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Two-sided search and perfect segregation with fixed search costs

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Abstract

This paper studies a two-sided search model with the following characteristics: there is a continuum of agents with different types in each population, match utility is nontransferable, and agents incur a fixed search cost in each period. When utility functions are additively separable in types and strictly increasing in the partner's type, there is a unique matching equilibrium. It exhibits perfect segregation as in Smith [Smith, The marriage model with search frictions, Working paper (1997) Department of Economics, MIT] and Burdett and Coles [Quarterly Journal of Economics, 112 (1997) 141]; i.e. agents form clusters and mate only within them. A simple sufficient condition on the match utility function and the density of types characterizes the duration of the search for each type of agent. The sufficiency of additive separability in the fixed search cost case is explained and contrasted with the discounted case; moreover, the results are generalized to a broader class of search cost functions that subsumes discounting and fixed search costs as special cases. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The class of problems that this paper studies can be illustrated as follows: suppose there is a large number of agents that belong from the outset to one of two disjoint populations. Also, suppose that in each population agents differ in their characteristics. Although it is costly for agents to locate potential mates from the population to which

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they do not belong, they prefer this search process to the prospect of remaining lonely forever. During each period they randomly meet in pairs with agents belonging to the other population, and they must decide if they will match forever or continue their searches. The goal of each agent is to find the best possible mate. Since agents differ, the value of the search is not the same for all, nor are their sets of acceptable matches.

In this decentralized setting with costly search and heterogeneity, I investigate the following questions: *In equilibrium, who matches with whom? Do people with better characteristics spend more or less time in the market before finding a suitable match? Does the way in which search frictions are modeled matter?*

This paper seeks to address these questions in a framework with a continuum of agents with different types in each population, with nontransferable utility that is additively separable in types, and where agents incur a fixed search cost in each period in order to locate potential mates.

Under these assumptions, I show that if the utility from a partner is increasing in his or her type, there exists a unique matching equilibrium in which agents partition into subpopulations and marriages occur only among agents from different populations who belong to the same element in the partition. In other words, the equilibrium exhibits the cross-sectional property that Smith (1997) calls *perfect segregation*, or *classes* as in Burdett and Coles (1997).

I then proceed to characterize the measure of agents in equilibrium subpopulations in terms of the properties of the density of types and the marginal utility of partners. I derive a simple sufficient condition that allows ordering of the sizes of these subpopulations as a function of the types they contain; this provides an easy way to determine the duration of the search as a function of the characteristic of each agent.

To illustrate these results, I fully solve the model for the case where types are uniformly distributed and match utility is simply the type of the potential mate. I then contrast the results with the discounted case.

Since I model search costs and match payoffs in a different manner from the approaches used in the aforementioned papers, I provide an intuitive explanation that clarifies why additive separability and multiplicative separability are sufficient conditions for perfect segregation in the fixed search cost and discounted cases, respectively. Moreover, I present a set of sufficient conditions that generalizes the perfect segregation result to a broader class of search cost functions that subsumes discounting and fixed search costs as special cases.

With a simple change of terminology, matching problems with search frictions and heterogeneity of the sort described above commonly arise in labor and marriage markets, as well as in the formation of partnerships in general. Therefore, the analysis conducted in this paper will shed light on the role that search and heterogeneity play in these situations.

This paper is a contribution to a recent literature that deals with matching problems with heterogeneous agents and costly search. In independent work, McNamara and Collins (1990), Burdett and Coles (1997), Smith (1997), and Bloch and Ryder (1999) derive perfect segregation results when the only search friction is discounting and except for Smith (1997), each agent's utility is linear and depends solely on the type of the

partner.¹ Morgan (1995, 1996) also assumes fixed search costs but with strictly supermodular match payoffs; he shows that the existence of complementarities leads to positive sorting in equilibrium, and he also analyzes the welfare properties of the model as well as the formation of clubs. However, his assumptions rule out the perfect segregation case, and he does not examine alternative search cost structures.

Unlike the papers described above, this is the first paper that (i) presents a detailed analysis of the perfect segregation result under fixed search costs; (ii) investigates the dependence of the size of equilibrium subpopulations (and therefore the duration of the search) on properties of the distribution of types and utility functions; and (iii) analyzes the effects of different modeling assumptions regarding search frictions. In this sense, the present paper represents a theoretical contribution that covers an important case that has not been heretofore analyzed, and reconciles the results obtained in the literature under alternative assumptions. It is easy to think about economic examples from marriage markets or entry-level labor markets where fixed search costs are important and complementarities of types in the payoff function are not. Moreover, the simplicity and tractability of the analysis and conditions presented in this paper make my model an attractive benchmark to use in empirical applications to labor and marriage markets. In fact, a recent paper by Wong (2000) develops a method for estimating a two-sided search model that exhibits perfect segregation, and presents some results for the US; the results derived in this paper expand the range of applicability of Wong's method.²

The present work is also related to the class of two-sided matching models, thoroughly surveyed by Roth and Sotomayor (1990), that focuses on the analysis and design of centralized matching mechanisms. Here, I focus on the effects of search frictions when matching proceeds in a decentralized fashion and there is replenishment when agents mate and exit the market. I present an example that illustrates the relationship between the two approaches.

The paper is organized as follows. Section 2 describes the basic model. The main results are presented in Section 3. Section 4 concludes. All proofs are collected in Appendix A.

2. The model

In this section I describe the main characteristics of the two-sided search model that will be analyzed in the next section. To fix the terminology, I cast the model in terms of a marriage market where women and men are (costly) searching for spouses. It is important to keep in mind that the problem analyzed in this paper arises in other markets as well.

¹See Smith (1997) for a complete account of the different papers that independently hit the perfect segregation result; Smith places Chade (1995), on which this paper is based, among them.

²See also the discussion in Pierret (1996) for the empirical relevance of this benchmark case in entry-level labor markets.

Time. Time is divided into discrete periods of length equal to one.

Agents. There are two populations of agents, one with a continuum of males and one with a continuum of females. Each female is characterized by a type $y \in [0, 1]$; types are distributed according to an atomless distribution $G(\cdot)$ and density $g(\cdot)$, $g(y) > 0$, $\forall y \in [0, 1]$. Similarly, each male is characterized by a type $x \in [0, 1]$; the corresponding distribution and (positive) density functions are denoted by $F(\cdot)$ and $f(\cdot)$, respectively.³ For simplicity, the measure of agents in each population is normalized to one.

Meeting technology. In each period, agents from opposite populations randomly meet in pairs.

Marriage game. When two agents meet, they *perfectly* observe each other's types, and then simultaneously announce *Accept* or *Reject*.⁴ A couple gets married only if both announce *Accept*, in which case they leave the market. In any other case, they go back to the pool of unmatched agents and wait for the next period's meetings.

Replenishment. When a mated pair exits the market, it is replaced by entrants of the same types. In other words, the size and the composition of the market are exogenously given.

Match payoffs. In each period, a single agent pays a fixed search cost $c > 0$ in order to locate a potential spouse. If a woman with type y marries a man with type x after t periods, her match utility is $u_f(x, y) = \alpha_1(x) + \alpha_2(y)$ minus search costs incurred ct ; $\alpha_1(\cdot)$ is nonnegative and strictly increasing, while $\alpha_2(\cdot)$ is an arbitrary real-valued function. Similarly, his utility from the match is $u_m(y, x) = \gamma_1(y) + \gamma_2(x)$ minus the expenditure in search costs ct ; $\gamma_1(\cdot)$ is nonnegative and strictly increasing, and $\gamma_2(\cdot)$ is any real-valued function.

Strategies. A stationary strategy for a woman with type y is a measurable set $A_f(y) \subseteq [0, 1]$, which contains the types of men she is willing to accept. It will be assumed that as a function of y , $A_f(y)$ is a Lebesgue-measurable set-valued map. $A_m(x)$ is defined in a similar way. To determine $A_f(y)$, y takes as given the set $\Omega_f(y) = \{x: y \in A_m(x)\}$, which contains the types of men that are willing to accept her.⁵ The *matching set* of woman y is the set $A_f(y) \cap \Omega_f(y)$, which contains the set of male types that she accepts and is

³The particular interpretation of the type of an agent depends on the problem under consideration. In marriage markets, the type of a woman (man) could be her (his) wealth, education, I.Q., income, beauty, etc.; see Becker's (1973, 1974) seminal papers, and Laitner (1991). In labor markets, the type of a worker could be his productivity and the type of the firm its 'quality'. The model in this paper applies to these cases so long as sorting is based on such characteristics and monetary transfers are restricted (e.g. fixed sharing rules in marriage, specified wages in entry-level labor markets, etc.).

⁴Chade (1999) analyzes the case in which types are observed with noise.

⁵That $\Omega_f(y)$ is measurable follows directly from the measurability of the set-valued map $A_m(x)$. See Aubin and Frankowska (1990, Theorem 8.1.4 ii).

accepted by. In other words, she can only mate with men with types belonging to this set. $A_m(x) \cap \Omega_m(x)$ is defined analogously.

Expected value of the search. Consider a woman with type y who is accepted by men with types in $\Omega_f(y)$ and who uses a stationary strategy given by $A_f(y)$. When $A_f(y) \cap \Omega_f(y)$ has positive measure, the expected payoff of $A_f(y)$ denoted by $\Phi_f(y, \Omega_f(y), A_f(y))$ is defined recursively by

$$\begin{aligned} \Phi_f(y, \Omega_f(y), A_f(y)) = & -c + \int_{A_f(y) \cap \Omega_f(y)} (\alpha_1(x) + \alpha_2(y))f(x) \, dx \\ & + \Phi_f(y, \Omega_f(y), A_f(y)) \int_{(A_f(y) \cap \Omega_f(y))^c} f(x) \, dx; \end{aligned} \tag{1}$$

when $A_f(y) \cap \Omega_f(y)$ has measure zero the expected payoff is $-\infty$ due to the accumulation of search costs. Explanation of (1) is straightforward: the agent pays the search cost and meets a man who either belongs to her matching set (an event whose probability is $\int_{A_f(y) \cap \Omega_f(y)} f(x) \, dx$ and expected payoff given by the first integral divided by this probability) or alternatively to the complement of the matching set, in which case she continues the search. From (1) we obtain

$$\Phi_f(y, \Omega_f(y), A_f(y)) = \frac{-c + \int_{A_f(y) \cap \Omega_f(y)} (\alpha_1(x) + \alpha_2(y))f(x) \, dx}{\int_{A_f(y) \cap \Omega_f(y)} f(x) \, dx}. \tag{2}$$

The expected value of $A_m(x)$ for a man with type x who is accepted by women whose types belong to $\Omega_m(x)$ is denoted by $\Phi_m(x, \Omega_m(x), A_m(x))$. It is defined in a similar way.

Equilibrium. A *matching equilibrium* is a profile of *stationary* strategies $(A_f(y), A_m(x))$, such that $\forall y \in [0, 1]$ and for any alternative strategy (stationary or not) $\sigma_f(y)$

$$\Phi_f(y, \Omega_f(y), A_f(y)) \geq \Phi_f(y, \Omega_f(y), \sigma_f(y)) \tag{3}$$

$$\Omega_f(y) = \{x: y \in A_m(x)\}, \tag{4}$$

and $\forall x \in [0, 1]$ and for any alternative strategy (stationary or not) $\sigma_m(x)$

$$\Phi_m(x, \Omega_m(x), A_m(x)) \geq \Phi_m(x, \Omega_m(x), \sigma_m(x)) \tag{5}$$

$$\Omega_m(x) = \{y: x \in A_f(y)\}. \tag{6}$$

Intuitively, each agent uses an optimal strategy given her (his) conjecture about the strategies chosen by agents on the other side of the market, and these conjectures are correct in equilibrium.

It is worth pointing out that the replenishment assumption, despite having been commonly used in matching models with search frictions (see, among others, Rubinstein and Wolinsky, 1985; Bloch and Ryder, 1999; Morgan, 1995; Burdett and Wright, 1998), is not satisfactory and is made here only in order to avoid a difficult existence proof. A better alternative is to explicitly model a flow of ‘new entrants’ in each period, which in steady state is equal in measure and composition to the flow of agents that leaves the pool of searchers in each period (Morgan, 1996; Burdett and Coles, 1997). Alternatively, one could assume that matches are dissolved exogenously, and ‘divorced’ agents go back to the pool of singles (Smith, 1997). In the next section, I will point out how the results may change if the exogenous replenishment assumption is relaxed.

3. Main results

3.1. Reservation-type strategies

Consider a woman with type y who is accepted by men whose types belong to $\Omega_f(y) = \{x: y \in A_m(x)\}$, with $A_m(x)$ stationary; her problem is to find an acceptance rule that maximizes her expected utility.

Let $\Phi_f^*(y)$ be the expected value of her search under an optimal strategy. It follows from standard results in optimal stopping theory (Shiryayev, 1978, Theorem 23, p. 94) that there exists an optimal acceptance rule described as follows: ‘Accept if $\alpha_1(x) + \alpha_2(y) \geq \Phi_f^*(y)$; otherwise, *Reject* and continue the search’.

Let $\underline{x}(y)$ be the (unique) solution to $\alpha_1(\underline{x}) + \alpha_2(y) = \Phi_f^*(y)$. The monotonicity of $\alpha_1(x)$ and the acceptance rule yield:

Lemma 1. *In equilibrium, the set of males that a woman with type y accepts is $A_f(y) = [\underline{x}(y), 1]$.*

When agents use reservation strategies as in Lemma 1, the set Ω_i , $i = f, m$, is bigger for higher types; therefore, we expect them to be more selective in their acceptance decisions. The following proposition confirms this intuition.

Lemma 2. *In equilibrium, the reservation-type $\underline{x}(y)$ is increasing in y .*

Similarly, the optimal strategy for a man with type x is of the form $A_m(x) = [\underline{y}(x), 1]$, and $\underline{y}(x)$ is increasing in x .

Consider a woman with type y ; given the monotonicity of $\underline{y}(x)$, there exists a type $\bar{x}(y)$ such that y is accepted by men with types $x \leq \bar{x}(y)$ and is rejected otherwise. In other words, $\bar{x}(y)$ is the highest type a woman with type y could end up being married to. This threshold is defined as

$$\bar{x}(y) = \sup\{x \in [0, 1]: y \geq \underline{y}(x)\}. \quad (7)$$

Hence, we have that for all $y \in [0, 1]$, $\Omega_f(y) = [0, \bar{x}(y)]$. Analogously, $\Omega_m(x) = [0, \bar{y}(x)]$ for all $x \in [0, 1]$.

The expected value of the search for a woman with type y who follows an optimal strategy can then be written as:

$$\Phi_f^*(y) = \max_{0 \leq a \leq \bar{x}(y)} \frac{-c + \int_a^{\bar{x}(y)} (\alpha_1(x) + \alpha_2(y))f(x) dx}{F(\bar{x}(y)) - F(a)}. \tag{8}$$

To find the optimal acceptance set $A_f(y) = [\underline{x}(y), 1]$, we simply need to find the reservation-type $0 \leq \underline{x}(y) \leq \bar{x}(y)$ that solves the maximization problem in (8).

Lemma 3. *If $\bar{x}(y) > 0$, then for each y there is a unique $\underline{x}(y) \in [0, \bar{x}(y)]$ that solves the maximization problem in (8).*

Since an analogous result holds for males, the optimal strategies are completely characterized by the functions $\underline{x}(y)$ and $\bar{y}(x)$, which, when $\int_0^{\bar{x}(y)} \alpha_1(x)f(x) dx > c$ and $\int_0^{\bar{y}(x)} \gamma_1(y)g(y) dy > c$, are implicitly and uniquely defined by

$$\int_{\underline{x}(y)}^{\bar{x}(y)} (\alpha_1(x) - \alpha_1(\underline{x}(y)))f(x) dx = c \tag{9}$$

and

$$\int_{\bar{y}(x)}^{\bar{y}(x)} (\gamma_1(y) - \gamma_1(\bar{y}(x)))g(y) dy = c, \tag{10}$$

and are equal to zero otherwise.

3.2. The matching equilibrium

Notice that there is always a trivial equilibrium where all agents choose to reject every match. As in Bloch and Ryder (1999) and Morgan (1996), this uninteresting case is ruled out by the assumption that agents announce *Accept* in cases of indifference.

The task now is to show that there exists an equilibrium that determines $\bar{x}(\cdot)$ and $\bar{y}(\cdot)$ from the strategies used by agents on each side of the market. The following theorem shows that a matching equilibrium exists, it is unique, and it has the cross-sectional property called *perfect segregation*.⁶

⁶The term perfect segregation is borrowed from Smith (1997), and it refers to the fact that in equilibrium, there is no ‘mixing’ among agents belonging to different clusters. In this sense, each cluster containing men and women is ‘perfectly segregated’ from the rest of the population.

Theorem 1. *There exists a unique matching equilibrium characterized by the formation of a finite number of disjoint subpopulations of men and women that partition the two populations. Marriages take place only between men and women belonging to corresponding subpopulations (e.g. men belonging to the first subpopulation of males mate with women from the first subpopulation of females, and so on).*

In other words, agents will mate with types ‘sufficiently similar’ to theirs, and *positive assortative mating* emerges endogenously in this decentralized environment. Notice that although mating occurs only among agents belonging to the same ‘class’ (corresponding subintervals), within each cluster there is ‘mixing’ between agents of different types. This is entirely due to the existence of search frictions; it is not difficult to show that at least in the symmetric case, the equilibrium converges to the identity matching as c tends to zero, and mating takes place only among agents with the same type. This is the unique *stable* matching of the model without search frictions; i.e. it has the property that one cannot find a pair of agents that are not married to each other and would prefer to be.

These results complement those obtained in the discounted case with no explicit search costs by McNamara and Collins (1990), Burdett and Coles (1997), and Smith (1997), and extend the results derived by Morgan (1996) with fixed search costs and strictly supermodular payoffs.

So far, the only result that depends on the replenishment assumption is the uniqueness of the matching equilibrium; as Burdett and Coles (1997) illustrate using an example with two types, with endogenous replenishment there could be multiple equilibria. However, all of these equilibria will exhibit the perfect segregation property. The fact that the lowest types in the population with the finer partition remain unmatched forever can also be a feature with explicit entry flows, so long as each agent dies with a positive probability in each period.

3.3. An example

Suppose that types are uniformly distributed on $[0, 1]$ and that $u_f(y, x) = x$ for a woman with type y and $u_m(x, y) = y$ for a man with type x . The clusters can be explicitly calculated in this case; the expected value of the search for a woman with type y is

$$\Phi_f(y, \bar{x}(y), \underline{x}(y)) = \frac{-c + \int_{\underline{x}(y)}^{\bar{x}(y)} x \, dx}{\bar{x}(y) - \underline{x}(y)}. \quad (11)$$

To avoid the trivial case where each agent accepts all matches, suppose that $\int_0^1 x \, dx = 1/2 > c$ so that $\underline{x}(y)$ is positive at least for the highest y . The first order conditions reveal that $\underline{x}(y)$ solves the following quadratic equation:

$$\underline{x}(y)^2 - 2\bar{x}(y)\underline{x}(y) + 2\left(\frac{\bar{x}(y)}{2} - c\right) = 0. \quad (12)$$

The only feasible solution for $\underline{x}(y)$ is

$$\underline{x}(y) = \bar{x}(y) - \sqrt{2c}, \tag{13}$$

and the reservation type is equal to zero if the nonnegativity constraint binds. Similarly, $\underline{y}(x)$ is given by

$$\underline{y}(x) = \bar{y}(x) - \sqrt{2c}, \tag{14}$$

and $\underline{y}(x)$ equals zero if the nonnegativity constraint binds. Now we can construct the matching equilibrium using Eqs. (13) and (14). Consider women with type $y = 1$; they are accepted by every man, and hence $\bar{x}(1) = 1$ and $\underline{x}(1) = 1 - \sqrt{2c}$. Men with types in $[1 - \sqrt{2c}, 1]$ are accepted by all women, and their common threshold is $\bar{y}(1) = 1 - \sqrt{2c}$. Therefore, we have constructed the first cluster containing the subintervals of men and women that are mutually acceptable; i.e. agents whose types belong to $[1 - \sqrt{2c}, 1]$. Iterating this argument on the remaining agents leads to the construction of the other clusters. The two populations are partitioned into the same number of subintervals, whose endpoints are given by the following sequence of bounds: $a_0 = 1$, $a_{n+1} = a_n - \sqrt{2c}$ if $a_n > \sqrt{2c}$ and $a_{n+1} = 0$ otherwise. Solving this recursion for $n \geq 1$ gives

$$a_{n+1} = 1 - n\sqrt{2c} \tag{15}$$

if positive, and zero otherwise.

The matching equilibrium has the following properties. Firstly, the two populations partition into subpopulations. Secondly, in each subpopulation agents set the same reservation-type a_{n+1} . Thirdly, there is only a finite number of clusters that form in equilibrium. This number is given by $N = \min\{n: 1 - n\sqrt{2c} \leq 0\}$. Fourthly, all subpopulations are of the same ‘size’ except the last one, which is smaller unless $c = 1/2n^2$. To see this, notice that the measure of people in each subpopulation (except the N th one) is given by $a_n - a_{n+1} = \sqrt{2c}$. Finally, notice the following important property: as c tends to zero, the length of each equilibrium subinterval tends to zero, and the equilibrium converges to the identity matching in which marriages take place only between agents of the same type.

This example has also been analyzed by Bloch and Ryder (1999) and Burdett and Coles (1997) in the discounted case with no fixed search costs, obtaining clusters that, as a function of types, are increasing in size. This suggests that the way in which search frictions are modeled affects the (potentially observable) predictions of the model; i.e. the duration of the search of agents with different attributes. In the next two sections these issues will be explored. After Theorem 2 is presented, the uniform example with fixed search costs will be revisited and the differences with its discounted counterpart explained.

3.4. The size of the clusters

I will now investigate the following question: *Do equilibrium subintervals with men and women of higher types contain larger or smaller measures of agents than the ones with agents of lower types?* The question is important not only from a theoretical

perspective, but also for potential empirical applications of the model to labor and marriage markets, where the duration of the search is of primary importance (Mortensen, 1986, pp. 861–869). To see this, consider a woman with type y who will eventually mate with a man with type $x \in [a, b]$. The probability that she meets someone in this interval in any given period is equal to the measure of men in the interval, $F(b) - F(a)$. Therefore, the random variable $T_y =$ ‘number of periods that y spends searching’ is geometrically distributed with probability of success equal to $F(b) - F(a)$. The expected value of this random variable is then

$$E[T_y] = \frac{1}{F(b) - F(a)}, \tag{16}$$

which is inversely proportional to the measure of men belonging to the matching set that y faces in equilibrium. In other words, the ‘average life’ of an agent in the market depends on the measure of people contained in the subinterval that constitutes his or her matching set in equilibrium. The question posed above is equivalent to the following: *Is the duration of the search increasing or decreasing in the agent’s type? Do higher types mate faster than lower types?*

Not surprisingly, the answer depends on the characteristics of the distribution of types and on the utility functions. For simplicity, it will be assumed that y and x are identically distributed F , and also that $\alpha_1(\cdot) = \gamma_1(\cdot) = h(\cdot)$, $h(\cdot)$ differentiable; then, in equilibrium, both populations partition in exactly the same way.

Consider an arbitrary cluster where women with types $y \in [a, b]$, $b \leq 1$ and $a \geq 0$ only accept men with types $x \in [a, b]$ and vice versa. Let $J(a, b) = \int_a^b (h(\xi) - h(a))f(\xi)d\xi$, $\xi = x, y$; since all women in the cluster use the same threshold $x = a$, it must be true that $J(a, b) - c = 0 \forall y \in [a, b]$. The same holds for men with types $x \in [a, b]$.

Lemma 4. *The size of the clusters is an increasing function of the agents’ types if for any $0 \leq a < b \leq 1$,*

$$\frac{F'(a)}{\int_a^b F'(\xi) d\xi} < \frac{h'(a)}{\int_a^b h'(\xi) d\xi}. \tag{17}$$

It is a decreasing function of the agents’ types if inequality (17) is reversed, and constant if (17) holds with equality. The only exception is the last cluster, which can have an arbitrarily small measure.

Intuitively, the role of condition (17) can be explained as follows. Given b , the optimal (common) threshold a set by agents accepted by types in $[0, b]$ is the unique solution to $J(a, b) = c$, which implicitly defines the (differentiable) function $a(b)$. Now, if b increases then a also increases and the change in $F(b) - F(a)$ is given by

$$\frac{d(F(b) - F(a))}{db} = F'(a) \left(\frac{F'(b)}{F'(a)} - a'(b) \right). \tag{18}$$

The term $F'(b)/F'(a)$ is the change in a that would be required in order to keep $F(b) - F(a)$ constant when b changes. Notice that $F(b) - F(a)$ is increasing in b if $a'(b)$ is smaller than $F'(b)/F'(a)$; this in turn will be true if condition (17) holds. Thus, it follows that given two equilibrium clusters, the ‘higher’ of the two will contain a larger measure of agents. The other two cases are explained in an analogous way.

Although the conditions stated in Lemma 4 may be difficult to check for a given density and given utility functions, the following sufficient condition is straightforward to compute:

Theorem 2. *If $f(\xi)/h'(\xi)$ is strictly increasing, strictly decreasing, or constant, then (17) will hold with $<$, $>$, or $=$, respectively.*

Notice that, in the special case where $h(\cdot)$ is a linear function, the sufficient conditions stated in Theorem 2 are simply that the cumulative distribution of types be strictly convex, concave, or linear, respectively. In particular, if the density is strictly increasing, then the increase in the optimal threshold when the best type in a cluster increases is smaller than the change needed to keep the measure of agents in the cluster constant. This makes the subpopulations containing better types have larger measure.

To illustrate these results, suppose that preferences are additive separable in types with $h(\xi) = d - e^{-\rho\xi}$, $d > 1$, $\rho > 0$, and $f(\xi) = \lambda e^{-\lambda\xi}$ for $\xi \geq 0$. The fact that the support of the density is unbounded above is not a problem, since all the results derived above carry over to this case, too. Theorems 1 and 2 predict that there is perfect segregation in equilibrium, and since $f(\xi)/h'(\xi) = (\lambda/\rho) e^{\xi(\rho-\lambda)}$, the duration of the search of an agent as a function of his or her type depends on the relation between ρ and λ .

As another example, consider the uniform case analyzed in the previous section. Obviously $f(\xi)/h'(\xi) = 1$, and by Theorem 2 the clusters have constant size (except the one with the lowest types). In the discounted case, Bloch and Ryder (1999) and Burdett and Coles (1997) showed that in the uniform case higher clusters contain a larger measure of agents. The difference is explained as follows: in the discounted case with multiplicatively separable utility functions, it is easy to show that the analogue to condition (17) for increasing clusters’ sizes is:

$$\frac{F'(a)}{\frac{(1-\beta)}{\beta} + \int_a^b F'(\xi) d\xi} < \frac{h'(a)}{\int_a^b h'(\xi) d\xi}, \tag{19}$$

where β is the (common) discount factor of the agents. Notice that whenever (17) is satisfied, so is (19). In particular, condition (19) is satisfied in the discounted case when $h(\cdot)$ is linear and $F(\cdot)$ is uniform, while (17) holds with equality under fixed search costs. Thus, in the former case higher clusters contain more agents, while in the latter the

size of the clusters is constant. This explains why the duration of the search as a function of types will be different in the two cases.⁷

3.5. The role of search costs

The sufficiency of additive separability as a condition for perfect segregation in equilibrium in the fixed search cost case can be intuitively explained as follows. The crucial property of perfect segregation is that all women belonging to the same subpopulation behave identically, as do all men in analogous circumstances. In other words, agents facing the same Ω_i choose identical A_i in equilibrium, $i = f, m$. This will happen if the reservation types of these agents depend on their types through Ω_i and *not* through the match payoff function. Consider a female with type y who is accepted by any man with a type in $[0, b)$ and suppose that her utility function is $u_f(x, y)$. Her problem boils down to finding the threshold \underline{x} that solves

$$\int_{\underline{x}}^b (u_f(x, y) - u_f(\underline{x}, y))f(x) dx = c, \quad (20)$$

which depends, in particular, on y and b . Under perfect segregation, within an equilibrium subpopulation women with different types but who face the same b choose the same \underline{x} . That is, given b , \underline{x} is independent of y . A sufficient condition for this to hold is the following: given any two types $y_1 > y_2$ facing the same b , and given any $\underline{x} \in (0, b)$, then

$$\int_{\underline{x}}^b (u_f(x, y_1) - u_f(\underline{x}, y_1) - u_f(x, y_2) + u_f(\underline{x}, y_2))f(x) dx = 0. \quad (21)$$

This will be satisfied if for every $y_1 > y_2$ facing the same b and $x > \underline{x}$,

$$u_f(x, y_1) + u_f(\underline{x}, y_2) = u_f(\underline{x}, y_1) + u_f(x, y_2), \quad (22)$$

which is equivalent to (Ross, 1983, pp. 6–7)

$$\frac{\partial^2 u_f(x, y)}{\partial x \partial y} = 0. \quad (23)$$

The solution to this simple partial differential equation is $u_f(x, y) = \alpha_1(x) + \alpha_2(y)$ for arbitrary functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$. Similarly, $u_m(x, y) = \gamma_1(y) + \gamma_2(x)$. When this *amodularity* property is supplemented with the monotonicity of $\alpha_1(\cdot)$ and $\gamma_1(\cdot)$, the equilibrium characterization derived in Theorem 1 follows.

⁷The results and techniques used in this section depend on the replenishment assumption. If entry flows are explicitly modeled, one should find conditions on the payoff function and the density of entrants that yield the properties of the steady state distribution implicit in (17) and (19).

The same analysis applies to the discounted case analyzed by Smith (1997). Let β be the common discount factor of men and women. The analogue of Eq. (20) is⁸

$$\int_{\underline{x}}^b \left(\frac{u_f(x, y)}{u_f(\underline{x}, y)} - 1 \right) f(x) dx = \frac{(1 - \beta)}{\beta}. \tag{24}$$

Following the same steps as before, a sufficient condition for perfect segregation is that if $y_1 > y_2$ face the same b and $\underline{x} \in (0, b)$, then

$$\int_{\underline{x}}^b \left(\frac{u_f(x, y_1)}{u_f(\underline{x}, y_1)} - \frac{u_f(x, y_2)}{u_f(\underline{x}, y_2)} \right) f(x) dx = 0. \tag{25}$$

In particular, this holds if

$$u_f(x, y_1)u_f(\underline{x}, y_2) = u_f(\underline{x}, y_1)u_f(x, y_2), \tag{26}$$

which is equivalent to

$$\frac{\partial^2 \ln u_f(x, y)}{\partial x \partial y} = 0. \tag{27}$$

Thus $u_f(x, y) = \alpha_1(x)\alpha_2(y)$ and similarly, $u_m(x, y) = \gamma_1(y)\gamma_2(x)$; if these functions are assumed positive with $\alpha_1(\cdot)$ and $\gamma_1(\cdot)$ strictly increasing, then we obtain the perfect segregation result derived by Smith (1997).

A simple but illuminating way to rederive and extend these results is by looking directly at the optimal stopping problems solved by men and women in equilibrium. Under fixed search costs, a woman with type y solves

$$\Phi_f^*(y) = \max_{\tau_y} E[u_f(x_{\tau_y}, y) - \tau_y c | \Omega_f(y)], \tag{28}$$

where the maximization is over the class of stopping times for which the expectation is well-defined. If $u_f(x, y)$ is additively separable, then

$$\Phi_f^*(y) = \alpha_2(y) + \max_{\tau_y} E[\alpha_1(x_{\tau_y}) - \tau_y c | \Omega_f(y)], \tag{29}$$

and the optimal stopping time τ_y^* depends on y *only* through $\Omega_f(y)$. A similar result holds for τ_x^* ; since the optimal stopping times are fully characterized by $\underline{x}(y)$ and $\underline{y}(x)$, perfect segregation follows.

If instead of fixed search costs we assume discounting and multiplicative separability, then

⁸This is simply the (manipulated) first order condition of the following problem:

$$\max_{0 \leq \underline{x} \leq b} \frac{\beta \int_{\underline{x}}^b u_f(x, y) f(x) dx}{(1 - \beta) + \beta(F(b) - F(\underline{x}))}.$$

$$\begin{aligned} \Phi_f^*(y) &= \max_{\tau_y} E[\beta^{\tau_y} u_f(x_{\tau_y}, y) | \Omega_f(y)] \\ &= \alpha_2(y) \max_{\tau_y} E[\beta^{\tau_y} \alpha_1(x_{\tau_y}) | \Omega_f(y)]. \end{aligned} \tag{30}$$

It is evident that the optimal stopping rule depends on y only through $\Omega_f(y)$, and similarly for men.

Although discounting and fixed search costs are the most common ways of modeling the ‘cost of time’ in matching models with frictions, one could consider a more general formulation with type-dependent cost functions and look for sufficient conditions for perfect segregation. Suppose that a woman with type y solves

$$\Phi_f^*(y) = \max_{\tau_y} E \left[\beta^{\tau_y} u_f(x_{\tau_y}, y) - \sum_{i=0}^{\tau_y} \beta^i c_f(x_i, y) | \Omega_f(y) \right]. \tag{31}$$

That is, payoffs are discounted and the cost per meeting depends not only on her type but also on the type of the man she meets (which is revealed in the meeting). Similarly, suppose that a man with type x solves an analogous problem but with $u_m(x, y)$, $c_m(x, y)$, and $\Omega_m(x)$.

Theorem 3. *Let $\alpha_1(\cdot)$ and $\gamma_1(\cdot)$ be strictly increasing functions, $\int_0^1 c_f(x, y) f(x) dx < \infty$ for all y , and $\int_0^1 c_m(x, y) g(y) dy < \infty$ for all x . Suppose that either*

- (i) $\beta \in (0, 1)$, $u_f(x, y) = \alpha_1(x)\alpha_2(y) > 0$, $u_m(x, y) = \gamma_1(y)\gamma_2(x) > 0$, $c_f(x, y) = c_f(x)\alpha_2(y) \geq 0$, and $c_m(x, y) = c_m(y)\gamma_2(x) \geq 0$; or
- (ii) $\beta = 1$, $u_f(x, y) = \alpha_1(x) + \alpha_2(y)$, $u_m(x, y) = \gamma_1(x) + \gamma_2(y)$, $c_f(x, y) = a_f c_f(x) > 0$, and $c_m(x, y) = a_m c_m(y) > 0$; or
- (iii) $\beta = 1$, $u_f(x, y) = \alpha_1(x)\alpha_2(y) > 0$, $u_m(x, y) = \gamma_1(y)\gamma_2(x) > 0$, $c_f(x, y) = a_f \alpha_2(y) > 0$, and $c_m(x, y) = a_m \gamma_2(x) > 0$.

Then the matching equilibrium exhibits perfect segregation.

The first set of sufficient conditions states that, under discounting and type-dependent search costs, perfect segregation emerges in equilibrium if both the match payoff function and the cost function are multiplicatively separable with a common term; this ensures that two agents that are accepted by the same set of types behave identically. A similar interpretation applies to the undiscounted cases contained in Theorem 3 (ii) and (iii).

Notice that Theorem 3 (i) subsumes the perfect segregation results of Smith (1997), Bloch and Ryder (1999), McNamara and Collins (1990), and Burdett and Coles (1997) as special cases. Similarly, the analysis that led to Theorem 1 is contained in (ii). Finally, (iii) shows that multiplicative separability is a sufficient condition for perfect segregation in the undiscounted case when search costs are type-dependent.

The general message of this section is the following: in order to derive ‘distribution free’ conditions that lead to equilibrium cross-sectional properties like perfect segrega-

tion (or more general forms of positive assortative mating), one has to pay careful attention to the way search costs are modeled. It is not sufficient to focus solely on the properties of the match payoff functions.

4. Concluding remarks

In this paper, I present a market containing two large populations of heterogeneous agents who costly and randomly search for mates belonging to the other population; mating is voluntary and requires the agreement of both parties. I prove that there is a unique matching equilibrium where each population partitions into intervals; mating occurs only among men and women belonging to corresponding intervals who segregate themselves from other agents. In other words, if we take a cross-section ‘slice’ of the market at any particular date, we will see that match formation occurs only within classes. Since the duration of the search is of primary importance in applications, I also provide sufficient conditions on the primitives of the model that yield a monotonic relationship between types and duration. Finally, I discuss why different assumptions on the match payoff function are needed for perfect segregation to arise depending on the way in which search frictions are modeled.

Besides being of theoretical interest, these results could prove useful in applications to labor markets, as well as to models that incorporate assortative mating in marriage markets when search frictions are present.

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Appendix

A.1. Proof of Lemma 1

Suppose that $x \in A_f(y)$; this means that $\alpha_1(x) + \alpha_2(y) \geq \Phi_f^*(y)$. Since $\alpha_1(\cdot)$ is strictly increasing, $x' > x$ implies that $\alpha_1(x') + \alpha_2(y) > \Phi_f^*(y)$, and x' also belongs to $A_f(y)$. Thus, $A_f(y) = [\underline{x}(y), 1]$, where $\underline{x}(y)$ solves $\alpha_1(\underline{x}(y)) + \alpha_2(y) = \Phi_f^*(y)$ if $\Phi_f^*(y) \geq 0$, and is equal to zero otherwise. \square

A.2. Proof of Lemma 2

I need to show that if $y' > y$, then $\underline{x}(y') \geq \underline{x}(y)$. The result is obvious if $\underline{x}(y) = 0$. If $\underline{x}(y) > 0$ and $\underline{x}(y') > 0$, then they are defined by $\alpha_1(\underline{x}(y)) + \alpha_2(y) = \Phi_f^*(y)$ and $\alpha_1(\underline{x}(y')) + \alpha_2(y') = \Phi_f^*(y')$, respectively. Since $\alpha_1(\cdot)$ is strictly increasing, it is enough to show that

$$\Phi_f^*(y') - \alpha_2(y') \geq \Phi_f^*(y) - \alpha_2(y). \tag{A.1}$$

Any x who accepts y will also accept y' ; i.e. $\Omega_f(y) \subseteq \Omega_f(y')$. The expression $\Phi_f^*(y) - \alpha_2(y)$ is given by

$$\begin{aligned} \Phi_f^*(y) - \alpha_2(y) &= -c + \int_{[\underline{x}(y), 1] \cap \Omega_f(y)} \alpha_1(x)f(x) \, dx \\ &\quad + (\Phi_f^*(y) - \alpha_2(y)) \int_{([\underline{x}(y), 1] \cap \Omega_f(y))^c} f(x) \, dx. \end{aligned} \tag{A.2}$$

Since $\Omega_f(y) \subseteq \Omega_f(y')$, $([\underline{x}(y), 1] \cap \Omega_f(y)) \subseteq ([\underline{x}(y), 1] \cap \Omega_f(y'))$ and $([\underline{x}(y), 1] \cap \Omega_f(y))^c \subseteq ([\underline{x}(y), 1] \cap \Omega_f(y'))^c$. But $\alpha_1(x) \geq \Phi_f^*(y) - \alpha_2(y)$ on $[\underline{x}(y), 1]$; hence

$$\begin{aligned} \Phi_f^*(y) - \alpha_2(y) &\leq -c + \int_{[\underline{x}(y), 1] \cap \Omega_f(y')} \alpha_1(x)f(x) \, dx \\ &\quad + (\Phi_f^*(y) - \alpha_2(y)) \int_{([\underline{x}(y), 1] \cap \Omega_f(y'))^c} f(x) \, dx. \end{aligned} \tag{A.3}$$

Rearranging terms gives

$$\Phi_f^*(y) - \alpha_2(y) \leq \frac{-c + \int_{[\underline{x}(y), 1] \cap \Omega_f(y')} \alpha_1(x)f(x) \, dx}{\int_{([\underline{x}(y), 1] \cap \Omega_f(y'))^c} f(x) \, dx}. \tag{A.4}$$

However, $[\underline{x}(y'), 1]$ is chosen by woman y' in an optimal way. Therefore

$$\begin{aligned} \Phi_f^*(y') - \alpha_2(y') &\geq -c + \int_{[\underline{x}(y'), 1] \cap \Omega_f(y')} \alpha_1(x)f(x) \, dx \\ &\quad + (\Phi_f^*(y') - \alpha_2(y')) \int_{([\underline{x}(y'), 1] \cap \Omega_f(y'))^c} f(x) \, dx, \end{aligned} \tag{A.5}$$

which can be rewritten as

$$\Phi_f^*(y') - \alpha_2(y') \geq \frac{-c + \int_{[\underline{x}(y), 1] \cap \Omega_f(y')} \alpha_1(x)f(x) \, dx}{\int_{[\underline{x}(y), 1] \cap \Omega_f(y')} f(x) \, dx}. \tag{A.6}$$

Hence,

$$\Phi_f^*(y') - \alpha_2(y') \geq \Phi_f^*(y) - \alpha_2(y). \tag{A.7}$$

Finally, if $\underline{x}(y') = 0$, then $\alpha_1(0) \geq \Phi_f^*(y') - \alpha_2(y')$ and $\underline{x}(y) = 0$ by (A.7). This completes the proof of the proposition. \square

A.3. Proof of Lemma 3

The Kuhn–Tucker conditions are:

$$\int_{\underline{x}(y)}^{\bar{x}(y)} (\alpha_1(x) - \alpha_1(\underline{x}(y)))f(x) \, dx - c \leq \lambda \frac{(F(\bar{x}(y)) - F(\underline{x}(y)))^2}{f(\underline{x}(y))} \tag{A.8}$$

$$\underline{x}(y) \geq 0, \tag{A.9}$$

with complementary slackness (CS1), and

$$\bar{x}(y) - \underline{x}(y) \geq 0 \tag{A.10}$$

$$\lambda \geq 0, \tag{A.11}$$

with complementary slackness. It is easy to show that $\lambda = 0$; if it were positive, then $\bar{x}(y) = \underline{x}(y) > 0$ and CS1 would read $-\bar{x}(y)c$, which is different from zero, a contradiction. Eq. (A.8) can then be rewritten as

$$\int_{\underline{x}(y)}^{\bar{x}(y)} (\alpha_1(x) - \alpha_1(\underline{x}(y)))f(x) \, dx \leq c. \tag{A.12}$$

Denote the left-hand side of (A.12) by $J(\underline{x}(y))$. Notice that $J(\bar{x}(y)) = 0$, $J(0) = \int_0^{\bar{x}(y)} \alpha_1(x)f(x) \, dx > 0$, and $J'(\underline{x}(y)) = -\alpha_1'(\underline{x}(y))(F(\bar{x}(y)) - F(\underline{x}(y))) < 0$. Therefore, if $\int_0^{\bar{x}(y)} \alpha_1(x)f(x) \, dx > c$ there is a unique $\underline{x}(y)$ such that $J(\underline{x}(y)) = c$; otherwise, the reservation type is equal to zero. In either case we have obtained a unique critical point. The constraint qualification is trivially met in this problem. Moreover, if $\int_0^{\bar{x}(y)} \alpha_1(x)f(x) \, dx \leq c$, then the objective function is always decreasing and $\underline{x}(y) = 0$ is a global maximum; if $\int_0^{\bar{x}(y)} \alpha_1(x)f(x) \, dx > c$, then the objective function is strictly increasing in a if $a < \underline{x}(y)$ and strictly decreasing if $a > \underline{x}(y)$, where $\underline{x}(y)$ is the unique solution to $J(\underline{x}(y)) = c$. Thus, the unique critical point is a global maximum, and the proof is complete. \square

A.4. Proof of Theorem 1

Consider the problem faced by women with the highest type $y = 1$; the existence of search frictions ensures that these women are accepted by any man, i.e. $\bar{x}(1) = 1$. The reservation type of any of these females is determined by

$$\int_{\underline{x}(1)}^1 (\alpha_1(x) - \alpha_1(\underline{x}(1)))f(x) dx = c, \tag{A.13}$$

if the solution is interior, and zero otherwise. Since $\underline{x}(y)$ is increasing, men whose types belong to $[\underline{x}(1), 1]$ will be accepted by all women. Therefore, $\bar{y} = 1$ for all of them, and since they face the same unconstrained problem as men with type $x = 1$, they use the same threshold as such men do. The common reservation type for these men is determined by

$$\int_{\underline{y}(1)}^1 (\gamma_1(y) - \gamma_1(\underline{y}(1)))g(y) dy = c, \tag{A.14}$$

if the solution is interior and zero otherwise. All females with types in $[\underline{y}(1), 1]$ are acceptable mates for all males. Hence, we have constructed the first cluster (subpopulations of men and women that are mutually acceptable): women with types in $[\underline{y}(1), 1]$ mate only with men with types in $[\underline{x}(1), 1]$ and vice versa.

If either $\underline{x}(1)$ or $\underline{y}(1)$ is zero, then the partition of each population is completed. Otherwise, consider the remaining agents whose types belong to $[0, \underline{y}(1))$ and $[0, \underline{x}(1))$, respectively. Take any woman with type $\underline{y}(1) - \epsilon$; for $\epsilon > 0$ small enough, this woman will be accepted by any man in $[0, \underline{x}(1))$. The reservation type for such a woman is determined by

$$\int_{\underline{x}(\underline{y}(1) - \epsilon)}^{\underline{x}(1)} (\alpha_1(x) - \alpha_1(\underline{x}(\underline{y}(1) - \epsilon)))f(x) dx = c, \tag{A.15}$$

if the solution is interior, and zero otherwise. Since $\underline{x}(y)$ is increasing, men whose types belong to $[\underline{x}(\underline{y}(1) - \epsilon), \underline{x}(1))$ will be accepted by all women with types in $[0, \underline{y}(1))$. Therefore, the common reservation type for these men is determined by

$$\int_{\underline{y}(\underline{x}(1) - \epsilon)}^{\underline{y}(1)} (\gamma_1(y) - \gamma_1(\underline{y}(\underline{x}(1) - \epsilon)))g(y) dy = c, \tag{A.16}$$

if the solution is interior and zero otherwise. Since ϵ was arbitrary, females with types in $[\underline{y}(\underline{x}(1)), \underline{y}(1))$ will mate only with men with types in $[\underline{x}(\underline{y}(1)), \underline{x}(1))$, and vice versa. This is the second equilibrium cluster. If either $\lim_{\epsilon \rightarrow 0} \underline{x}(\underline{y}(1) - \epsilon)$ or $\lim_{\epsilon \rightarrow 0} \underline{y}(\underline{x}(1) - \epsilon)$ is zero, then the partition of each population is completed. Otherwise, continue with the construction of the next cluster.

Iterating this reasoning repeatedly leads to the formation of all remaining clusters. The process stops when the nonnegativity constraints on the thresholds bind; a simple argument will show that this occurs after a finite number of iterations. Let $J(a, b) = \int_a^b (\alpha_1(x) - \alpha_1(a))f(x) dx$, and define $G(a, b) = \alpha(1)(b - a)f(\hat{x})$, where $f(\hat{x}) = \max_{x \in [0, 1]} f(x)$; notice that $J(a, b) \leq G(a, b)$, with equality only at $a = b$. Moreover, there is a unique solution $\tilde{a} = b - (c/\alpha(1)f(\hat{x}))$ that solves $G(\tilde{a}, b) = c$, with $b - \tilde{a} = c/(\alpha_1(1)f(\hat{x})) > 0$. Since $J(a, b) < G(a, b)$ for all $a \in [0, b)$, it follows that $b - a \geq c/(\alpha_1(1)f(\hat{x})) > 0$. This provides a lower bound for the length that a subpopulation of men can have in a matching equilibrium. The corresponding lower bound for women is $c/\gamma_1(1)g(\hat{y})$, where $g(\hat{y}) = \max_{y \in [0, 1]} g(y)$. Let $\lambda = (c/\alpha_1(1)f(\hat{x})) \wedge (c/\gamma_1(1)g(\hat{y}))$; then, the integer part of $1/\lambda$ is an upper bound for the number of subpopulations that form in equilibrium.

Notice that each agent is using a strategy that is optimal given the strategy followed by the other agents. Therefore, a matching equilibrium has been constructed. Uniqueness follows easily from the construction process described and the fact that agents accept when indifferent. \square

A.5. Proof of Lemma 4

We will show that, if (17) holds, then given any two clusters $[a_{n+k}, a_{n+k-1})$ and $[a_{n+1}, a_n)$, $0 \leq a_{n+k} < a_{n+k-1} \leq a_{n+1} < a_n \leq 1$, $k \geq 2$,

$$F(a_n) - F(a_{n+1}) > F(a_{n+k-1}) - F(a_{n+k}). \tag{A.17}$$

Since the lower endpoint of an equilibrium subpopulation is determined by the reservation type set by the highest type in the cluster, it is sufficient to show that given any interval $[a, b]$, with $0 \leq a < b \leq 1$ and a determined by the unique solution to

$$\int_a^b (h(\xi) - h(a))f(\xi) d\xi = c, \tag{A.18}$$

the quantity $F(b) - F(a)$ is increasing in b if (17) holds.

Eq. (A.18) determines implicitly a function $a(b)$, whose derivative is:

$$a'(b) = \frac{F'(b)(h(b) - h(a))}{h'(a)(F(b) - F(a))}. \tag{A.19}$$

Therefore,

$$\begin{aligned} \frac{d(F(b) - F(a))}{db} &= F'(b) - F'(a)a'(b) \\ &= F'(b) \left(1 - \frac{F'(a)(h(b) - h(a))}{h'(a)(F(b) - F(a))} \right). \end{aligned} \tag{A.20}$$

If (17) holds, then the expression between brackets is positive, and $F(b) - F(a)$ is increasing in b . The opposite happens if the inequality in (17) is reversed, resulting in clusters that are decreasing in size for higher types. The only exception is the cluster

containing the lowest types, which can be arbitrarily small. Finally, if (17) holds with equality, then all clusters (except the one containing the lowest types) will have the same measure. \square

A.6. Proof of Theorem 2

Only the first part will be proved. By hypothesis, if $\xi_1 < \xi_2$, then

$$\frac{f(\xi_2)}{h'(\xi_2)} > \frac{f(\xi_1)}{h'(\xi_1)}. \tag{A.21}$$

Rearranging this expression and integrating over $[\xi_1, b]$ gives

$$\int_{\xi_1}^b f(\xi_2) d\xi_2 > \int_{\xi_1}^b f(\xi_1) \frac{h'(\xi_2)}{h'(\xi_1)} d\xi_2, \tag{A.22}$$

which is equivalent to

$$\frac{f(\xi_1)}{\int_{\xi_1}^b f(\xi_2) d\xi_2} < \frac{h'(\xi_1)}{\int_{\xi_1}^b h'(\xi_2) d\xi_2}. \tag{A.23}$$

This completes the proof. \square

A.7. Proof of Theorem 3

Only (i) will be proved since the proofs of (ii) and (iii) are analogous. Consider a female with type y ; following the same steps that led to (25) yields the sufficient condition

$$\int_{\underline{x}}^b \left(\frac{u_f(x, y_1)}{u_f(x, y_1)} - \frac{u_f(x, y_2)}{u_f(x, y_2)} \right) f(x) dx = \int_0^1 \left(\frac{c_f(x, y_1)}{u_f(x, y_1)} - \frac{c_f(x, y_2)}{u_f(x, y_2)} \right) f(x) dx. \tag{A.24}$$

It is immediate to check that the functional forms stated in (i) satisfy this equation for any pair (y_1, y_2) that face the same b , and any $\underline{x} \in (0, b)$. A similar analysis holds for men. The construction of the clusters follows in the same way as in Theorem 1 (or as in Proposition 2 in Smith, 1997). \square

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