\sigma_{\mathbf{x}}$, and he observes a signal θ in his current meeting. The functional equation that describes his problem can be written as follows

$$v(\theta, y) = \max\{a(\theta, y)\gamma(\theta, y) + (1 - a(\theta, y))\, \beta\Psi(y), \beta\Psi(y)\}, \tag{1}$$

where

$$\begin{aligned}
a(\theta, y) &= \int_{\underline{x}}^{\overline{x}} \int_{\Lambda_x} n(\omega \mid y) k(x \mid \theta)\, d\omega\, dx, \\
\Lambda_x &= \{\omega \in [\underline{\omega}, \overline{\omega}] : \sigma_x(\omega) = 1\}, \\
k(x \mid \theta) &= \frac{m(\theta \mid x) f(x)}{m(\theta)}, \\
m(\theta) &= \int_{\underline{x}}^{\overline{x}} m(\theta \mid x) f(x)\, dx, \\
\gamma(\theta, y) &= E\left[\frac{x}{1-\beta} \mid \theta, \sigma_x(\omega) = 1, y\right] \\
&= \int_{\underline{x}}^{\overline{x}} \frac{x}{1-\beta} \frac{\left(\int_{\Lambda_x} n(\omega \mid y) d\omega\right) k(x \mid \theta)}{a(\theta, y)}\, dx, \\
\Psi(y) &= \int_{\underline{\theta}}^{\overline{\theta}} v(\theta, y) m(\theta)\, d\theta.
\end{aligned}$$

The interpretation of (1) is as follows. After observing θ, he updates his beliefs about the current partner's type x using Bayes' rule, which yields a posterior density $k(x \mid \theta)$.

If the man announces reject, his expected payoff is $\beta\Psi(y)$, which is the value of continuing the search from next period onward under an optimal strategy, and where the expectation is taken with respect to the unconditional density of θ.

If he announces accept, his expected payoff is given by the first term inside the max operator. To understand this expression, consider a woman of type x. Since she only accepts signals in the set Λ_x, the probability that she accepts him is $\int_{\Lambda_x} n(\omega \mid y)\, d\omega$. Thus, conditional on his current partner being of type x, the expected payoff of announcing accept is

$$\left(\int_{\Lambda_x} n(\omega \mid y)\, d\omega\right) \frac{x}{1-\beta} + \left(1 - \int_{\Lambda_x} n(\omega \mid y)\right) \beta\Psi(y).$$

Integrating over x using $k(x \mid \theta)$ yields the expected payoff of announcing accept after observing θ, given by

$$\int_{\underline{x}}^{\overline{x}} \left(\left(\int_{\Lambda_x} n(\omega \mid y)\, d\omega\right) \frac{x}{1-\beta} + \left(1 - \int_{\Lambda_x} n(\omega \mid y)\right) \beta\Psi(y)\right) k(x \mid \theta)\, dx. \tag{2}$$

Straightforward algebraic manipulation of (2) allows us to rewrite it as $a(\theta, y)\gamma(\theta, y) + (1 - a(\theta, y))\beta\Psi(y)$, where $a(\theta, y)$ is the probability of being accepted in the current meeting and $\gamma(\theta, y)$ is the expected discounted payoff of being accepted, given $\sigma_{\mathbf{x}}$, θ, and y. It is worth pointing out that $\gamma(\theta, y)$ includes both the information about x conveyed by θ *and* the information about x that is revealed in the event that the man is accepted by his current partner.[6]

The following proposition shows that $v(\theta, y)$ is uniquely defined and that the optimal policy exhibits the 'reservation-signal' property: i.e., a man of type y only accepts values of θ above a threshold.

Proposition 1. *Let $\sigma_{\mathbf{x}}$ be a stationary strategy profile used by females. Then,*

(i) *there exists a unique nonnegative bounded function that satisfies the functional equation* (1);

(ii) *if the set of types x for which Λ_x is nonempty has positive measure, then the optimal strategy for a man with type $y \in [\underline{y}, \overline{y}]$ is*

$$\sigma_y(\theta) = \begin{cases} 1 & \text{if } \theta \geqslant \hat{\theta}(y), \\ 0 & \text{if } \theta < \hat{\theta}(y), \end{cases}$$

where $\underline{\theta} \leqslant \hat{\theta}(y) < \overline{\theta}$ for every $y \in [0, 1]$.

Proof. (i) Let B be the space of nonnegative bounded functions $\xi : [\underline{\theta}, \overline{\theta}] \times [\underline{y}, \overline{y}] \longrightarrow R$ endowed with the sup norm. It is well known that this is a complete metric space. Define the operator T on B by

$$T\xi(\theta, y) = \max\{a(\theta, y)\gamma(\theta, y) + (1 - a(\theta, y))\beta E[\xi(\theta, y)],\ \beta E[\xi(\theta, y)]\},$$

where $E[\xi(\theta, y)] = \int_{\underline{\theta}}^{\overline{\theta}} \xi(\theta, y)m(\theta)\,d\theta$. It is easy to show that $T : B \longrightarrow B$, and also that T satisfies Blackwell's sufficient conditions for a contraction. It then follows from the Contraction Mapping Theorem that there is a unique $v \in B$ such that $v = Tv$.

(ii) Let $C \subseteq [\underline{x}, \overline{x}]$ be the set of types x such that $\Lambda_x \neq \emptyset$. Let $h(x \mid \theta, y) = \frac{\left(\int_{\Lambda_x} n(\omega|y)\,d\omega\right)k(x|\theta)}{a(\theta,y)}$ be the conditional density of x given θ and the event that y is accepted by x; obviously, $h(x \mid \theta, y)$ is positive on C and zero elsewhere. Take $x' > x$, and $\theta' > \theta$, with $x', x \in C$. It is easy to see that $\frac{h(x'|\theta',y)}{h(x|\theta',y)} > \frac{h(x'|\theta,y)}{h(x|\theta,y)}$ if and only if $\frac{k(x'|\theta')}{k(x|\theta')} > \frac{k(x'|\theta)}{k(x|\theta)}$, and that the latter follows from the strict MLRP of $m(\theta \mid x)$. Thus, the family $\{h(x \mid \theta, y) : x \in C,\ \theta \in [\underline{\theta}, \overline{\theta}]\}$ satisfies the strict MLRP in (x, θ), and this implies that $\gamma(\theta, y) = \int_C \frac{x}{1-\beta} h(x \mid \theta, y)\,dx$ is strictly increasing in θ [14, Proposition 4]. Given the assumptions made on the densities of types and signals, it follows from an application of the Lebesgue Dominated Convergence Theorem (LDCT) that $\gamma(\theta, y)$ is continuous in θ. Moreover, $\gamma(\overline{\theta}, y) > \beta\Psi(y)$, since drawing $\overline{\theta}$ and being accepted is the best possible

[6] To simplify the notation, throughout the paper we omit from the functions' arguments the strategy used by the other population.

outcome. Therefore, if $\gamma(\underline{\theta}, y) < \beta\Psi(y)$, then there is a unique $\underline{\theta} < \hat{\theta}(y) < \overline{\theta}$ that solves $\gamma(\hat{\theta}, y) = \beta\Psi(y)$. Otherwise, the optimal reservation signal is $\hat{\theta}(y) = \underline{\theta}$. □

Proposition 1 shows that the set of signals a man of type y accepts is given by the nonempty interval $[\hat{\theta}(y), \overline{\theta}]$. When the solution is interior, $\hat{\theta}(y)$ is the unique value that solves $\gamma(\hat{\theta}, y) = \beta\Psi(y)$, and is equal to $\underline{\theta}$ otherwise. That is, $\hat{\theta}(y)$ is the signal that leaves an agent with type y indifferent between continuing the search under an optimal strategy and accepting a partner when the signal observed is $\hat{\theta}$ *and* the partner accepts him. The acceptance event contains information about the current partner's type that an optimal strategy *must* take into account.

The reservation-signal property reveals that the functional form of the value function for a man with type y who observes a realization θ in his current meeting is

$$v(\theta, y) = \begin{cases} a(\theta, y)\gamma(\theta, y) + (1 - a(\theta, y))\beta\Psi(y) & \text{if } \theta \geqslant \hat{\theta}(y), \\ \beta\Psi(y) & \text{if } \theta < \hat{\theta}(y). \end{cases} \tag{3}$$

Eq. (3) gives the value function in terms two unknowns, $\Psi(y)$ and $\hat{\theta}(y)$. Using (3) and the definitions of $\Psi(y)$ and $k(x \mid \theta)$ we obtain, after some manipulation, the following expression for $\Psi(y)$, the maximum expected discounted utility of y given the strategy of women

$$\beta\Psi(y) = \frac{\int_{\hat{\theta}(y)}^{\overline{\theta}} a(\theta, y)\gamma(\theta, y)m(\theta)\, d\theta}{\delta + \int_{\hat{\theta}(y)}^{\overline{\theta}} a(\theta, y)m(\theta)\, d\theta}, \tag{4}$$

where $\delta = \frac{1-\beta}{\beta}$.

Similar results hold for women. Consider the decision problem that a woman of type x faces when men use a stationary strategy profile $\sigma_{\mathbf{y}}$, and she observes a signal ω in her current meeting. The functional equation that describes her problem is

$$z(\omega, x) = \max\{b(\omega, x)\alpha(\omega, x) + (1 - b(\omega, x))\beta\Phi(x),\ \beta\Phi(x)\}, \tag{5}$$

where

$$b(\omega, x) = \int_{\underline{y}}^{\overline{y}} \int_{\Xi_y} m(\theta \mid x) l(y \mid \omega)\, d\theta\, dy,$$

$$\Xi_y = \{\theta \in [\underline{\theta}, \overline{\theta}] : \sigma_y(\theta) = 1\},$$

$$l(y \mid \omega) = \frac{n(\omega \mid y)g(y)}{n(\omega)},$$

$$n(\omega) = \int_{\underline{y}}^{\overline{y}} n(\omega \mid y)g(y)\, dy,$$

$$\alpha(\omega, x) = E\left[\frac{y}{1-\beta} \mid \omega, \sigma_y(\theta) = 1, x\right]$$

$$= \int_{\underline{y}}^{\overline{y}} \frac{y}{1-\beta} \frac{\left(\int_{\Xi_y} m(\theta \mid x)\, d\theta\right) l(y \mid \omega)}{b(\omega, x)}\, dy,$$

$$\Phi(x) = \int_{\underline{\omega}}^{\overline{\omega}} z(\omega, x) n(\omega)\, d\omega.$$

If Ξ_y is nonempty on a set of types y of positive measure, then the optimal strategy for x is given by the reservation-signal $\underline{\omega} \leqslant \hat{\omega}(x) < \overline{\omega}$ satisfying $\alpha(\hat{\omega}, x) = \beta\Phi(x)$ at an interior solution; i.e., the acceptance set of a woman with type x is the nonempty interval $[\hat{\omega}(x), \overline{\omega}]$. Proceeding as before, we obtain

$$\beta\Phi(x) = \frac{\int_{\hat{\omega}(x)}^{\overline{\omega}} b(\omega, x)\alpha(\omega, x) n(\omega)\, d\omega}{\delta + \int_{\hat{\omega}(x)}^{\overline{\omega}} b(\omega, x) n(\omega)\, d\omega}. \tag{6}$$

3.2. Existence of equilibrium

In order to prove the existence of a matching equilibrium, we pursue a line of attack that consists of the following steps. First, note that we can reinterpret the dynamic matching model with two heterogeneous populations as a two-player game with incomplete information and a continuum of types, in which each player chooses a type-contingent strategy. To wit, each type of a player chooses a *function*, which can be a complicated object if nonstationary strategies are allowed, thereby making the computation of expected utilities a formidable task. Second, since we are restricting attention to equilibria in stationary strategies, each type of a player chooses an indicator function on the interval of signals. This allows us to compute in closed form the expected utility of each player for any strategy profile given by indicator functions. Notice, however, that each type continues to choose a function. Third, the results of the previous section reveal that the indicator function chosen by each type is completely summarized by a type-dependent reservation-signal. Hence, without loss of generality, we can assume that each type of a player chooses a *scalar*, namely, the reservation signal above which potential partners are accepted. Fourth, note that we can now reinterpret the dynamic matching model with two heterogeneous populations as a two-player game with incomplete information, in which types and actions belong to compact intervals of the real line. Thus, we can use the results available on existence of an equilibrium in this class of games to prove existence of a matching equilibrium in pure strategies.

We now proceed to formalize the steps outlined above. We need to show that there exists a strategy profile $(\sigma^*_{\mathbf{x}}, \sigma^*_{\mathbf{y}})$ that constitutes a matching equilibrium. By Proposition 1, the task amounts to proving that there exists a pair of functions $(\hat{\theta}^*(y), \hat{\omega}^*(x))$ such that, for every man of type y, $\hat{\theta}^*(y)$ maximizes his expected discounted utility given $\hat{\omega}^*(x)$, and for every woman of type x, $\hat{\omega}^*(x)$ maximizes her expected discounted utility given $\hat{\theta}^*(y)$.

When types are *perfectly* observable and automatic replenishment is assumed, there is a nice constructive proof of existence of a matching equilibrium that has been used repeatedly in the literature.[7] It proceeds by first solving the decision problem faced by the highest-type agents in each population, who are accepted by everybody from the other population. This step defines a subset of types of women and a subset of types of men that are mutually acceptable. The proof then continues with the decision problems faced by agents with lower types, who tend to be accepted by a lower subset of types from the other population.

In the present setting with noise, however, each type is accepted with positive probability by *every* other type: agents with different types only differ in their *probabilities* of being accepted, but not in the *subset* of agents that accepts them. This renders the constructive argument outlined above inapplicable, and hence the existence problem calls for an alternative fixed point argument.

Consider the following two-player Bayesian game with a continuum of actions, a continuum of types, and simultaneous moves. There are two players, 1 and 2; player 1 has a type $y \in [\underline{y}, \overline{y}]$ distributed according to an atomless density $g(y)$, and he chooses $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$. Player 2 has a type $x \in [\underline{x}, \overline{x}]$ distributed independently from y according to an atomless density $f(x)$, and she chooses $\hat{\omega} \in [\underline{\omega}, \overline{\omega}]$. The players' payoff functions are $U_1(\hat{\theta} \mid y, \hat{\omega}(\cdot))$ and $U_2(\hat{\omega} \mid x, \hat{\theta}(\cdot))$, respectively, where

$$U_1(\hat{\theta} \mid y, \hat{\omega}(\cdot)) = \frac{\int_{\hat{\theta}}^{\overline{\theta}} a(\theta, y)\gamma(\theta, y)m(\theta)\, d\theta}{\delta + \int_{\hat{\theta}}^{\overline{\theta}} a(\theta, y)m(\theta)\, d\theta}, \tag{7}$$

$$U_2(\hat{\omega} \mid x, \hat{\theta}(\cdot)) = \frac{\int_{\hat{\omega}}^{\overline{\omega}} b(\omega, x)\alpha(\omega, x)n(\omega)\, d\omega}{\delta + \int_{\hat{\omega}}^{\overline{\omega}} b(\omega, x)n(\omega)\, d\omega}. \tag{8}$$

A pure strategy equilibrium of the Bayesian game is a pair of functions $(\hat{\theta}^*(y), \hat{\omega}^*(x))$ such that, given $\hat{\omega}^*(x)$, then $\hat{\theta}^*(y)$ solves, for each y,

$$\max_{\underline{\theta} \leqslant \hat{\theta} \leqslant \overline{\theta}} U_1(\hat{\theta} \mid y, \hat{\omega}(\cdot)) \tag{9}$$

and given $\hat{\theta}^*(y)$, then $\hat{\omega}^*(x)$ solves, for each x,

$$\max_{\underline{\omega} \leqslant \hat{\omega} \leqslant \overline{\omega}} U_2(\hat{\omega} \mid x, \hat{\theta}(\cdot)). \tag{10}$$

We are now ready to show that proving the existence of a matching equilibrium is tantamount to proving the existence of a pure strategy equilibrium in this Bayesian game.

Proposition 2. $(\hat{\theta}^*(y), \hat{\omega}^*(x))$ *is a matching equilibrium if and only if it is a pure strategy equilibrium of the two-player Bayesian game defined above.*

Proof. The result obviously holds in the case in which men and women reject all potential partners.

[7] See the proofs in [5–9,13,16,21].

If both sides use strategies such that the probability of being accepted is positive for all types, then Proposition 1 shows that optimal strategies will exhibit the reservation-signal property. The expected discounted utility of any threshold $\hat{\theta}$ for a man of type y is given by

$$\frac{\int_{\hat{\theta}}^{\overline{\theta}} a(\theta, y)\gamma(\theta, y)m(\theta)\, d\theta}{\delta + \int_{\hat{\theta}}^{\overline{\theta}} a(\theta, y)m(\theta)\, d\theta}.$$

In equilibrium, the optimal threshold chosen by y must satisfy

$$\begin{aligned} \beta\Psi(y) &= \max_{\underline{\theta} \leqslant \hat{\theta} \leqslant \overline{\theta}} \frac{\int_{\hat{\theta}}^{\overline{\theta}} a(\theta, y)\gamma(\theta, y)m(\theta)\, d\theta}{\delta + \int_{\hat{\theta}}^{\overline{\theta}} a(\theta, y)m(\theta)\, d\theta} \\ &= \max_{\underline{\theta} \leqslant \hat{\theta} \leqslant \overline{\theta}} U_1(\hat{\theta} \mid y, \hat{\omega}(\cdot)). \end{aligned} \tag{11}$$

Similarly, the optimal threshold for a woman of type x must satisfy

$$\begin{aligned} \beta\Phi(x) &= \frac{\int_{\hat{\omega}}^{\overline{\omega}} b(\omega, x)\alpha(\omega, x)n(\omega)\, d\omega}{\delta + \int_{\hat{\omega}}^{\overline{\omega}} b(\omega, x)n(\omega)\, d\omega} \\ &= \max_{\underline{\omega} \leqslant \hat{\omega} \leqslant \overline{\omega}} U_2(\hat{\omega} \mid x, \hat{\theta}(\cdot)). \end{aligned} \tag{12}$$

It now follows from (11) and (12) that $(\hat{\theta}^*(y), \hat{\omega}^*(x))$ is a matching equilibrium such that $\hat{\theta}^*(y) \neq \overline{\theta}$ and $\hat{\omega}^*(x) \neq \overline{\omega}$ on sets of types of positive measure, if and only if $(\hat{\theta}^*(y), \hat{\omega}^*(x))$ solves problems (9) and (10) for every y and x. □

It is well known that a pure strategy equilibrium need not exist in Bayesian games with a continuum of actions and types (see the discussion in [2] and the references therein). In a recent contribution, Athey [2] presents an elegant existence proof for a class of Bayesian games whose payoff functions satisfy the following *Single Crossing Condition* (SCC) for games with incomplete information: whenever every opponent of a player uses an increasing strategy, the player's objective function satisfies the *Single Crossing Property of Incremental Returns* (SCP-IR) in his action and type. [8] This condition implies that each player has a best response strategy that is increasing whenever his opponents use increasing strategies.

Athey shows that if a Bayesian game satisfies SCC and some regularity conditions, [9] then there exists a pure strategy equilibrium in increasing strategies [2, Theorem 2 and Corollary 2.1]. We use Athey's theorem in the proof of the following result:

Theorem 1. *There exists a matching equilibrium in strategies that are increasing in types.*

The proof of Theorem 1 makes extensive use of the following identity, which is quite helpful in signing expressions containing log-spm terms.

[8] A function $h : \mathbb{R}^2 \to \mathbb{R}$ satisfies SCP-IR in $(x; t)$ if, for all $x_H > x_L$ and all $t_H > t_L$, $h(x_H, t_L) - h(x_L, t_L) > 0$ implies $h(x_H, t_H) - h(x_L, t_H) > 0$, and $h(x_H, t_L) - h(x_L, t_L) \geqslant 0$ implies $h(x_H, t_H) - h(x_L, t_H) \geqslant 0$.

[9] To wit, type-densities are bounded and atomless, actions belong to a closed interval, and expected payoffs are well defined and continuous when opponents use increasing strategies.

Lemma 1. *Let f, g, h, and z be real-valued, integrable (with respect to the Lebesgue measure) functions defined on* $[a, b]$. *Then*

$$\int_a^b fg \int_a^b zh - \int_a^b fh \int_a^b gz = \iint_{s<t} (f(s)z(t) - f(t)z(s))(g(s)h(t) - g(t)h(s))\, ds\, dt,$$

where $s < t$ *denotes the set* $\{(s, t) \in [a, b] \times [a, b] : s < t\}$.[10]

Proof. Appendix. □

Proof of Theorem 1. The proof consists of showing that the two-player Bayesian game defined above satisfies the sufficient conditions for existence of a pure strategy equilibrium stated in [2, Corollary 2.1].

Note that type densities are bounded and atomless by assumption, and action sets are closed and bounded intervals. It is easy to see that expected payoffs are well-defined and finite. Moreover, an application of the LDCT shows that $U_1(\hat{\theta}_n(y) \mid y, \hat{\omega}_n(\cdot))$ converges to $U_1(\hat{\theta}(y) \mid y, \hat{\omega}(\cdot))$ whenever $\hat{\theta}_n(y)$ converges to $\hat{\theta}(y)$ and $\hat{\omega}_n(x)$ converges to $\hat{\omega}(x)$ for almost every $x \in [\underline{x}, \overline{x}]$. A similar property holds for player 2.

The only remaining condition to check is that payoff functions satisfy the SCC for games with incomplete information. Since none of the sufficient conditions for SCC in [2, Section 3.2] hold in our case, we show SCC directly using its definition.

Consider player 1, and suppose that player 2 uses a strategy $\hat{\omega}(x)$ that is increasing in x. Let $\rho(\theta, y) = a(\theta, y)\gamma(\theta, y)$. We need to show that, for all $\hat{\theta}_{\mathrm{H}} > \hat{\theta}_{\mathrm{L}}$ and for all $y_{\mathrm{H}} > y_{\mathrm{L}}$, if

$$\frac{\int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} \rho(\theta, y_{\mathrm{L}}) m(\theta)\, d\theta}{\delta + \int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} a(\theta, y_{\mathrm{L}}) m(\theta)\, d\theta} - \frac{\int_{\hat{\theta}_{\mathrm{L}}}^{\overline{\theta}} \rho(\theta, y_{\mathrm{L}}) m(\theta)\, d\theta}{\delta + \int_{\hat{\theta}_{\mathrm{L}}}^{\overline{\theta}} a(\theta, y_{\mathrm{L}}) m(\theta)\, d\theta} \geqslant 0, \tag{13}$$

then

$$\frac{\int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} \rho(\theta, y_{\mathrm{H}}) m(\theta)\, d\theta}{\delta + \int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} a(\theta, y_{\mathrm{H}}) m(\theta)\, d\theta} - \frac{\int_{\hat{\theta}_{\mathrm{L}}}^{\overline{\theta}} \rho(\theta, y_{\mathrm{H}}) m(\theta)\, d\theta}{\delta + \int_{\hat{\theta}_{\mathrm{L}}}^{\overline{\theta}} a(\theta, y_{\mathrm{H}}) m(\theta)\, d\theta} \geqslant 0 \tag{14}$$

and similarly when the inequalities are strict. (See Eqs. (7) and (8).)

If player 2's strategy is $\hat{\omega}(x) = \overline{\omega}$ for all $x \in [\underline{x}, \overline{x}]$, then (13) and (14) are satisfied, for the left side of both inequalities is equal to zero. Hence, SCC holds in this case.

[10] This is an integral version of an identity that appears in [15, p. 42]

$$\left(\sum_{i=1}^n a_i c_i\right)\left(\sum_{i=1}^n b_i d_i\right) - \left(\sum_{i=1}^n a_i d_i\right)\left(\sum_{i=1}^n b_i c_i\right) = \sum_{1 \leqslant i < j \leqslant n} (a_i b_j - a_j b_i)(c_i d_j - c_j d_i).$$

Suppose now that $\hat{\omega}(x) \neq \overline{\omega}$ on a set of types with positive measure. Notice that (13) holds if and only if

$$\left(\delta + \int_{\hat{\theta}_{\mathrm{L}}}^{\overline{\theta}} a(\theta, y_{\mathrm{L}}) m(\theta)\, d\theta\right) \int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} \rho(\theta, y_{\mathrm{L}}) m(\theta)\, d\theta - \left(\delta + \int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} a(\theta, y_{\mathrm{L}}) m(\theta)\, d\theta\right) \int_{\hat{\theta}_{\mathrm{L}}}^{\overline{\theta}} \rho(\theta, y_{\mathrm{L}}) m(\theta)\, d\theta \geqslant 0.$$

Since $\int_{\hat{\theta}_{\mathrm{L}}}^{\overline{\theta}} = \int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} + \int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}}$, after some manipulation this expression becomes

$$\left(\int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} a(\theta, y_{\mathrm{L}}) m(\theta)\, d\theta\right) \left(\frac{\int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} \rho(\theta, y_{\mathrm{L}}) m(\theta)\, d\theta \int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} a(\theta, y_{\mathrm{L}}) m(\theta)\, d\theta}{\int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} \rho(\theta, y_{\mathrm{L}}) m(\theta)\, d\theta \int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} a(\theta, y_{\mathrm{L}}) m(\theta)\, d\theta} - 1\right) \geqslant \delta. \tag{15}$$

Similarly, (14) is equivalent to

$$\left(\int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} a(\theta, y_{\mathrm{H}}) m(\theta)\, d\theta\right) \left(\frac{\int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} \rho(\theta, y_{\mathrm{H}}) m(\theta)\, d\theta \int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} a(\theta, y_{\mathrm{H}}) m(\theta)\, d\theta}{\int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} \rho(\theta, y_{\mathrm{H}}) m(\theta)\, d\theta \int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} a(\theta, y_{\mathrm{H}}) m(\theta)\, d\theta} - 1\right) \geqslant \delta. \tag{16}$$

To show that SCC holds, following from (15)–(16) it suffices to prove that the left-hand side of (16) is greater than the left-hand side of (15) or, equivalently, that

$$\left(\int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} a(\theta, y) m(\theta)\, d\theta\right) \left(\frac{\int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} \rho(\theta, y) m(\theta)\, d\theta \int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} a(\theta, y) m(\theta) d\theta}{\int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} \rho(\theta, y) m(\theta)\, d\theta \int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} a(\theta, y) m(\theta)\, d\theta} - 1\right) \tag{17}$$

is an increasing function of y.

It will be more convenient to rewrite (17) as

$$\frac{\int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} \rho \int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} a - \int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} \rho \int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} a}{\int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} \rho}, \tag{18}$$

where we have omitted the density and the arguments of the functions to simplify the notation. The derivative of (18) with respect to y is equal to

$$\frac{\left(\int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} \rho \int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} \rho \int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} a_y + \int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} \rho \int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} a \int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} \rho_y\right) - \left(\left(\int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} \rho\right)^2 \int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} a_y + \int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} \rho_y \int_{\hat{\theta}_{\mathrm{H}}}^{\overline{\theta}} \rho \int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} a\right)}{\left(\int_{\hat{\theta}_{\mathrm{L}}}^{\hat{\theta}_{\mathrm{H}}} \rho\right)^2}. \tag{19}$$

The sign of (19) is determined by the sign of the numerator, which is of the form $(C + D) - (A + B)$. From the identity

$$(C + D) - (A + B) = \frac{(D - A)(D - B) + (CD - AB)}{D},$$

it follows that the numerator of (19) is nonnegative if $(D - A)(D - B)$ and $(CD - AB)$ are nonnegative. Simple algebra reveals that

$$D - A = \int_{\hat{\theta}_L}^{\hat{\theta}_H} \rho \left(\int_{\hat{\theta}_L}^{\hat{\theta}_H} a \int_{\hat{\theta}_H}^{\overline{\theta}} \rho_y - \int_{\hat{\theta}_L}^{\hat{\theta}_H} \rho \int_{\hat{\theta}_H}^{\overline{\theta}} a_y \right), \tag{20}$$

$$D - B = \int_{\hat{\theta}_L}^{\hat{\theta}_H} a \left(\int_{\hat{\theta}_L}^{\hat{\theta}_H} \rho \int_{\hat{\theta}_H}^{\overline{\theta}} \rho_y - \int_{\hat{\theta}_L}^{\hat{\theta}_H} \rho_y \int_{\hat{\theta}_H}^{\overline{\theta}} \rho \right), \tag{21}$$

$$CD - AB = \left(\int_{\hat{\theta}_L}^{\hat{\theta}_H} \rho \right)^2 \int_{\hat{\theta}_L}^{\hat{\theta}_H} a \int_{\hat{\theta}_H}^{\overline{\theta}} \rho \left(\int_{\hat{\theta}_L}^{\hat{\theta}_H} a_y \int_{\hat{\theta}_H}^{\overline{\theta}} \rho_y - \int_{\hat{\theta}_L}^{\hat{\theta}_H} \rho_y \int_{\hat{\theta}_H}^{\overline{\theta}} a_y \right). \tag{22}$$

Consider (20); the sign of this expression depends on the sign of the term in parentheses. The definitions of $a(\theta, y)$ and $\rho(\theta, y)$ allow us to write this term as follows:

$$\begin{aligned}
&\int_{\underline{x}}^{\overline{x}} (1 - N(\hat{\omega}(x) \mid y))\mu(x, \hat{\theta}_L, \hat{\theta}_H)\, dx \int_{\underline{x}}^{\overline{x}} \frac{x}{1-\beta} \left(-\frac{\partial N(\hat{\omega}(x) \mid y)}{\partial y} \right) \\
&\quad \times \mu(x, \hat{\theta}_H, \overline{\theta})\, dx - \int_{\underline{x}}^{\overline{x}} \frac{x}{1-\beta} (1 - N(\hat{\omega}(x) \mid y))\mu(x, \hat{\theta}_L, \hat{\theta}_H)\, dx \\
&\quad \times \int_{\underline{x}}^{\overline{x}} \left(-\frac{\partial N(\hat{\omega}(x) \mid y)}{\partial y} \right) \mu(x, \hat{\theta}_H, \overline{\theta})\, dx,
\end{aligned} \tag{23}$$

where $\mu(x, \hat{\theta}_L, \hat{\theta}_H) = \int_{\hat{\theta}_L}^{\hat{\theta}_H} k(x \mid \theta)m(\theta)\, d\theta$ and $\mu(x, \hat{\theta}_H, \overline{\theta}) = \int_{\hat{\theta}_H}^{\overline{\theta}} k(x \mid \theta)m(\theta)\, d\theta$. Eq. (23) is equivalent to

$$\begin{aligned}
&\iint\limits_{s<t} \left(\frac{s-t}{1-\beta} \right) \left(\frac{-\frac{\partial N(\hat{\omega}(s)|y)}{\partial y}}{1 - N(\hat{\omega}(s) \mid y)} \mu(t, \hat{\theta}_L, \hat{\theta}_H)\mu(s, \hat{\theta}_H, \overline{\theta}) - \frac{-\frac{\partial N(\hat{\omega}(t)|y)}{\partial y}}{1 - N(\hat{\omega}(t) \mid y)} \right. \\
&\quad \left. \times\, \mu(s, \hat{\theta}_L, \hat{\theta}_H)\mu(t, \hat{\theta}_H, \overline{\theta}) \right) (1 - N(\hat{\omega}(s) \mid y))(1 - N(\hat{\omega}(t) \mid y))\, ds\, dt,
\end{aligned} \tag{24}$$

where we have applied Lemma 1 with $f(x) = \frac{x}{1-\beta}$, $g(x) = -\frac{\partial N(\hat{\omega}(x)|y)}{\partial y}\mu(x, \hat{\theta}_H, \overline{\theta})$, $z(x) = 1$, and $h(x) = (1 - N(\hat{\omega}(x) \mid y))\mu(x, \hat{\theta}_L, \hat{\theta}_H)$.

Notice that if $\hat{\omega}(x)$ is increasing in x, then $1 - N(\hat{\omega}(x) \mid y)$ is log-spm. For,

$$1 - N(\hat{\omega}(x) \mid y) = \int_{\underline{\omega}}^{\overline{\omega}} I_{[\hat{\omega}(x), \overline{\omega}]}(\omega) n(\omega \mid y)\, d\omega,$$

where $I_{[\hat{\omega}(x),\overline{\omega}]}(\omega)$ is the indicator function of the set $[\hat{\omega}(x), \overline{\omega}]$. When $\hat{\omega}(x)$ is increasing, $[\hat{\omega}(x), \overline{\omega}]$ is increasing in x in the strong set order, and hence $I_{[\hat{\omega}(x),\overline{\omega}]}(\omega)$ is log-spm (see

[3]).[11] Since the product of log-spm functions is log-spm (see [10]), and this property is preserved by integration, it follows that $1 - N(\hat{\omega}(x) \mid y)$ is log-spm.

The log-supermodularity of $1 - N(\hat{\omega}(x) \mid y)$ implies that $\frac{-\frac{\partial N(\hat{\omega}(x)|y)}{\partial y}}{1-N(\hat{\omega}(x)|y)}$ is increasing in x. Therefore, to prove that (24) is nonnegative it suffices to show that if $s < t$, then

$$\mu(t, \hat{\theta}_{\mathrm{L}}, \hat{\theta}_{\mathrm{H}})\mu(s, \hat{\theta}_{\mathrm{H}}, \overline{\theta}) - \mu(s, \hat{\theta}_{\mathrm{L}}, \hat{\theta}_{\mathrm{H}})\mu(t, \hat{\theta}_{\mathrm{H}}, \overline{\theta}) \leqslant 0. \tag{25}$$

A simple argument reveals that (25) is satisfied. Let $A(\tau)$ be a correspondence such that, for each $\tau \in \mathbb{R}$, $A(\tau)$ is a subset of $[\underline{\theta}, \overline{\theta}]$, and let $\tau_{\mathrm{L}} < \tau_{\mathrm{H}}$, $A(\tau_{\mathrm{L}}) = [\hat{\theta}_{\mathrm{L}}, \hat{\theta}_{\mathrm{H}}]$, and $A(\tau_{\mathrm{H}}) = [\hat{\theta}_{\mathrm{H}}, \overline{\theta}]$. Notice that the set $A(\tau_{\mathrm{H}})$ is greater than $A(\tau_{\mathrm{L}})$ in the strong set order. Then, (25) can be written as follows:

$$\begin{aligned} &\int_{\underline{\theta}}^{\overline{\theta}} I_{A(\tau_{\mathrm{L}})}(\theta) k(t \mid \theta) m(\theta)\, d\theta \int_{\underline{\theta}}^{\overline{\theta}} I_{A(\tau_{\mathrm{H}})}(\theta) k(s \mid \theta) m(\theta)\, d\theta \\ &\quad - \int_{\underline{\theta}}^{\overline{\theta}} I_{A(\tau_{\mathrm{L}})}(\theta) k(s \mid \theta) m(\theta)\, d\theta \int_{\underline{\theta}}^{\overline{\theta}} I_{A(\tau_{\mathrm{H}})}(\theta) k(t \mid \theta) m(\theta)\, d\theta, \end{aligned} \tag{26}$$

where $I_{A(\tau_i)}(\theta)$, $i = \mathrm{L}, \mathrm{H}$, is the indicator function of the set $A(\tau_i)$. Ref. [3, Lemma 3], and the preservation of log-supermodularity by multiplication and integration, imply that (26) is negative. Hence, $D - A \geqslant 0$.

Consider (21); the sign of this expression depends on the sign of the term in parentheses, which is equal to

$$\begin{aligned} &\int_{\underline{x}}^{\overline{x}} \frac{x}{1-\beta}(1 - N(\hat{\omega}(x) \mid y))\mu(x, \hat{\theta}_{\mathrm{L}}, \hat{\theta}_{\mathrm{H}})\, dx \int_{\underline{x}}^{\overline{x}} \frac{x}{1-\beta}\left(-\frac{\partial N(\hat{\omega}(x) \mid y)}{\partial y}\right) \\ &\quad \times \mu(x, \hat{\theta}_{\mathrm{H}}, \overline{\theta})\, dx - \int_{\underline{x}}^{\overline{x}} \frac{x}{1-\beta}(1 - N(\hat{\omega}(x) \mid y))\mu(x, \hat{\theta}_{\mathrm{H}}, \overline{\theta})\, dx \int_{\underline{x}}^{\overline{x}} \frac{x}{1-\beta} \\ &\quad \times \left(-\frac{\partial N(\hat{\omega}(x) \mid y)}{\partial y}\right)\mu(x, \hat{\theta}_{\mathrm{L}}, \hat{\theta}_{\mathrm{H}})\, dx. \end{aligned} \tag{27}$$

Eq. (27) is equivalent to

$$\begin{aligned} &\iint_{s<t} \frac{st}{(1-\beta)^2}\left(\frac{-\frac{\partial N(\hat{\omega}(t)|y)}{\partial y}}{1 - N(\hat{\omega}(t) \mid y)} - \frac{-\frac{\partial N(\hat{\omega}(s)|y)}{\partial y}}{1 - N(\hat{\omega}(s) \mid y)}\right)(\mu(s, \hat{\theta}_{\mathrm{L}}, \hat{\theta}_{\mathrm{H}})\mu(t, \hat{\theta}_{\mathrm{H}}, \overline{\theta}) \\ &\quad -\mu(t, \hat{\theta}_{\mathrm{L}}, \hat{\theta}_{\mathrm{H}})\mu(s, \hat{\theta}_{\mathrm{H}}, \overline{\theta}))(1 - N(\hat{\omega}(s) \mid y))(1 - N(\hat{\omega}(t) \mid y))\, ds\, dt, \end{aligned} \tag{28}$$

where we have applied Lemma 1 with $f(x) = \frac{x}{1-\beta}(1 - N(\hat{\omega}(x) \mid y))$, $g(x) = \mu(x, \hat{\theta}_{\mathrm{L}}, \hat{\theta}_{\mathrm{H}})$, $z(x) = \frac{x}{1-\beta}\left(-\frac{\partial N(\hat{\omega}(x)|y)}{\partial y}\right)$, and $h(x) = \mu(x, \hat{\theta}_{\mathrm{H}}, \overline{\theta})$. It follows from (25) and the log-supermodularity of $1 - N(\hat{\omega}(x) \mid y)$ that the integrand in (28) is nonnegative. Hence, $D - B \geqslant 0$.

[11] A set A is greater than B in the strong set order if for any $a \in A$ and $b \in B$, $a \vee b \in A$ and $a \wedge b \in B$.

Finally, consider (22). The sign of this expression depends on the sign of the second term in parentheses, which is equal to

$$\int_{\underline{x}}^{\overline{x}} \left(-\frac{\partial N(\hat{\omega}(x) \mid y)}{\partial y}\right) \mu(x, \hat{\theta}_{\mathrm{L}}, \hat{\theta}_{\mathrm{H}})\, dx \int_{\underline{x}}^{\overline{x}} \frac{x}{1-\beta} \left(-\frac{\partial N(\hat{\omega}(x) \mid y)}{\partial y}\right) \times \mu(x, \hat{\theta}_{\mathrm{H}}, \overline{\theta})\, dx - \int_{\underline{x}}^{\overline{x}} \frac{x}{1-\beta} \left(-\frac{\partial N(\hat{\omega}(x) \mid y)}{\partial y}\right) \mu(x, \hat{\theta}_{\mathrm{L}}, \hat{\theta}_{\mathrm{H}})\, dx \times \int_{\underline{x}}^{\overline{x}} \left(-\frac{\partial N(\hat{\omega}(x) \mid y)}{\partial y}\right) \mu(x, \hat{\theta}_{\mathrm{H}}, \overline{\theta})\, dx. \tag{29}$$

Eq. (29) is equivalent to

$$\iint_{s<t} \left(\frac{s-t}{1-\beta}\right) \left(\mu(t, \hat{\theta}_{\mathrm{L}}, \hat{\theta}_{\mathrm{H}})\mu(s, \hat{\theta}_{\mathrm{H}}, \overline{\theta}) - \mu(s, \hat{\theta}_{\mathrm{L}}, \hat{\theta}_{\mathrm{H}})\mu(t, \hat{\theta}_{\mathrm{H}}, \overline{\theta})\right) \times \frac{\partial N(\hat{\omega}(s) \mid y)}{\partial y} \frac{\partial N(\hat{\omega}(t) \mid y)}{\partial y}\, ds\, dt, \tag{30}$$

where we have applied Lemma 1 with $f(x) = \frac{x}{1-\beta}$, $g(x) = -\frac{\partial N(\hat{\omega}(x)|y)}{\partial y}\mu(x, \hat{\theta}_{\mathrm{H}}, \overline{\theta})$, $z(x) = 1$, and $h(x) = -\frac{\partial N(\hat{\omega}(x)|y)}{\partial y}\mu(x, \hat{\theta}_{\mathrm{L}}, \hat{\theta}_{\mathrm{H}})$. It follows from (25) and $s < t$ that the integrand in (30) is nonnegative. Hence, $CD - AB \geqslant 0$.

The nonnegativity of (20)–(22) implies that (17) is increasing in *y*. Thus, the payoff function of player 1 satisfies SCC. A similar analysis reveals that SCC holds for player 2 as well. It then follows from [2, Corollary 2.1] that there exists a pure strategy equilibrium in increasing strategies in this Bayesian game. Hence, by Proposition 2, there exists a matching equilibrium in strategies that are increasing in types, thereby completing the proof of the theorem. □

One potential weakness of Theorem 1 is the existence of a *trivial* equilibrium in our model, in which $\hat{\theta}(y) = \overline{\theta}$ for every $y \in [\underline{y}, \overline{y}]$ and $\hat{\omega}(x) = \overline{\omega}$ for every $x \in [\underline{x}, \overline{x}]$; i.e., agents reject every potential partner. Notice that SCC and the other conditions of Athey's Theorem are trivially satisfied in this case; hence, the trivial equilibrium is an equilibrium of the two-player Bayesian game analyzed in Theorem 1.

Our next result addresses this issue. The proof, which is contained in the Appendix, consists of three steps. First, we show that if there is a finite number of actions, then the trivial equilibrium can be eliminated and the Bayesian game analyzed in Theorem 1 has a nontrivial equilibrium in increasing strategies. Second, we appeal to [2, Theorem 2] to show that the Bayesian game with a continuum of actions has an equilibrium in increasing strategies that is the limit of a sequence of nontrivial equilibria of games with a finite number of actions. Third, we show that the limit equilibrium is nontrivial as well.

Proposition 3. *The two-player Bayesian game analyzed in Theorem* 1 *has an equilibrium in increasing strategies that is nontrivial.*

Proof. Appendix. □

3.3. Assortative mating

Given a nontrivial matching equilibrium in which $(\hat{\theta}^*(y), \hat{\omega}^*(x))$ are increasing functions, it is easy to show that, although match formation is based on noisy signals, the equilibrium exhibits positive assortative mating in the following stochastic sense: the higher the type of an agent, the 'better' (in FOSD sense) the distribution over the types of partners the agent can end up marrying.

Theorem 2. *A nontrivial matching equilibrium in which* $\hat{\theta}^*(y)$ *and* $\hat{\omega}^*(x)$ *are increasing functions exhibits positive assortative mating.*

Proof. Consider a man with type y; his equilibrium probability of marriage is

$$\int_{\hat{\theta}(y)}^{\overline{\theta}} a(\theta, y)m(\theta)d\theta = \int_{\underline{x}}^{\overline{x}} (1 - N(\hat{\omega}^*(x) \mid y))(1 - M(\hat{\theta}^*(y) \mid x)) f(x)\, dx.$$

Therefore, the conditional density over the types of y's potential spouses is positive on $[\underline{x}, \overline{x}]$ and is given by

$$\pi^*(x \mid y) = \frac{(1 - N(\hat{\omega}^*(x) \mid y))(1 - M(\hat{\theta}^*(y) \mid x)) f(x)}{\int_{\underline{x}}^{\overline{x}} (1 - N(\hat{\omega}^*(x) \mid y))(1 - M(\hat{\theta}^*(y) \mid x)) f(x)\, dx}.$$

When strategies are increasing, $\pi^*(x \mid y)$ satisfies the strict MLRP, and hence the associated c.d.f. $\Pi^*(x \mid y)$ is a strictly decreasing function of y; i.e., if $y' > y''$, then $\Pi^*(x \mid y')$ dominates $\Pi^*(x \mid y'')$ in the sense of FOSD. Obviously, an analogous result holds for women. □

In order to understand the intuition underlying the assortative mating result described in Theorem 2, consider the following properties of $a(\theta, y)$, $\gamma(\theta, y)$, and $\Psi(y)$:

Proposition 4. (i) *$a(\theta, y)$ is increasing in y.*
(ii) *If $\hat{\omega}(x)$ is increasing in x, then $\gamma(\theta, y)$ is increasing in y, and $\gamma(\theta, y) \leqslant E[\frac{x}{1-\beta} \mid \theta]$.*
(iii) *If $\hat{\omega}(x)$ is increasing in x, then $\Psi(y)$ is increasing in y.*

Proof. Appendix. □

(i) says that, the larger a man's type, the higher his probability of being accepted by women, given any value of θ. (ii) reveals that, when women use an increasing strategy, then men suffer an acceptance curse (i.e., $\gamma(\theta, y) \leqslant E[\frac{x}{1-\beta} \mid \theta]$), which is more severe for men of lower types, since $\gamma(\theta, y)$ is increasing in y. Finally, consider (iii); $\Psi(y)$ is increasing if and only if

$$\int_{\hat{\theta}}^{\overline{\theta}} a_y(\theta, y)(\gamma(\theta, y) - \gamma(\hat{\theta}, y))m(\theta)\, d\theta + \int_{\hat{\theta}}^{\overline{\theta}} a(\theta, y)\gamma_y(\theta, y)m(\theta)\, d\theta \tag{31}$$

is positive. Loosely speaking, the first term in (31) is the change in the value of continuing the search attributable to a larger probability of being accepted when y increases. The second term in (31) is the change in the expected discounted payoff from a match attributable to the change in the informational content of being accepted when y increases; the sign of this effect depends on the strategy chosen by women. When $\hat{\omega}(x)$ is increasing in x, the two effects *reinforce* each other, thereby making the option value of the search increasing in y.

Consider a matching equilibrium in increasing strategies, and let $y' > y''$ be two types of men. Then the acceptance curse is worse for y'' than for y'. If the option value of the search were the same for both types, then y' would be *less* selective than y'', for a lesser acceptance curse implies that $\gamma(\theta, y') \geqslant \gamma(\theta, y'')$ for any signal θ. The option value of the search, however, is larger for y' than for y'', inducing y' to be *more* selective than y''. [12]

Theorem 1 implies that the best response for men to an increasing strategy is increasing as well. Therefore, when women use an increasing strategy, the option value effect prevails, and hence y' is more selective than y'' in his acceptance decision; i.e., $\hat{\theta}^*(y') \geqslant \hat{\theta}^*(y'')$. Similar results hold for women, thereby explaining the assortative mating property characterized in Theorem 2. [13]

It would be interesting to know if there exist equilibria in strategies that are *not* increasing everywhere, for this would imply a violation of positive assortative mating due exclusively to the information contained in the acceptance event.

To see how such an equilibrium could arise, it is instructive to consider the case in which $\hat{\omega}(x)$ is *strictly decreasing* in x. Since women use a strictly decreasing strategy, men enjoy an acceptance 'blessing' (i.e., $\gamma(\theta, y) \geqslant E[\frac{x}{1-\beta} \mid \theta]$), and this effect is stronger for men of lower types (i.e., $\gamma_y < 0$). Intuitively, men of low types tend to send low signals and—given that $\hat{\omega}(x)$ is strictly decreasing in x—they are relatively more likely to be accepted by women of high types. Therefore, if the option value of the search were independent of y, then men of higher types would be *more* selective than men of lower types. But the option value of the search also depends on y, and the sign of Ψ_y is ambiguous. The first term in (31) is always positive, for high types are accepted more often. The second term in (31), however, is now negative because of the shape of the strategy chosen by women, which generates a smaller acceptance blessing for higher types. If the second term prevails, then $\Psi_y < 0$, inducing agents with higher types to be *less* selective than agents with lower types. In this case, if the option value effect were strong enough, it would yield an optimal strategy $\hat{\theta}(y)$ that is also strictly decreasing in y.

The following result shows that such an equilibrium does not exist.

Proposition 5. *A nontrivial matching equilibrium in monotone strategies must be in increasing strategies.*

[12] It is worth emphasizing that this trade-off emerges as a consequence of the existence of the acceptance curse. If agents did not take into account the acceptance curse effect, i.e., if they only considered $E[\frac{x}{1-\beta} \mid \theta]$, then $\hat{\theta}(y)$ would *trivially* be monotone increasing, for only the continuation value would depend on the agent's type.

[13] The reader may wonder if one could not derive the monotonicity property of equilibrium strategies using a result contained in [21], which shows that if types are perfectly observable and payoff functions are increasing in types and log-spm, then equilibrium strategies are increasing in types. The answer is negative, for in the present context $\gamma(\theta, y)$ need *not* be log-spm, and although $a(\theta, y)\gamma(\theta, y)$ is log-spm, it is *not* increasing in θ.

Proof. It suffices to show that, if $\hat{\omega}(x)$ is decreasing in x and nontrivial, then $\hat{\theta}(y)$ is increasing in y.

From Proposition 1 and Eq. (4), it follows that the threshold $\hat{\theta}(y)$ solves (at an interior solution)

$$\gamma(\hat{\theta}, y) = \frac{\int_{\hat{\theta}}^{\overline{\theta}} a(\theta, y)\gamma(\theta, y)m(\theta)\, d\theta}{\delta + \int_{\hat{\theta}}^{\overline{\theta}} a(\theta, y)m(\theta)\, d\theta}, \tag{32}$$

which can be written as

$$\delta = \int_{\hat{\theta}}^{\overline{\theta}} a(\theta, y)\left(\frac{\gamma(\theta, y)}{\gamma(\hat{\theta}, y)} - 1\right) m(\theta)\, d\theta. \tag{33}$$

It is easy to show that (33) has a unique solution. The right-hand side of (33) is positive, greater than δ at $\hat{\theta} = \underline{\theta}$ (assuming an interior solution), zero at $\hat{\theta} = \overline{\theta}$, strictly decreasing, and continuous. Therefore, there is a unique $\underline{\theta} < \hat{\theta} < \overline{\theta}$ that satisfies (33). Obviously, if δ is greater than the right-hand side of (33) evaluated at $\hat{\theta} = \underline{\theta}$, then the optimal threshold is $\hat{\theta} = \underline{\theta}$.

Eq. (33) reveals that a sufficient condition for $\hat{\theta}(y)$ to be increasing in y is that $a(\theta, y)\left(\frac{\gamma(\theta,y)}{\gamma(\hat{\theta},y)} - 1\right)$ be increasing in y, for any $\theta > \hat{\theta}$.

Let $y_H > y_L$, and consider

$$a(\theta, y_H)\left(\frac{\gamma(\theta, y_H)}{\gamma(\hat{\theta}, y_H)} - 1\right) - a(\theta, y_L)\left(\frac{\gamma(\theta, y_L)}{\gamma(\hat{\theta}, y_L)} - 1\right). \tag{34}$$

Expression (34) is nonnegative if and only if

$$\begin{aligned}&\rho(\hat{\theta}, y_L)\left(\rho(\theta, y_H)a(\hat{\theta}, y_H) - \rho(\hat{\theta}, y_H)a(\theta, y_H)\right)\\ &\quad -\rho(\hat{\theta}, y_H)\left(\rho(\theta, y_L)a(\hat{\theta}, y_L) - \rho(\hat{\theta}, y_L)a(\theta, y_L)\right)\end{aligned} \tag{35}$$

is nonnegative. Using the definition of a and ρ, one can show after some algebraic manipulation that (35) is equivalent to

$$\begin{aligned}&\int_r \iint_{s>t} r(s-t)A(r, s, t, y_L, y_H)k(r \mid \hat{\theta})k(s \mid \hat{\theta})k(t \mid \hat{\theta})\\ &\quad \times \left(\frac{k(s \mid \theta)}{k(s \mid \hat{\theta})} - \frac{k(t \mid \theta)}{k(t \mid \hat{\theta})}\right) dt\, ds\, dr,\end{aligned} \tag{36}$$

where

$$\begin{aligned}A(r, s, t, y_L, y_H) = {}&(1 - N(\hat{\omega}(r) \mid y_L))(1 - N(\hat{\omega}(s) \mid y_H))(1 - N(\hat{\omega}(t) \mid y_H))\\ &-(1 - N(\hat{\omega}(r) \mid y_H))(1 - N(\hat{\omega}(s) \mid y_L)) \times (1 - N(\hat{\omega}(t) \mid y_L)).\end{aligned} \tag{37}$$

Notice that $s - t > 0$, and $\frac{k(s|\theta)}{k(s|\hat{\theta})} - \frac{k(t|\theta)}{k(t|\hat{\theta})} > 0$ by the MLRP. The sign of $A(r, s, t, y_L, y_H)$, however, can be positive or negative.

The following argument shows that, despite the ambiguity in the sign of $A(r, s, t, y_L, y_H)$, (36) is nonnegative.

The integrand in (36) contains three kinds of terms, namely, those with (i) $r \geqslant s > t$, (ii) $s > r \geqslant t$, and (iii) $s > t \geqslant r$.

Consider (i) $r \geqslant s > t$. If $r = s$, then $A(r, s, t, y_L, y_H)$ is positive since $N(\hat{\omega}(t) \mid y_L) > N(\hat{\omega}(t) \mid y_H)$. If $r > s$, then $1 - N(\omega \mid y)$ log-spm and $\hat{\omega}(x)$ decreasing in x imply that $(1 - N(\hat{\omega}(r) \mid y_L))(1 - N(\hat{\omega}(s) \mid y_H)) > (1 - N(\hat{\omega}(r) \mid y_H))(1 - N(\hat{\omega}(s) \mid y_L))$. Since $N(\hat{\omega}(t) \mid y_L) > N(\hat{\omega}(t) \mid y_H)$, it follows that $A(r, s, t, y_L, y_H)$ is positive. Hence, all the terms in the integrand in which $r \geqslant s > t$ are positive.

Consider (ii) $s > r \geqslant t$. This case is analogous to case (i) and its proof follows along the same lines. Hence, all the terms in the integrand in which $s > r \geqslant t$ are positive.

Finally, consider (iii) $s > t \geqslant r$. If $t = r$, then $A(r, s, t, y_L, y_H)$ is positive since $N(\hat{\omega}(s) \mid y_L) > N(\hat{\omega}(s) \mid y_H)$. If $t > r$, then the sign of $A(r, s, t, y_L, y_H)$ is ambiguous. We claim that for any term in the integrand such that $s > t > r$, there is a positive term $s > r' > t'$, with $r' = t$ and $t' = r$, such that the sum of the two terms is positive. That is, we need to show that

$$\begin{aligned} & r'(s - t')A(r', s, t', y_L, y_H)k(r' \mid \hat{\theta})k(s \mid \hat{\theta})k(t' \mid \hat{\theta}) \left(\frac{k(s \mid \theta)}{k(s \mid \hat{\theta})} - \frac{k(t' \mid \theta)}{k(t' \mid \hat{\theta})} \right) \\ & \quad + r(s - t)A(r, s, t, y_L, y_H)k(r \mid \hat{\theta})k(s \mid \hat{\theta})k(t \mid \hat{\theta}) \left(\frac{k(s \mid \theta)}{k(s \mid \hat{\theta})} - \frac{k(t \mid \theta)}{k(t \mid \hat{\theta})} \right) \end{aligned} \tag{38}$$

is positive. Notice that $r'(s - t') > r(s - t)$ if and only if $t(s - r) > r(s - t)$, which holds since $t > r$. Similarly, $\frac{k(s|\theta)}{k(s|\hat{\theta})} - \frac{k(t'|\theta)}{k(t'|\hat{\theta})} > \frac{k(s|\theta)}{k(s|\hat{\theta})} - \frac{k(t|\theta)}{k(t|\hat{\theta})}$ if and only if $\frac{k(t|\theta)}{k(t|\hat{\theta})} > \frac{k(t'|\theta)}{k(t'|\hat{\theta})}$, which holds by the MLRP and $t' = r < t$. Therefore, (38) is bigger than

$$\begin{aligned} & r(s - t)k(r \mid \hat{\theta})k(s \mid \hat{\theta})k(t \mid \hat{\theta}) \left(\frac{k(s \mid \theta)}{k(s \mid \hat{\theta})} - \frac{k(t \mid \theta)}{k(t \mid \hat{\theta})} \right) \\ & \quad \times \big(A(r', s, t', y_L, y_H) + A(r, s, t, y_L, y_H) \big) \end{aligned}$$

and this expression is positive since

$$\begin{aligned} & A(r', s, t', y_L, y_H) + A(r, s, t, y_L, y_H) \\ & \quad = \big((1 - N(\hat{\omega}(r) \mid y_L))(1 - N(\hat{\omega}(t) \mid y_H)) + (1 - N(\hat{\omega}(t) \mid y_L)) \\ & \qquad \times (1 - N(\hat{\omega}(r) \mid y_H)) \big) \big(N(\hat{\omega}(s) \mid y_L) - N(\hat{\omega}(s) \mid y_H) \big) \geqslant 0. \end{aligned} \tag{39}$$

Hence, any negative term in case (iii) is dominated by a positive term in the integrand belonging to case (ii).

We have thus shown that (36) is positive, which in turn implies that, given a decreasing strategy $\hat{\omega}(x)$, the best response $\hat{\theta}(y)$ is increasing in y. This completes the proof. □

In other words, monotone equilibria *must* be in increasing strategies and therefore exhibit positive assortative mating. It would be interesting to know if there exists an equilibrium in nonincreasing strategies, or to have a general proof that rules out such an equilibrium.

To be sure, this is a challenging problem, for we have shown that the model reduces to an asymmetric two-player Bayesian game with a continuum of types, a continuum of actions, and a complicated payoff structure that makes it extremely difficult either to explicitly construct equilibria using differential equation methods, or to prove general characterization results. [14]

The most natural way to proceed would be to extend Proposition 5 and prove that a player's best response is an increasing function given *any* strategy of the other player, thereby showing that *all* matching equilibria are in increasing strategies. [15]

The following example shows that an increasing strategy is *not* always a player's best response. Let $0 < x_1 < x_2 < x_3 < 1$, $0 < y_1 < y_2 < y_3 < 1$, $f(x_i) = \frac{1}{3}$, $g(y_i) = \frac{1}{3}$, $i = 1, 2, 3$, $\theta \in \{\underline{\theta}, \overline{\theta}\}$, $\omega \in \{\underline{\omega}, \overline{\omega}\}$, $m(\overline{\theta} \mid x_i) = x_i$, $n(\overline{\omega} \mid y_i) = y_i$, $i = 1, 2, 3$. Assume that x_1 and x_3 accept both signals, while x_2 only accepts the high one. After some simple but lengthy algebraic manipulation, one can show that (36) is equal to

$$(y_{\mathrm{H}} - y_{\mathrm{L}})\Big((x_1(1-x_1) + x_3(1-x_3))((x_2-x_1)^2 + (x_3-x_2)^2) \\ - x_2(1-x_2)(x_3-x_1)^2\Big),$$

which is negative if, for instance, $x_1 = 0.1$, $x_2 = 0.5$, and $x_3 = 0.9$. Thus, the best response to such strategy is a *decreasing* one. [16]

4. An example

As an illustration, consider a symmetric version of the model with a continuum of types and binary signals. More precisely, let $f(x)$ be the density of women's types, $\underline{x} = \underline{y}$, $\overline{x} = \overline{y}$, $g(y) = f(y)$, $\theta \in \{\underline{\theta}, \overline{\theta}\}$, $\underline{\theta} < \overline{\theta}$, $m(\overline{\theta} \mid x) = \varepsilon(x)$, with $\varepsilon(\underline{x}) \geqslant 0$, $\varepsilon(\overline{x}) \leqslant 1$, and $\varepsilon'(x) > 0$. Also, let $\omega \in \{\underline{\omega}, \overline{\omega}\}$, $\underline{\omega} < \overline{\omega}$, $n(\overline{\omega} \mid y) = \varepsilon(y)$, with $\varepsilon(\underline{y}) \geqslant 0$, $\varepsilon(\overline{y}) \leqslant 1$, and $\varepsilon'(y) > 0$.

Consider type y and suppose women use an increasing strategy. In this setting, an increasing strategy is characterized by a threshold x^*, such that women whose types are below x^* accept both signals, and otherwise they accept only the high signal. Then,

$$\gamma(\overline{\theta}, y) = \frac{\int_{\underline{x}}^{x^*} \frac{x}{1-\beta}\varepsilon(x) f(x)\,dx + \varepsilon(y)\int_{x^*}^{\overline{x}} \frac{x}{1-\beta}\varepsilon(x) f(x)\,dx}{\int_{\underline{x}}^{x^*} \varepsilon(x) f(x)\,dx + \varepsilon(y)\int_{x^*}^{\overline{x}} \varepsilon(x) f(x)\,dx}$$

$$\gamma(\underline{\theta}, y) = \frac{\int_{\underline{x}}^{x^*} \frac{x}{1-\beta}(1-\varepsilon(x)) f(x)\,dx + \varepsilon(y)\int_{x^*}^{\overline{x}} \frac{x}{1-\beta}(1-\varepsilon(x)) f(x)\,dx}{\int_{\underline{x}}^{x^*} (1-\varepsilon(x)) f(x)\,dx + \varepsilon(y)\int_{x^*}^{\overline{x}} (1-\varepsilon(x)) f(x)\,dx}.$$

[14] It is difficult to characterize the equilibrium set even for well-known Bayesian games such as the first-price auction with a single unit and asymmetric bidders. See [12] and the general proof of existence of a monotone equilibrium in [19] for two recent contributions on this class of games.

[15] If one mimics the proof of Proposition 5 for any given $\hat{\omega}(x)$, then in cases (i)–(iii) analyzed in the proof, there are six subcases to consider, namely the six ways in which $\hat{\omega}(r)$, $\hat{\omega}(s)$, and $\hat{\omega}(t)$ can be ordered. Among them, the only one in which the argument breaks down is when $s > r > t$ and either $\hat{\omega}(r) > \hat{\omega}(s) > \hat{\omega}(t)$ or $\hat{\omega}(r) > \hat{\omega}(t) > \hat{\omega}(s)$. Obviously, these subcases do not arise if $\hat{\omega}(x)$ is monotone.

[16] Notice, however, that this cannot constitute a matching equilibrium, for Proposition 5 shows that the best response to a decreasing strategy is an increasing one.

Note that if x^* is either equal to $\underline{x}$ or equal to $\overline{x}$, then $\gamma(\underline{\theta}, y) = E[x \mid \underline{\theta}]$ and $\gamma(\overline{\theta}, y) = E[x \mid \overline{\theta}]$. Intuitively, the 'pooling' strategy used by women in those cases imply that being accepted does not confer any additional information. If x^* is interior, however, then $\gamma(\underline{\theta}, y) < E[x \mid \underline{\theta}]$ and $\gamma(\overline{\theta}, y) < E[x \mid \overline{\theta}]$.

Easy algebra reveals that y accepts both signals or only the high one according to whether

$$\delta \gtrless m(\overline{\theta})a(\overline{\theta}, y)\left(\frac{\gamma(\overline{\theta}, y)}{\gamma(\underline{\theta}, y)} - 1\right). \tag{40}$$

Since the right-hand side of (40) is increasing in y by Proposition 5, there exists a threshold $y^* \in [\underline{y}, \overline{y}]$ such that all $y < y^*$ accept both signals and all $y \geqslant y^*$ accept only the high signal. Therefore, nontrivial (symmetric) matching equilibria in increasing strategies have a simple structure: they are characterized by a threshold η, the same for both men and women, such that agents whose types are below η accept both signals and agents whose types are above η accept only the high signal.

There are three equilibrium configurations: (i) everyone accepts both signals (i.e., $\eta = \overline{x}$); (ii) everyone accepts only the high signal (i.e., $\eta = \underline{x}$); (iii) some types accept both signals and the rest accept only the high one (i.e., $\underline{x} < \eta < \overline{x}$).

Suppose that $\eta = \overline{x}$ for women. Then the right-hand side of (40) is equal to

$$\int_{\underline{x}}^{\overline{x}} \varepsilon(x) f(x)\, dx \left(\frac{\int_{\underline{x}}^{\overline{x}} x\varepsilon(x) f(x)\, dx \int_{\underline{x}}^{\overline{x}} (1-\varepsilon(x)) f(x)\, dx}{\int_{\underline{x}}^{\overline{x}} x(1-\varepsilon(x)) f(x)\, dx \int_{\underline{x}}^{\overline{x}} \varepsilon(x) f(x)\, dx} - 1\right). \tag{41}$$

Denote (41) by A. Since $\delta = \frac{1-\beta}{\beta}$, it follows from (40) that equilibrium (i) exists if $\beta \leqslant (1 + A)^{-1}$.

Suppose that $\eta = \underline{x}$. Then the right-hand side of (40) is equal to

$$\varepsilon(y) \int_{\underline{x}}^{\overline{x}} \varepsilon(x) f(x)\, dx \left(\frac{\int_{\underline{x}}^{\overline{x}} x\varepsilon(x) f(x)\, dx \int_{\underline{x}}^{\overline{x}} (1-\varepsilon(x)) f(x)\, dx}{\int_{\underline{x}}^{\overline{x}} x(1-\varepsilon(x)) f(x)\, dx \int_{\underline{x}}^{\overline{x}} \varepsilon(x) f(x)\, dx} - 1\right), \tag{42}$$

which is obviously increasing in y. Thus, men accept only the high signal if and only if (42) evaluated at $y = \underline{y}$, denoted by B, is bigger than δ. It follows from (40) that equilibrium (ii) exists if $\beta \geqslant (1 + B)^{-1}$.

Suppose that $\underline{x} < \eta < \overline{x}$. For equilibrium (iii) to exist, we must show that

$$\delta = m(\overline{\theta})a(\overline{\theta}, \eta)\left(\frac{\gamma(\overline{\theta}, \eta)}{\gamma(\underline{\theta}, \eta)} - 1\right) \tag{43}$$

has a solution. Notice that the right-hand side of (43) is equal to A if $\eta = \overline{x}$, equal to B if $\eta = \underline{x}$, and continuous in η. Moreover, $B < A$. Therefore, if $\beta \in ((1 + A)^{-1}, (1 + B)^{-1})$, then there exists an equilibrium with $\underline{x} < \eta < \overline{x}$.

Let us further specialize the model to the case in which there are only two types of men and women. The tractability that ensues will be exploited in the next section to shed light on potential extensions of the model.

Formally, let $x \in \{x_1, x_2\}$, $0 < x_1 < x_2$, $y \in \{y_1, y_2\}$, $0 < y_1 < y_2$, $x_1 = y_1$, $x_2 = y_2$, $f(x_2) = g(y_2) = \lambda$, $0 < \lambda < 1$, and, with some abuse of notation, let $\varepsilon(x_2) = \varepsilon$, $\varepsilon(x_1) = 1 - \varepsilon$, with $\varepsilon > \frac{1}{2}$.

Suppose women of type x_1 accept both signals, while women of type x_2 accept only the high signal, and consider the optimal response for men. Then,

$$\gamma(\underline{\theta}, y_1) = \frac{(1-\lambda)\varepsilon \frac{x_1}{1-\beta} + \lambda(1-\varepsilon)^2 \frac{x_2}{1-\beta}}{(1-\lambda)\varepsilon + \lambda(1-\varepsilon)^2},$$

$$\gamma(\overline{\theta}, y_1) = \frac{(1-\lambda)(1-\varepsilon)\frac{x_1}{1-\beta} + \lambda\varepsilon(1-\varepsilon)\frac{x_2}{1-\beta}}{(1-\lambda)(1-\varepsilon) + \lambda\varepsilon(1-\varepsilon)},$$

$$\gamma(\underline{\theta}, y_2) = \frac{(1-\lambda)\frac{x_1}{1-\beta} + \lambda(1-\varepsilon)\frac{x_2}{1-\beta}}{(1-\lambda) + \lambda(1-\varepsilon)},$$

$$\gamma(\overline{\theta}, y_2) = \frac{(1-\lambda)(1-\varepsilon)\frac{x_1}{1-\beta} + \lambda\varepsilon^2 \frac{x_2}{1-\beta}}{(1-\lambda)(1-\varepsilon) + \lambda\varepsilon^2}.$$

Following the same steps as before, one can show that configuration (i) is the unique equilibrium for low values of β, (ii) is the unique equilibrium for high values of β, and (iii) is the unique equilibrium for intermediate values of β. As $\varepsilon \to \frac{1}{2}$, only equilibrium (i) survives. As $\varepsilon \to 1$, the set of matching equilibria converges to the one with observable types, in which both types accept each other if $\beta < (\lambda \frac{x_2}{x_1} + (1-\lambda))^{-1}$, while high types only accept high types if the inequality is reversed.

5. Extensions

In order to derive the results, we make several simplifying assumptions. We now discuss extensions of the analysis and difficulties that ensue in each case.

5.1. Payoff complementarities

We assume that an agent's per-period payoff from a match is simply the type of his or her partner. All the results, however, extend to the case in which payoff functions are increasing and log-spm. Formally, suppose that a single agent's per period utility is 0, but if a man of type y marries a woman of type x, then his per-period payoff is $p(x, y)$, while hers is $q(x, y)$, where the functions p and q are positive, strictly increasing and differentiable in each argument, and log-spm.

Under these assumptions, the proof of Theorem 1 needs to be modified as follows. Notice that $\rho(\theta, y)$ becomes

$$\rho(\theta, y) = \int_{\underline{x}}^{\overline{x}} \frac{p(x, y)}{1-\beta}(1 - N(\hat{\omega}(x) \mid y))k(x \mid \theta)\, dx, \tag{44}$$

and hence

$$\rho_y(\theta, y) = \int_{\underline{x}}^{\overline{x}} \frac{p_y(x, y)}{1-\beta}(1 - N(\hat{\omega}(x) \mid y))k(x \mid \theta)\, dx + \int_{\underline{x}}^{\overline{x}} \frac{p(x, y)}{1-\beta}\left(-\frac{\partial N(\hat{\omega}(x) \mid y)}{\partial y}\right) k(x \mid \theta)\, dx. \tag{45}$$

For notational simplicity, let ρ'_y be the first term and ρ''_y be the second term in (45). If we insert (44) and (45) in (19), then simple algebra reveals that the numerator can be written as the sum of two expressions. The first is identical to the numerator of (19) but with ρ''_y instead of ρ_y; exactly the same argument used in the proof of Theorem 1 shows that it is positive. The second is

$$\int_{\hat{\theta}_L}^{\hat{\theta}_H} a \left(\int_{\hat{\theta}_L}^{\hat{\theta}_H} \rho \int_{\hat{\theta}_H}^{\overline{\theta}} \rho'_y - \int_{\hat{\theta}_H}^{\overline{\theta}} \rho \int_{\hat{\theta}_L}^{\hat{\theta}_H} \rho'_y \right) \tag{46}$$

and an application of Lemma 1—with $f(x) = \frac{p(x,y)}{1-\beta}(1 - N(\hat{\omega}(x) \mid y))$, $g(x) = \mu(x, \hat{\theta}_L, \hat{\theta}_H)$, $z(x) = \frac{p_y(x,y)}{1-\beta}(1 - N(\hat{\omega}(x) \mid y))$, and $h(x) = \mu(x, \hat{\theta}_H, \overline{\theta})$—shows that this expression is positive as well. Thus, SCC holds and Theorem 1 extends to this case. The other results, namely, the ones contained in Propositions 1–5 and Theorem 2, go through with only notational changes. [17]

The intuition underlying this extension is as follows. The analysis of Section 3 shows that, if signals are noisy and each agent's per-period payoff while matched is equal to his or her partner's type, then the best response to an increasing strategy is increasing as well, and thus positive assortative mating emerges in equilibrium. In particular, this means that, as a function of an agent's type, the option value of continuing the search, $\beta\Psi(y)$, rises faster than the expected payoff of accepting a match, $\gamma(\theta, y)$. This is also true without noise if each agent's per-period payoff while matched is log-spm and strictly increasing in both partners' types (see [21]). Thus, the addition of payoff complementarities to the model makes $\beta\Psi(y)$ rise even faster than $\gamma(\theta, y)$ than before, thereby *reinforcing* all the results of Section 3.

5.2. *Entry flows*

Instead of automatic replenishment of agents, we could endogeneize the distribution of types of unmatched agents in the market either by assuming that there is a measure of new agents entering the market in each period with exogenously given densities of types (as in [6]), or by allowing matches to be exogenously destroyed, with agents in those matches reentering the pool of singles (as in [21]). In either case, we would need to state explicitly

[17] As a referee pointed out to us, when complementarities are present, $E[\frac{p(x,y)}{1-\beta} \mid \theta] - \gamma(\theta, y)$ need not be decreasing in y. For example, in the two-type case considered in Section 4, $E[\frac{p(x,y_2)}{1-\beta} \mid \theta] - \gamma(\theta, y_2) < E[\frac{p(x,y_1)}{1-\beta} \mid \theta] - \gamma(\theta, y_1)$ if and only if $(1-\lambda+\lambda\varepsilon)(1-\varepsilon)^2(p(x_2, y_2) - p(x_1, y_2)) < (\varepsilon(1-\lambda)(1-\varepsilon) + \varepsilon^3\lambda)(p(x_2, y_1) - p(x_1, y_1))$, which need not hold if complementarities are strong enough. Notice that none of the results depend on this monotonicity property; it was only used to provide an intuitive explanation of the assortative mating result of Theorem 2.

the balanced flow conditions that determine the number of singles present in the market and their distributions of types. This introduces substantial technical complications to the analysis without affecting the main insights of the paper.

For instance, consider the model with an exogenous inflow of new singles. Formally, let e be the measure of new single men and new single women who enter the market in each period, and let their distribution of types be given by continuous densities $r(x)$ and $s(y)$ over $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$, respectively. A matching equilibrium with entry flows consists of a strategy profile $(\hat{\theta}(y), \hat{\omega}(x))$, a measure u of single women (equal to the measure of single men) present in the market, and densities $f(x)$ and $g(y)$, such that (i) given u, $f(x)$, and $g(y)$, the strategy profile $(\hat{\theta}(y), \hat{\omega}(x))$ is a matching equilibrium, and (ii) given $(\hat{\theta}(y), \hat{\omega}(x))$, the following balanced flow conditions are satisfied:

$$\left(\int_{\underline{y}}^{\overline{y}} (1 - N(\hat{\omega}(x) \mid y))(1 - M(\hat{\theta}(y) \mid x))g(y)\,dy\right) f(x)u = er(x) \tag{47}$$

for every $x \in [\underline{x}, \overline{x}]$, and

$$\left(\int_{\underline{x}}^{\overline{x}} (1 - N(\hat{\omega}(x) \mid y))(1 - M(\hat{\theta}(y) \mid x))f(x)\,dx\right) g(y)u = es(y) \tag{48}$$

for every $y \in [\underline{y}, \overline{y}]$. The left-hand side of (47) and (48) is the number of unmatched agents of each type that exits the pool of singles (measure of each type times his or her marriage probability), while the right-hand side is the number of singles of each type that enters the market.

Note that the results derived in Section 3 show that, if the distributions of types present in the market are atomless, then there exists an equilibrium in increasing strategies. This corresponds to (i) of the definition stated above.

With entry flows, we must also show that (ii) of the definition is fulfilled: to wit, we need to show that those strategies are consistent with atomless distributions of types that satisfy the balanced flow conditions. In general, this is a nontrivial fixed point problem, for $\hat{\theta}(y)$ and $\hat{\omega}(x)$ are functions of $f(x)$ and $g(y)$ as well.

A special case in which matching equilibria with entry flows can be fully characterized in an elementary fashion is the case with two types and two signals analyzed in Section 4.

In the matching equilibrium in which both types accept both signals, the balanced flow conditions are $(1 - \lambda)u = e(1 - r)$ and $\lambda u = e\,r$. Hence, $\lambda = r$ and the number of singles in the market is $u = e$, as intuition suggests.

In the matching equilibrium in which the low type accepts both signals and the high type accepts only the high signal, the balanced flow conditions are $(\lambda\varepsilon^2 + (1-\lambda)(1-\varepsilon))\lambda u = e\,r$ and $(\lambda(1 - \varepsilon) + (1 - \lambda))(1 - \lambda)u = e\,(1 - r)$, which imply that

$$r = \frac{\lambda^2\varepsilon^2 + \lambda(1 - \lambda)(1 - \varepsilon)}{\lambda^2\varepsilon^2 + 2\lambda(1 - \lambda)(1 - \varepsilon) + (1 - \lambda)^2}. \tag{49}$$

The right-hand side is equal to 0 at $\lambda = 0$, to 1 at $\lambda = 1$, and is strictly increasing in λ. Thus, for any given value of $0 < r < 1$, there exists a unique value of λ, say λ^*, that satisfies (49). Inserting λ^* in any of the balanced flow equations yields $u = \frac{e\,r}{\lambda^{*2}\varepsilon^2+\lambda^*(1-\lambda^*)(1-\varepsilon)}$.

Finally, in the matching equilibrium in which both types accept only the high signal, the balanced flow conditions are $(\lambda\varepsilon^2 + (1-\lambda)\varepsilon(1-\varepsilon))\lambda u = er$ and $(\lambda\varepsilon(1-\varepsilon) + (1-\lambda)(1-\varepsilon)^2)(1-\lambda)u = e(1-r)$, which imply that

$$r = \frac{\lambda^2\varepsilon^2 + \lambda(1-\lambda)\varepsilon(1-\varepsilon)}{\lambda^2\varepsilon^2 + 2\lambda(1-\lambda)\varepsilon(1-\varepsilon) + (1-\lambda)^2(1-\varepsilon)^2}. \tag{50}$$

The right-hand side is equal to 0 at $\lambda = 0$, to 1 at $\lambda = 1$, and is strictly increasing in λ. Thus, for any given value of $0 < r < 1$, there exists a unique value of λ, say λ^{**}, that satisfies (50). Inserting λ^{**} in any of the balanced flow equations yields $u = \frac{er}{\lambda^{**2}\varepsilon^2 + \lambda^{**}(1-\lambda^{**})\varepsilon(1-\varepsilon)}$.

An interesting feature that emerges from the analysis is that $\lambda^* > \lambda^{**}$ or, equivalently, $1-\lambda^{**} > 1-\lambda^*$. Intuitively, the number of low-type singles present in the market is larger in the matching equilibrium in which they only accept the high signal, for this decreases the speed at which they exit the market.

5.3. Divorce

We assume that agents who get married leave the market. An alternative would be to allow married agents to continue accumulating information about their partners' types and to break up if they decide to do so, thereby reentering the pool of singles. The simplest version of this extension is to assume that spouses' types are revealed one period after they are married, at which point they simultaneously announce whether they want to continue with the match or get divorced. An educated guess is that the insights of the paper regarding match formation will continue to hold with divorce, and that couples will stay together if their types belong to the same 'class,' as in the complete information version of the model (see [6]). The boundaries of the classes, however, will have to be modified to account for the existence of imperfect information about types when the marriage decision takes place.

One technical problem with this extension is that the strategy of each agent is summarized by two thresholds instead of one. Therefore, we cannot appeal to Athey's theorem in this case. [18]

5.4. Controlled signals

Note that agents do not control the signal observed by their partners. An equivalent interpretation is that agents exert effort that shifts the distribution of signals in the sense FOSD, and the cost of effort is zero; in the equilibrium characterized in this paper, everyone exerts the maximum feasible level of effort. The problem becomes more complicated if effort is costly, for agents must not only choose how much effort to exert in each meeting, but they must also consider the effort strategy of potential partners as an additional piece of information about the type of a potential partner. This extension is also subject to the same technical problem mentioned above, for the strategy of each agent can no longer be summarized by a scalar.

[18] McAdams [11] has recently extended Athey's theorem to cases with multidimensional types and actions.

5.5. Observable own-signal

We assume that, in each meeting, agents privately observe a signal about their current partners. Alternatively, we could assume that paired agents observe both signals, in which case an agent's acceptance decision can be conditioned on the two signals observed.

Under this alternative assumption, each agent chooses a type-dependent set containing realizations (θ, ω) that induce him or her to announce 'accept'; this set consists of all realizations (θ, ω) for which the expected discounted payoff from accepting a match is greater than the option value of continuing the search. The set, however, need not be easily describable by a threshold function, for the expected discounted payoff from acceptance need not be an increasing function of a potential partner's signal. The difficulty is that a potential partner's signal affects both the support of types that accept an agent and, given that support, the distribution of those types.

Suppose we restrict attention to strategies with the reservation-signal property; i.e., there are threshold functions $(\hat{\theta}(y, \omega), \hat{\omega}(x, \theta))$ such that a male with type y accepts his current partner if (θ, ω) satisfies $\theta \geqslant \hat{\theta}(y, \omega)$, and a woman with type x accepts her current partner if (θ, ω) satisfies $\omega \geqslant \hat{\omega}(x, \theta)$.

Let women use a strategy $\hat{\omega}(x, \theta)$, which is increasing in x and decreasing in θ. That is, for a given value of θ, women are more selective as their types increase, and for a given type x, women are more willing to accept a match the better is the value of the signal they have sent in their current meeting. Then one can show that the best response for men in this case has similar properties, namely $\hat{\theta}(y, \omega)$ is increasing in y and decreasing in ω.

Monotonicity of the strategies as a function of types follows easily in this case. Since agents now observe the signals sent by themselves and their partners in each meeting, they do not have to calculate the type-dependent average probability of being accepted. Consequently, the expected payoff of accepting a match is *independent* of types, and only the option value of the search is type dependent. Monotonicity with respect to signals, however, is a more subtle property as it uses the assumption that potential partners follow a strategy with the aforementioned properties.

If a matching equilibrium in which $(\hat{\theta}(y, \omega), \hat{\omega}(x, \theta))$ are increasing in types and decreasing in signals exists, then it will exhibits a stochastic form of positive assortative mating. Proving existence, however, calls for different techniques than the ones used in Section 3, for each type now chooses a *function* instead of a scalar, and hence Athey's theorem cannot be applied.

6. Concluding remarks

This paper studies matching with both search and information frictions. In this context, the optimal strategy for an agent must take into account not only information about a partner's type contained in the signal observed, but also information contained in the event that he or she is accepted by a partner.

We show that if potential partners choose increasing strategies, then each agent will find it optimal to respond with a reservation-signal that is also increasing in his or her type.

This property allows us to reduce the model to a two-player Bayesian game, and then use a theorem in [2] to prove the existence of a nontrivial equilibrium in strategies that are increasing functions of the agents' types.

The equilibrium characterized exhibits a stochastic form of positive assortative mating: the higher an agent's type, the better (in FOSD sense) the distribution over the types of partners the agent can end up marrying. Hence, agents with similar types tend to marry each other, but only stochastically.

The positive assortative mating result depends on subtle interplay between two opposite forces driven by the acceptance curse effect. First, the higher an agent's type the lesser the acceptance curse. This prompts agents to become less selective as a function of their types. Second, the higher the agent's type, the larger his or her option value of being single, as the agent suffers less from the acceptance curse and is accepted with higher probability by potential partners. This force, which is stronger than the first, induces agents to become more selective as a function of their types, and positive assortative mating ensues.

We also show that any monotone equilibrium must be in increasing strategies, and consequently must exhibit positive assortative mating. Whether or not an equilibrium in non-increasing strategies exists remains an open question.

Finally, we discuss several potential extensions of the model, namely, the introduction of payoff complementarities, entry flows, divorce, and alternative assumptions regarding the information structure of the model. Although the main insights of this paper appear to be robust to these extensions, their study could generate some interesting additional results about marriage markets, and they therefore constitute important topics for future research.

Acknowledgments

I thank two anonymous referees for their insightful comments and suggestions, which substantially improved the paper, and also Bart Taub, Alejandro Manelli, Ed Schlee, and Gustavo Ventura for their comments on earlier versions of this paper. This revision was conducted while I was visiting the Department of Economics at the University of Michigan. I am grateful for its hospitality, and I especially thank Lones Smith for numerous conversations on the subject.

Appendix

Proof of Lemma 1. Note that

$$\begin{aligned}\int_a^b fg \int_a^b zh &= \int_a^b \int_a^b f(s)g(s)z(t)h(t)\,ds\,dt \\ &= \iint\limits_{s<t} f(s)g(s)z(t)h(t)\,ds\,dt + \iint\limits_{s>t} f(s)g(s)z(t)h(t)\,ds\,dt \\ &\quad + \iint\limits_{s=t} f(s)g(s)z(t)h(t)\,ds\,dt\end{aligned}$$

$$= \iint_{s<t} (f(s)g(s)z(t)h(t) + f(t)g(t)z(s)h(s))\, ds\, dt + \iint_{s=t} f(s)g(s)z(t)h(t)\, ds\, dt. \tag{51}$$

Similarly,

$$\int_a^b fh \int_a^b gz = \iint_{s<t} (f(s)h(s)g(t)z(t) + f(t)h(t)g(s)z(s))\, ds\, dt + \iint_{s=t} f(s)h(s)g(t)z(t)\, ds\, dt. \tag{52}$$

Subtracting (52) from (51) and manipulating yields

$$\int_a^b fg \int_a^b zh - \int_a^b fh \int_a^b gz = \iint_{s<t} (f(s)z(t) - f(t)z(s))(g(s)h(t) - g(t)h(s))\, ds\, dt,$$

and this completes the proof of the lemma. □

Proof of Proposition 3. The proof consists of three steps. First, we show that if there is only a finite number of actions, then the two-player Bayesian game analyzed in Theorem 1 has a nontrivial equilibrium in increasing strategies. Second, we appeal to [2, Corollary 2.1] to show that the two-player Bayesian game with a continuum of actions has an equilibrium in increasing strategies that is the limit of a sequence of nontrivial equilibria of finite-action games. Third, we show that the limit equilibrium is nontrivial as well.

Consider the two-player Bayesian game of Theorem 1 with a finite number of actions. Formally, suppose $\hat{\theta} \in \{\theta_0, \theta_1, \ldots, \theta_M\}, \theta_0 < \theta_1 < \cdots < \theta_M$, and $\hat{\omega} \in \{\omega_0, \omega_1, \ldots, \omega_M\}$, $\omega_0 < \omega_1 < \cdots < \omega_M$, with $\theta_0 = \underline{\theta}$, $\theta_M = \overline{\theta}$, $\omega_0 = \underline{\omega}$, and $\omega_M = \overline{\omega}$.

Define the following sets:

$$\mathbf{Z_1} = \{\mathbf{z^1} \in [\underline{y}, \overline{y}]^{M+2} : z^1_0 = \underline{y}, z^1_1 \leqslant z^1_2 \leqslant \cdots \leqslant z^1_M, z^1_{M+1} = \overline{y}\},$$
$$\mathbf{Z_2} = \{\mathbf{z^2} \in [\underline{x}, \overline{x}]^{M+2} : z^2_0 = \underline{x}, z^2_1 \leqslant z^2_2 \leqslant \cdots \leqslant z^2_M, z^2_{M+1} = \overline{x}\},$$
$$\mathbf{Z} = \mathbf{Z_1} \times \mathbf{Z_2}.$$

Since $\mathbf{Z_i}$ is a compact set, $i = 1, 2$, it follows that $\mathbf{Z}$ is a compact set.

Athey [2] showed that an increasing strategy for player i can be represented by a vector $\mathbf{z^i} \in \mathbf{Z_i}$, $i = 1, 2$. In our case, an increasing strategy $\hat{\theta}(y)$ can be represented by a vector $\mathbf{z^1} \in \mathbf{Z^1}$, and an increasing strategy $\hat{\omega}(x)$ can be represented by a vector $\mathbf{z^2} \in \mathbf{Z_2}$. In particular, $\hat{\omega}(x) = \omega_M$ for all $x \in [\underline{x}, \overline{x}]$ is represented by $\mathbf{z^2} = \underline{\mathbf{z}}^2 = (\underline{x}, \underline{x}, ..., \underline{x}, \overline{x})$.

Let $\mathbf{z} = (\mathbf{z^1}, \mathbf{z^2}) \in \mathbf{Z}$ be a strategy profile, and consider player 1's decision problem. He solves

$$\max_{\hat{\theta} \in \{\theta_0, \theta_1, \ldots, \theta_M\}} U_1(\hat{\theta} \mid y, \mathbf{z}),$$

where

$$U_1(\hat{\theta} \mid y, \mathbf{z}) = \frac{\sum_{m=0}^{M} \int_{z_m^2}^{z_{m+1}^2} \frac{x}{1-\beta}(1 - N(\hat{\omega}_m \mid y))(1 - M(\hat{\theta} \mid x)) f(x)\, dx}{\delta + \sum_{m=0}^{M} \int_{z_m^2}^{z_{m+1}^2} (1 - N(\hat{\omega}_m \mid y))(1 - M(\hat{\theta} \mid x)) f(x)\, dx}. \tag{53}$$

Let $\Theta(y \mid \mathbf{z}) = \arg\max_{\hat{\theta} \in \{\theta_0, \theta_1, \ldots, \theta_M\}} U_1(\hat{\theta} \mid y, \mathbf{z})$ be the correspondence of maximizers. It follows from the Theorem of the Maximum that $\Theta(y \mid \mathbf{z})$ is nonempty and upper-hemicontinuous. Moreover, since $U_1(\hat{\theta} \mid y, \mathbf{z})$ satisfies the SCC in $(\hat{\theta}; y)$, it follows that $\Theta(y \mid \mathbf{z})$ is increasing in y in the strong set order [2, p. 867].

A property of $\Theta(y \mid \mathbf{z})$ that is important for our purposes is the following. Notice that, given any $y \in [\underline{y}, \overline{y}]$, if $\mathbf{z}$ is such that $\mathbf{z^2} \neq \underline{\mathbf{z}}^{\mathbf{2}}$, then $U(\theta_M \mid y, \mathbf{z}) = 0 < U(\theta_m \mid y, \mathbf{z})$, $m = 0, 1, \ldots M - 1$, and thus $\theta_M \notin \Theta(y \mid \mathbf{z})$. And if $\mathbf{z}$ is such that $\mathbf{z^2} = \underline{\mathbf{z}}^{\mathbf{2}}$, then $\Theta(y \mid \mathbf{z}) = \{\theta_0, \theta_1, \ldots, \theta_M\}$ for all $y \in [\underline{y}, \overline{y}]$. That is, the *only* case in which θ_M is in the best response correspondence of player 1 is when player 2 uses the strategy $\hat{\omega}(x) = \omega_M$ for all $x \in [\underline{x}, \overline{x}]$, and in this case *all* the actions available to player 1 are best responses, no matter what his type is.

Define

$$\Theta'(y \mid \mathbf{z}) = \begin{cases} \Theta(y \mid \mathbf{z}) & \text{if } \mathbf{z}^2 \neq \underline{\mathbf{z}}^2, \\ \{\theta_0, \theta_1, \ldots, \theta_{M-1}\} & \text{if } \mathbf{z}^2 = \underline{\mathbf{z}}^2. \end{cases}$$

Notice that $\Theta'(y \mid \mathbf{z})$ differs from $\Theta(y \mid \mathbf{z})$ only at points in which the strategy of player 2 is $\hat{\omega}(x) = \omega_M$ for all $x \in [\underline{x}, \overline{x}]$; at such points we have deleted θ_M from the set of player 1's correspondence of maximizers. Notice also that all of the properties of $\Theta(y \mid \mathbf{z})$ stated above hold for $\Theta'(y \mid \mathbf{z})$ as well. In particular, $\Theta'(y \mid \mathbf{z})$ is increasing in y in the strong set order.

Similarly, for player 2 we can define $\Omega(x \mid \mathbf{z}) = \arg\max_{\hat{\omega} \in \{\omega_0, \omega_1, \ldots, \omega_M\}} U_2(\hat{\omega} \mid x, \mathbf{z})$ and

$$\Omega'(x \mid \mathbf{z}) = \begin{cases} \Omega(x \mid \mathbf{z}) & \text{if } \mathbf{z}^1 \neq \underline{\mathbf{z}}^1, \\ \{\omega_0, \omega_1, \ldots, \omega_{M-1}\} & \text{if } \mathbf{z}^1 = \underline{\mathbf{z}}^1. \end{cases}$$

Let

$$\begin{aligned} \Gamma_1'(\mathbf{z}) &= \{\mathbf{z}^1 \in \mathbf{Z}^1 : \exists\, \hat{\theta}(y) \ \textit{consistent with}\ \mathbf{z}^1, \hat{\theta}(y) \in \Theta'(y \mid \mathbf{z}), \forall y \in [\underline{y}, \overline{y}]\}, \\ \Gamma_2'(\mathbf{z}) &= \{\mathbf{z}^2 \in \mathbf{Z}^2 : \exists\, \hat{\omega}(x) \ \textit{consistent with}\ \mathbf{z}^2, \hat{\omega}(x) \in \Omega'(x \mid \mathbf{z}), \forall x \in [\underline{x}, \overline{x}]\}, \\ \Gamma'(\mathbf{z}) &= \Gamma_1'(\mathbf{z}) \times \Gamma_2'(\mathbf{z}). \end{aligned}$$

The set $\mathbf{Z}$ is nonempty, compact, and convex, and $\Gamma'(\mathbf{z})$ is nonempty. Since, for each $\mathbf{z}$, $\Theta'(y \mid \mathbf{z})$ and $\Omega'(x \mid \mathbf{z})$ are increasing in y in the strong set order, it follows from [2, Lemma 2] that $\Gamma_1'(\mathbf{z})$ and $\Gamma_2'(\mathbf{z})$ are convex-valued. Thus, $\Gamma'(\mathbf{z})$ is convex-valued. The proof of Athey [2, Lemma 3] applies with only notational changes to this case, thereby showing that the graph of $\Gamma'(\mathbf{z})$ is closed; i.e., if $(\mathbf{z}_n, \hat{\mathbf{z}}_n)$ converges to $(\mathbf{z}, \hat{\mathbf{z}})$, with $\hat{\mathbf{z}}_n \in \Gamma'(\mathbf{z}_n)$, then $\hat{\mathbf{z}} \in \Gamma'(\mathbf{z})$.

From Kakutani's fixed point theorem, it follows that the correspondence $\Gamma'(\mathbf{z})$ has a fixed point $\mathbf{z} \in \Gamma'(\mathbf{z})$. Moreover, the definition of $\Theta'(y \mid \mathbf{z})$ and $\Omega'(x \mid \mathbf{z})$ ensures that any fixed point $\mathbf{z}$ is such that $\mathbf{z}^1 \neq \underline{\mathbf{z}}^1$ and $\mathbf{z}^2 \neq \underline{\mathbf{z}}^2$ (e.g., if $\mathbf{z}^1 = \underline{\mathbf{z}}^1$, then the definition of $\Omega'(x \mid \mathbf{z})$ implies that $\mathbf{z}^2 \neq \underline{\mathbf{z}}^2$, which in turn implies that $\mathbf{z}^1 = \underline{\mathbf{z}}^1$ cannot be a best response to $\mathbf{z}^2$).

Since each vector $\mathbf{z}$ represents a profile of increasing strategies, it follows that the finite-action Bayesian game has a nontrivial equilibrium $(\theta^*(y), \omega^*(x))$ in increasing strategies.

We can now appeal to Athey [2, Corollary 2.1], which shows that, if we take a sequence of finite-action sets that increases to the continuum-action case, then the corresponding sequence of nontrivial equilibria contains a subsequence, say $(\hat{\theta}_n^*(y), \hat{\omega}_n^*(x))$, which converges to an equilibrium in increasing strategies in the continuum-action game.

It remains to show that the limit equilibrium is nontrivial as well. Suppose not; i.e., suppose that, as the action sets converge to the continuum case, the sequence of nontrivial equilibria for finite-action games converges to the trivial equilibrium in which $\hat{\theta}^*(y) = \overline{\theta}$, for all $y \in [\underline{y}, \overline{y}]$, and $\hat{\omega}^*(x) = \overline{\omega}$, for all $x \in [\underline{x}, \overline{x}]$. The following argument provides a contradiction. Consider any type y of player 1 and suppose for a moment that $\hat{\theta}$ is a continuous variable in (53). Fixed any n along the sequence of equilibria. Then the Kuhn–Tucker conditions for his problem can be written as follows:

$$\begin{aligned} \gamma(\hat{\theta}, y, \hat{\omega}_n^*(\cdot)) &\geqslant U_1(\hat{\theta} \mid y, \hat{\omega}_n^*(\cdot)) \\ \hat{\theta} - \underline{\theta} &\geqslant 0, \end{aligned} \tag{54}$$

with complementary slackness. Now, as the strategy of player 2 converges to $\hat{\omega}^*(x) = \overline{\omega}$ for all $x \in [\underline{x}, \overline{x}]$, the right-hand side of (54) goes to zero, while the left-hand side is bounded below by $\frac{\underline{x}}{1-\beta} > 0$ for all n. Hence, as n goes to infinity, there exists an integer N, such that, for all $n \geqslant N$, the maximum of (53) is achieved at $\hat{\theta} = \underline{\theta}$. Since $\underline{\theta}$ belongs to player 1's action set for all n, this result holds if $\hat{\theta}$ assumes only a finite number of values.

We have thus shown that, if player 2's strategy converges to $\hat{\omega}^*(x) = \overline{\omega}$ for all $x \in [\underline{x}, \overline{x}]$, player 1's strategy cannot converge to $\hat{\theta}^*(y) = \overline{\theta}$ for all $y \in [\underline{y}, \overline{y}]$. Hence, the limit of the aforementioned subsequence of nontrivial equilibria is nontrivial as well. This completes the proof. □

Proof of Proposition 4. (i) $a_y(\theta, y) = -\int_{\underline{x}}^{\overline{x}} \frac{\partial N(\hat{\omega}(x)|y)}{\partial y} k(x \mid \theta)\, dx \geqslant 0$, since $\frac{\partial N(\hat{\omega}(x)|y)}{\partial y} \leqslant 0$ by the MLRP.

(ii) By definition, $\gamma(\theta, y) = \frac{\rho(\theta,y)}{a(\theta,y)}$ and hence $\gamma_y = \frac{\rho_y a - \rho a_y}{a^2}$. Thus, it suffices to show that $\rho_y a - \rho a_y \geqslant 0$. Applying Lemma 1 with $f(x) = \frac{x}{1-\beta}$, $g(x) = -\frac{\partial N(\hat{\omega}(x)|y)}{\partial y} k(x \mid \theta)$, $z(x) = 1$, and $h(x) = (1 - N(\hat{\omega}(x) \mid y))k(x \mid \theta)$, we obtain that $\rho_y a - \rho a_y$ is equal to

$$\iint_{s<t} \left(\frac{s-t}{1-\beta}\right) k(s \mid \theta)k(t \mid \theta)(1 - N(\hat{\omega}(s) \mid y))(1 - N(\hat{\omega}(t) \mid y)) \times \left(\frac{-\frac{\partial N(\hat{\omega}(s)|y)}{\partial y}}{1 - N(\hat{\omega}(s) \mid y)} - \frac{-\frac{\partial N(\hat{\omega}(t)|y)}{\partial y}}{1 - N(\hat{\omega}(t) \mid y)} \right) ds\, dt.$$

Since $s < t$ and $1 - N(\hat{\omega}(x) \mid y)$ is log-spm, it follows that the first and the last terms in the integrand are nonpositive and therefore the integral is nonnegative. Thus, $\gamma_y(\theta, y) \geqslant 0$.

Using the definition of $\gamma(\theta, y)$ and $E[\frac{x}{1-\beta} \mid \theta]$ we obtain

$$E\left[\frac{x}{1-\beta} \mid \theta\right] - \gamma(\theta, y) = \int_{\underline{x}}^{\overline{x}} \frac{x}{1-\beta} H(x \mid y) k(x \mid \theta)\, dx,$$

where

$$H(x \mid y) = 1 - \frac{1 - N(\hat{\omega}(x) \mid y)}{\int_{\underline{x}}^{\overline{x}} (1 - N(\hat{\omega}(x) \mid y)) k(x \mid \theta)\, dx}.$$

Notice that $\int_0^1 H(x \mid y) k(x \mid \theta)\, dx = 0$, and if $x' > x''$, then $H(x'' \mid y) > 0$ implies $H(x' \mid y) > 0$. It follows from [17, Lemma 1] that $E[\frac{x}{1-\beta} \mid \theta] - \gamma(\theta, y) \geqslant 0$.

(iii) If we differentiate (4) with respect to y and use $\gamma(\hat{\theta}(y), y) = \beta\Psi(y)$, we obtain that $\beta\Psi_y(y) \geqslant 0$ if and only if

$$\int_{\hat{\theta}}^{\overline{\theta}} a_y(\theta, y)(\gamma(\theta, y) - \gamma(\hat{\theta}, y)) m(\theta)\, d\theta + \int_{\hat{\theta}}^{\overline{\theta}} a(\theta, y) \gamma_y(\theta, y) m(\theta)\, d\theta \geqslant 0$$

and this inequality follows from (i) and (ii). □

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