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Stable coalitions in a continuous-time model of risk sharing

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Abstract

In an economy with a continuum of individuals, each individual has a stochastic, continuously evolving endowment process. Individuals are risk-averse and would therefore like to insure their endowment processes. It is feasible to obtain insurance by pooling endowments across individuals because the processes are mutually independent. We characterize the payoff from an insurance contracting scheme of this type, and we investigate whether such a scheme would survive as an equilibrium in a noncooperative setting. We focus on the stability of cooperative arrangements with respect to the dynamic formation of coalitions. The economy "crystallizes" into a collection of coalitions in equilibrium.

Keywords: Coalitions; Risk sharing; Brownian motion; Optimal stopping

JEL classification: C71; C72; C73; D80

1. Introduction

In this paper we explore the effect of a dynamic individual rationality constraint on the degree of risk sharing that is feasible in an economy. Our analysis builds on our paper (Taub and Chade, 2002), which has the following structure. There is a continuum of

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individuals with constant absolute risk aversion preferences. Each has a Brownian motion endowment process that is independent of all others, so risk sharing is possible and desirable. At each instant the endowment processes are evenly distributed on the real line. Risk sharing in which there is equal division of the endowments within an interval or segment is physically feasible. Individuals whose endowments wander outside of a segment are excluded from the insurance until they return, but they can participate in the risk-sharing contract associated with other segments.

Ex ante welfare increases if these segments are large, but they cannot grow without bound, because individuals whose current endowments are much larger than the average of the segment have an incentive to defect from the contract because their contractual consumption is much lower than their endowment, and because Brownian motion is persistent, any gains from defecting in order to exploit this difference will persist. Autarky, or temporary autarky, becomes more attractive than the contract, despite the higher risk inherent in autarky. We found that this incentive to defect can be defeated if there are infinitely many segments, and that a sufficient number of them participate in punishing defection.

In this paper we extend our analysis to allow coalitions of these segments to form and investigate whether there is a positive and finite segment size that is stable (that is, unblocked) with respect to this coalition formation process. We do in fact find such a segment size. Furthermore, we find that this segment size increases with the degree of patience of the agents, and increased segment size facilitates cooperation.

In order to further motivate our dynamic model, we first explore a static model of risk-sharing among heterogeneous agents in Section 2. We begin with a model in which there is a discrete set of agents, and then we show how a continuum model differs from the discrete case. The continuum model highlights a fundamental instability of static risk-sharing models: we show how the instability is removed in a dynamic model. In order to lay the groundwork for the dynamic model, we recapitulate the framework of our model from Taub and Chade in Section 3 before analyzing stable coalition formation. This restates the essential features of Sections 2 through 5 of our previous paper. Because the basic structure of the model is fully analyzed there, we do not include proofs in this section. In Sections 4 and 5 we move beyond our previous paper and analyze coalitions. We end with a brief discussion in Section 6.

2. A static model

We intuitively motivate the segmentation and equal division assumptions that we impose in the sequel by considering a one-period setting in which there is a fundamental motivation to form groups in order to share risk. There is a collection of agents denoted by M; an agent $m \in M$ has a stochastic endowment y_m drawn from a Gaussian distribution $N(m; \sigma^2)$. The agents are risk averse and have von Neumann–Morgenstern utility function $u(c) = -e^{-ac}$. Because the variances are common across individuals, the endowment distributions are ordered by first order stochastic dominance, so that under autarky individuals with a high m are better off. Using the distributional assumption, the autarkic utility of agent m is given by $-e^{-am+\frac{\sigma^2 a^2}{2}}$.

Contracts are possible prior to the draw of endowments, and despite the ordering of preferences over autarkic endowments, risk-sharing contracts are desirable. The one restriction that will be imposed on contracts is that they be of the equal division form, as in Farrell and Scotchmer (1988): all realized endowment within a contracting group must be split evenly among its members. While this restriction is ad hoc at this point, we provide a more compelling motivation in our dynamic model.

In Taub and Chade (2002) we analyze the finite case in which $M=\{m_1,\ldots, m_n\}$, $m_1>m_2>\ldots>m_n$. If a coalition $G\in M$ decides to pool their members' endowments and share them equally, each member receives an expected utility equal to

$$-e^{\left(-a\frac{\sum_{m\in G^m}}{|G|}+\frac{a^2\sigma^2}{2|G|}\right)}$$

since

$$\frac{\sum_{m\in G^{y_m}}}{|G|}$$

is Gaussian

$$N\left(\frac{\sum_{m\in G^m}}{|G|},\ \frac{\sigma^2}{|G|}\right).$$

The size and composition of a coalition affects the welfare of its members in two ways: first, due to the common variance, independent of the composition of the group, the larger the group, the smaller the risk that each agent bears. Second, the expected consumption of each member is affected by the composition of the group; groups containing agents with lower expected endowments enjoy lower levels of expected consumption. These effects reveal a trade-off: large coalitions allow for a better diversification of risks but for more heterogeneity, which affects the expected consumption of their members. Agents with better expected endowments may be reluctant to share risks with agents that have low expected endowments, in which case we would expect the formation of risk pooling groups containing "similar" agents.

We define a *stable coalition structure* as a partition of the set of players into coalitions such that no new coalition could form and make its players better off. In our previous paper we demonstrate that there is a unique stable coalition structure consisting of consecutive coalitions. It is characterized by an acute form of segmentation by income: high-income agents share risks among themselves, and they do not mingle with the rest of the population; the same is true for other groups, although everybody wishes to belong to a higher-mean coalition. Again, the equal-division rule makes it optimal for high-income agents to separate themselves from the rest of the population, since a better diversification of risk cannot be achieved without introducing more heterogeneity.

Now consider a modified setting in which there is a continuum of agents, i.e., $M=[0, 1]$. Let $G\in M$ be a measurable set with (Lebesgue) measure $v(G)>0$. If a coalition G forms, then by the law of large numbers each member receives an expected utility equal to

$$-e^{-a\int_G m\frac{\mathrm{d}m}{v(G)}}$$

unlike the case with a finite number of agents, it is easy to show that there is no stable coalition structure in which coalitions have positive measure. Suppose otherwise and consider a coalition G containing a continuum of agents $m=1$ (there must be at least one such G); without loss of generality, let G be an interval. Then for any $\epsilon>0$ small enough, agents in the subgroup $[1-\epsilon, 1]$ will be better off if they defect and share risks among themselves, blocking the proposed coalition structure. The problem is that aggregate uncertainty is eliminated within any group with positive measure. This destroys the trade-off between risk and return that is crucial for the results with a finite number of agents.

The defection problem in the continuum model can be resolved by re-examining contract formation in a dynamic setting. We construct a model without commitment and with mobility of agents across coalitions conditional on their endowment state, like the model of our previous paper. This restores the trade-off between risk and return: agents who defect into smaller coalitions enjoy the resulting risk-pooling for a shorter duration. In order to complete the analysis, we first briefly recapitulate the dynamic model in the next section.

3. The dynamic model

In the dynamic setting the endowment process for each individual is $y(t)$, a non-geometric, driftless diffusion described by

$$\mathrm{d}y(t) = \mathrm{d}z(t)$$

where $z(t)$ is a standard Brownian motion. Each individual is indexed at time t by his endowment process $y(t)$. In some initial period $t=0$, there is a uniform distribution of individuals, and therefore of endowments, on the entire real line R. The measure ν of individuals in an interval A is simply uniform Lebesgue measure.[1]

3.1. Preferences

Each individual maximizes discounted expected utility, with exponential instantaneous return function:

$$-E\left[\int_0^\infty e^{-\rho t}e^{-ac(t)}\mathrm{d}t|y(0)=y\right] \quad \rho>\frac{a^2}{2}$$

where ρ is the subjective rate of time preference, a is the coefficient of absolute risk aversion, and $c(t)$ is the consumption process, and the condition $\rho>a^2/2$ is an assumption of our model needed for convergence of the integral. We restrict attention to these

[1]The measure of agents is infinite but we focus only on segments with a finite measure of agents.

preferences in conjunction with the non-geometric and driftless diffusion processes because it is easy to construct contracts due to the absence of wealth effects.

Definition 1. A coalition S is a bounded and measurable set of endowments such that agents who enter the coalition and whose endowments reside in the set pool their endowments and share them equally.

We will focus on the simplest possible coalition, a *segment*.

Definition 2. A coalition S is a segment if it is an interval $[\underline{y}, \bar{y})$.

A segment has fixed boundaries but is fluid in the sense that individuals are constantly flowing into it as their endowments hit the entry point or exit points, and also they are constantly flowing out as their endowments go beyond the boundaries $\underline{y}$ and $\bar{y}$. For tractability we assume that individuals enter the segment at the midpoint of the boundaries $\underline{y}$ and $\bar{y}$. Because endowments diverge in a balanced fashion thereafter, it is feasible to set consumption equal to the endowment at this entry point.

Individuals can enter and leave the segment an indefinite number of times. Also, an individual's endowment might lie in the interval $[\underline{y}, \bar{y})$, yet the individual will not participate in the segment contract because his endowment will not yet have attained an entry point. Because of this structure, stopping times will be used to characterize contracts and also the objectives and constraints facing individuals.

In our previous paper we show that if a single segment offers an equal-sharing contract to individuals that insures their endowments while they remain in the segment, then no matter how patient they are, there are always individuals who prefer to renege on the contract and consume their endowments. In order to prevent defection, it is necessary to increase the size of the carrot-insurance-that can be removed. This requires the presence of multiple segments providing insurance and coordinating the punishments of defectors. We next describe the mechanics of multiple-segment contracts.

3.2. Multiple segments

We now set out the notation for multiple segments. All the individuals with current endowments in a finite interval (risk pooling group) $[y_{k-1}, y_{k+1})$ cooperate in the sense that they pool their endowments. In return, each receives the average endowment of the group as consumption. Since the distribution is uniform, this contractual consumption is $y_k = (y_{k+1} + y_{k-1})/2$. We refer to the interval and its associated contract as the kth *segment*, S_k. Individuals within the segment in the initial instant t can give up their endowment at time $s > t$ in exchange for the same payoff y_k regardless of how their individual endowments evolve, because each individual's endowment evolves independently, and the average endowment remains at y_k according to the law of large numbers; this is true even if endowments evolve beyond the boundaries of the segment.

Each individual belongs to S_k until the first time his endowment goes beyond the upper or lower bound of the segment. Denote this stopping time by T_k; it is actually the minimum $T_{y_{k-1}} \wedge T_{y_{k+1}}$. Thus individuals leak out of this contract as their endowments hit the boundary. Nevertheless, they will still value membership because of the temporary insurance value. This type of contract will be our central focus.

Definition 3. A contract is defined to be the following object:

$$\left\{\{S_k\}_{k=-\infty}^{\infty},\ \{I_k(t)\}_{k=-\infty}^{\infty},\ \{y(i,t)\}_{i\in I_k(t)},\ \{a(i,t)\}_{i\in I_k(t)},\ \Psi\right\}$$

where S_k is a segment; $I_k(t)$ is a set of individuals; $y(i, t)$ is an endowment process realization for an individual and time that can be related to S_k; $a(i, t)$ is an action process in which $a \in \{\text{cooperate; defect}\}$; and Ψ is a rule that maps endowment and action histories to consumption allocations for those histories.

We restrict attention to mappings of the following kind: the mapping Ψ converts endowment for agent i in segment S_k into the average endowment y_k in that segment, conditional on the agent having entered the segment ($T_{yk}<t$) and not yet exited ($t<T_{y_{k-1}}\wedge T_{y_{k+1}}$), if the action 'cooperate' was observed for $s\in[0, t]$; for any other action history the agent is punished with autarky.[2] More technically,

$$\Psi\left[y(i,t),\ \{a(i,s)\}^t\right] = \begin{cases} y_k & T_{y_k}<t<T_{y_{k-1}}\wedge T_{y_{k+1}},\ a(i,s)=\text{cooperate}, \qquad 0\le s\le t \\ y(i,t) & \text{otherwise} \end{cases}$$

for all k. Before the game commences at $t=0$, the decision to implement the contract takes place, taking account of individual rationality and the incentives to form segments at all t.

3.3. Boundary behavior

Because Brownian motion has unbounded variation, the endowment path after the escape time T_k will cross the boundary infinitely often after T_k in any finite time interval. We can define contracts recursively by using a hysteresis approach: an individual entering segment $S_{k+1}\equiv[y_k, y_{k+2}]$ does so at endowment y_{k+1}; he receives that endowment until his escape from S_{k+1}. If the escape is into the lower segment S_k, it is at y_k, the contractual consumption level of S_k. He will thus consume y_k until his escape from $S_k\equiv[y_{k-1}, y_{k+1}]$, and so on. Thus the segments S_k and S_{k+1} share the interval $[y_k, y_{k+1}]$, but an individual whose endowment process is in that overlapping range can belong either to S_k or to S_{k+1} depending on the recent history of his endowment.

Definition 4. A feasible segment contract is one such that $\int_{S_k}\Psi\left[y(i,t),\{a(i,s)\}^t\right]\mathrm{d}v(i) = \int_{S_k} y(i,t)\mathrm{d}v(i)$ for all k and $t>0$.

In other words, there must be a balance between consumption and endowment within each segment. Note that a more general contract notion could transfer resources across segments. We do not examine this possibility.

Definition 5. A contract is individually rational if for all t no positive measure of individuals prefers autarky to the local segment S_k of the contract.

[2]Obviously, this stopping-time structure can encompass more complicated contracts that have multiple entry and exit points.

The logic behind this definition is that if a positive measure of high-endowment individuals defect, then the contract, which requires pooling of high endowments with low endowments in order to implement equal sharing, will be infeasible.

3.4. Convergence in the infinite extent model

We now contemplate an arbitrary number of segments. Because of our assumptions of exponential utility (that is, CARA) and Brownian endowment, we focus on segments that all have the same width $2D$: that is, $y_k - y_{k-1} = D$ for all k. There are infinitely many, so each individual's consumption process jumps at discrete times and is constant in between jumps.

Locally eliminating risk does not fully eliminate risk: although constant consumption is feasible within segments because each has a continuum of individuals, risk is not eliminated because of the jumps in consumption that occur in transitions across segments, and this risk is a decreasing function of segment size, analogous to the decreasing risk of coalition size in the static discrete model.

Let there be a segment collection $\{S_k\}_{k=-\infty}^{\infty}$, defined by $S_k \equiv [y_{k-1}, y_{k+1}]$, and with the width of the segments defined by $y_{k+1} - y_{k-1} = 2D$. Thus each coalition is defined by a central endowment level y_k, and two boundary levels of endowment y_{k+1} and y_{k-1}. If an individual's endowment process $y(t)$ hits y_k, he then enters the kth segment and his consumption is constant and equal to y_k until he hits one of the boundaries y_{k+1} or y_{k-1}. At that time he enters either the $k+1$ or the $k-1$ segment respectively. The segments therefore overlap.[3]

Define the following stopping times: $T_{y,y_{k+1}}$ is the time to go from y to y_{k+1}. Also, $T_{y,y_{k-1}} \wedge T_{y_{k+1}}$ is a double boundary escape time starting at $y \in S_k$. The following value function recursion then holds:

$$\upsilon(y,\ S_k) = -\, e^{-ay_k}\, \frac{1 - E\left[e^{-\rho T_{y,y_{k-1}} \wedge T_{y,y_{k+1}}}\right]}{\rho} + E\left[e^{-\rho T_{y,y_{k-1}} \wedge T_{y,y_{k+1}}}\, \upsilon\left(y\left(T_{y,y_{k-1}} \wedge T_{y,y_{k+1}}\right)\right)\right],$$
$$y \in S_k, \qquad k = \ldots, -1, 0, 1, \ldots \tag{1}$$

If we impose the requirement that there be no defection, the value recursion can be solved at the segment boundaries using difference equation methods. Define $\upsilon_k = \upsilon(y_k, S_k)$ and

$$\lambda = e^{-\sqrt{2\rho}D} < 1.$$

[3]More complicated contracts could be comprised of segments that are of different sizes. In that case, there might be gaps between the exit points and entry points between adjacent or nearby segments, requiring individuals to sojourn temporarily in autarky between segments. Alternatively, complicated arrangements of segments could be constructed to bridge gaps of this sort. Our equal-length overlapping segment construction avoids these situations. To be more concrete, contracts consisting of the union of disjoint segments are unstable. Consider a proposed coalition of segments consisting of two disjoint segments $[0, 2D)$ and $[3D, 5D)$. Clearly, the upper segment $[3D, 5D)$ can gain by forming a coalition of segments with the next lower adjacent segment, $[2D, 4D)$: the insurance benefit is the same (during sojourns in the coalition) but the contractual consumption is higher. Therefore segments do not have "holes".

After algebraic manipulation and appropriate application of boundary conditions the solution is

$$\upsilon_k = -\frac{1}{\rho}\frac{(1-\lambda)^2}{(1-\lambda e^{aD})(1-\lambda e^{-aD})}e^{-aDk} \tag{2}$$

which can be used to calculate $\upsilon(y, S_k)$.

In our previous paper, we analyze whether everybody pooling their endowments at each point in time could be the outcome of a Subgame Perfect Equilibrium. We show that the answer is affirmative if the punishment is autarky forever after a defection; more importantly, we show that temporary punishments suffice so long as enough segments cooperate in punishing the defectors.

4. Blocking coalitions

Since subgame perfection considers only unilateral deviations, it does not rule out the possibility of deviations of coalitions of agents. We now explore such coalitional deviation. We examine the smallest feasible deviation: the union of two adjacent segments. In this section we establish our main technical result, namely that such two-segment coalitions are unstable when segment size is small, but are stable for large segment sizes. We use this result as a foundation for our general results, which we set out in the next section.

We first define a coalition of segments.

Definition 6. A segment coalition is a finite collection of segments $\{S_{k_1}, \ldots, S_{k_N}\}_{i=1}^{N}$ that follows the equal sharing rule for all members S_{k_i}.

The assumptions underlying our analysis are as follows. First, we are assuming that we have an equilibrium with multiple segments of size $2D$, such that no individual has incentives to revert to autarky (i.e., the discount rate is sufficiently low). Second, we are implicitly assuming that the underlying punishment for an individual who defects is autarky forever.[4] Third, we are only allowing equilibrium segments of size D to merge; if a subgroup of a segment wants to form a separate coalition, then the grim punishment applies and we know from our previous paper that this is sufficient to deter such deviation. Fourth, we only allow for the possibility of the current and future members of the two segments to merge temporarily and revert to the original segment equilibrium once they exit the coalition.

We assume that individuals in the coalition revert to the no-coalition state after exiting the expanded-segment contract, in order to avoid the asymmetric calculations that would otherwise ensue. If the gain from this temporary deviation is positive, the gain from a permanent deviation that is comprised of a sequence of such temporary deviations would also be positive. We must emphasize that the converse does not necessarily hold: there might be complicated permanent deviations that result in positive gains despite a temporary negative payoff.

[4]In our previous paper we established that an equilibrium could be sustained by a finite number N^* of adjacent segments participating in punishments. But N^* depends on the segment size D, and a coalition of segments alters D: in order to avoid this complication we focus only on grim punishments here.

Each individual in the deviating coalition of segments reverts to the no-deviation contract at different times. Despite these different exit times, the resources to fund the pooling of endowments continue to exist because there is a continuum of agents in the deviating contract at all times even as their measure shrinks as exit occurs. We also assume that individuals can join the deviating coalition only when their endowment processes reach the midpoint of the equilibrium contract segment.

In the following proposition we analyze the simplest deviation, namely when two adjacent segments, S_k and S_{k+1}, contemplate deviation. Mechanically, this deviation works by individuals from the two segments deciding to join the deviating coalition when their endowments reach either y_k or y_{k+1}, the original entry points for the separate segments S_k and S_{k+1}. The deviating coalition is initially devoid of members but individuals flow in (if they wish to join) over time. We emphasize that the measure of individuals that will flow in has positive measure at every instant by the construction we described in our previous paper. Therefore we can use the law of large numbers and be assured that although the density of individuals is initially concentrated around the entry points, the average of endowment is well-defined and fixed at $(y_k+y_{k+1})/2$.

Proposition 1. *There exists a unique D^* such that for $D<D^*$, the incentive to form two-segment coalitions is positive, and for $D>D^*$ the incentive to form coalitions is negative.*

Proof. At time $t=0$ let there be a segment equilibrium in which all segments have identical size $2D$ and in which the individuals in the economy are distributed evenly, i.e. the measure v of individuals indexed by their endowment in an interval $[a, b)$ is $b-a$.

We wish to compare the payoffs for an individual in the segment equilibrium, with contractual consumption initially at y_{k+1}, whose endowment can exit at y_{k+2} or y_k followed by y_{k-1}, with the payoffs for an individual in the deviating segment coalition comprised of $S_k \cup S_{k+1}$ who consumes $y_k+D/2$ and who exits at either y_{k+2} or y_{k-1} without the intervening exit at y_k. (An individual starting in the lower segment at y_k will definitely gain from the coalition so we do not have to check that case.)

The payoff calculation is similar to the calculations for the temporary deviations by individuals: we compare only the payoffs during the deviation. When an individual exits the coalesced segments of the deviating coalition, he is assumed to revert to the equilibrium segment-contract. Because there are no wealth effects, we can without loss of generality let $k=1$; then $y_k=y_1=D$. It then suffices to analyze the incentives of an individual at $y_{k+1}=2D$. The temporary payoff for an individual at $y_{k+1}=2D$ in the deviating coalition centered at y_k is

$$E\left[\int_0^{T_0 \wedge T_{3D}} e^{-\rho t}\left(-e^{-\frac{3}{2}aD}\right)\mathrm{d}t \,|\, y(0)=2D\right] = -e^{-\frac{3}{2}aD}\,\frac{1-\dfrac{\cosh\left(\left(2D-\frac{3}{2}D\right)\sqrt{2\rho}\right)}{\cosh\left(\frac{3}{2}D\sqrt{2\rho}\right)}}{\rho} \tag{3}$$

whereas the payoff without being in the coalition is characterized by a recursion. That is, we must allow for a return to $2D$ before the exit at 0; the recursive expression of this is contained in the following pair of equations:

$$\begin{aligned} w(2D) &= E\left[\int_0^{T_D\wedge T_{3D}} e^{-\rho t}\left(-e^{-a2D}\right)\mathrm{d}t\right] + E\left[1_{\{T_D<T_{3D}\}}e^{-\rho T_D}\right]w(D) \\ &= -e^{-2aD}\frac{1-E\left[e^{-\rho T_D\wedge T_{3D}}\right]}{\rho} + E\left[1_{\{T_D<T_{3D}\}}e^{-\rho T_D}\right]w(D) \\ &= -\frac{e^{-2aD}}{\rho}\left(1-\frac{\cosh(0)}{\cosh(D\sqrt{2\rho})}\right) + \frac{\sinh(D\sqrt{2\rho})}{\sinh(2D\sqrt{2\rho})}w(D) \end{aligned} \tag{4}$$

and

$$\begin{aligned} w(D) &= E\left[\int_0^{T_0\wedge T_{2D}} e^{-\rho t}\left(-e^{-aD}\right)\mathrm{d}t\right] + E\left[1_{\left\{T_{2D}<T_0\right\}}e^{-\rho T_{2D}}\right]w(2D) \\ &= -\frac{e^{-aD}}{\rho}\left(1-\frac{\cosh(0)}{\cosh(D\sqrt{2\rho})}\right) + \frac{\sinh(D\sqrt{2\rho})}{\sinh(2D\sqrt{2\rho})}w(2D) \end{aligned} \tag{5}$$

where we have substituted the two-boundary exit time formulas from Karatzas and Shreve (1991).

Now recall that $\sinh(2x)/\sinh(x)=\cosh(x)$. Therefore the recursions (4.2) and (4.3) reduce to

$$w(2D) = -\frac{e^{-2aD}}{\rho}\left(1-\frac{1}{\cosh(D\sqrt{2\rho})}\right) + \frac{1}{\cosh(D\sqrt{2\rho})}w(D)$$

$$w(D) = -\frac{e^{-aD}}{\rho}\left(1-\frac{1}{\cosh(D\sqrt{2\rho})}\right) + \frac{1}{\cosh(D\sqrt{2\rho})}w(2D)$$

Define $b=\frac{1}{\cosh(D\sqrt{2\rho})}$. Then in matrix form

$$\begin{pmatrix} w(2D) \\ w(D) \end{pmatrix} = \begin{pmatrix} -\dfrac{e^{-2aD}+be^{-aD}}{\rho(1+b)} \\ -\dfrac{e^{-aD}+be^{-2aD}}{\rho(1+b)} \end{pmatrix}$$

We want the solution for $w(2D)$:

$$-\frac{e^{-2aD}+be^{-aD}}{\rho(1+b)}$$

The segmented equilibrium is blocked by a segment coalition if

$$-\frac{e^{-2aD}+be^{-aD}}{\rho(1+b)} < -e^{-\frac{3}{2}aD}\frac{1-\dfrac{\cosh\left(\left(2D-\frac{3}{2}D\right)\sqrt{2\rho}\right)}{\cosh\left(\frac{3}{2}D\sqrt{2\rho}\right)}}{\rho}$$

Recall the definitions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}; \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

Thus the inequality is in more detail

$$-\frac{\cosh(D\sqrt{2\rho})e^{-2aD} + e^{-aD}}{\rho(\cosh(D\sqrt{2\rho}) + 1)} < -e^{-\frac{3}{2}aD}\frac{1 - \dfrac{\cosh\left(\left(2D - \frac{3}{2}D\right)\sqrt{2\rho}\right)}{\cosh\left(\frac{3}{2}D\sqrt{2\rho}\right)}}{\rho}$$

Further reduction allows the left hand side to be expressed as a weighted sum:

$$\frac{\cosh(D\sqrt{2\rho})}{\cosh(D\sqrt{2\rho}) + 1}\left(-e^{-2aD}\right) + \frac{1}{\cosh(D\sqrt{2\rho}) + 1}\left(-e^{-aD}\right) < -e^{-\frac{3}{2}aD}\left(1 - \frac{\cosh\left(\left(2D - \frac{3}{2}D\right)\sqrt{2}\rho\right)}{\cosh\left(\frac{3}{2}D\sqrt{2\rho}\right)}\right) \tag{6}$$

Defining

$$b \equiv \frac{1}{\cosh(\sqrt{2\rho}D)} \quad b(0) = 1 \quad b(\infty) = 0$$

and rearranging, there will be a unique D^* if there is a unique solution of the equation

$$\frac{e^{-\frac{1}{2}aD} + be^{\frac{1}{2}aD}}{1 + b} = 1 - \frac{\cosh\sqrt{2\rho}\frac{D}{2}}{\cosh\sqrt{2\rho}\frac{3D}{2}}$$

The right hand side of this equation, RHS(D), is monotone increasing and

$$\mathrm{RHS}(0) = 0 \quad \mathrm{RHS}(\infty) = 1 \quad \mathrm{RHS}'(D) > 0, \qquad D > 0$$

The behavior of the left hand side is more complicated; it is a convex combination of two terms, $e^{-\frac{1}{2}aD}$ and $e^{\frac{1}{2}aD}$ defining

$$\phi(D) \equiv \frac{b(D)}{1 + b(D)}$$

we can write the left-hand side as

$$(1 - \phi(D))e^{-\frac{1}{2}aD} + \phi(D)e^{\frac{1}{2}aD}$$

The left hand side is approximated by $\cosh 1/2aD$ for small values of D (so that $b \approx 1$ and $\phi=1/2$) and by $e^{-\frac{1}{2}aD}$ for large values of D (so that $b \approx 0$ and $\phi=0$). This follows because b (and therefore ϕ) dominates $e^{\frac{1}{2}aD}$ by the assumption that $\rho > a^2/2$ or $\sqrt{2\rho}>a$. Thus, the term

$$\phi(D)e^{\frac{1}{2}aD}$$

is monotone decreasing beyond a critical value of D, and monotone increasing below that threshold. We also have

$$\mathrm{LHS}(0)=1 \quad \mathrm{LHS}(\infty)=0$$

but it is not monotone: since LHS approximates $\cosh 1/2aD$ at $D=0$, $\mathrm{LHS}''(0)>0$, and $\mathrm{LHS}'(\infty)=0$. We can therefore be assured that that there is a $\tilde{D}>0$ such that $\mathrm{LHS}(D)>1$ for $D<\tilde{D}$, and that $\mathrm{LHS}'(D)<0$ for all $D>\tilde{D}$. Thus, the following property holds:

$$\mathrm{LHS}'(D)<0, \quad \mathrm{LHS}(D)<1, \quad D>\tilde{D}$$

and this ensures that there is a single crossing point with the RHS. □

5. Sufficiency of two-segment coalitions

In this section we establish that all temporary coalitional deviations can be characterized by two-segment coalitions, and therefore ruling out two-segment coalitions is sufficient to rule out all finite-segment temporary coalitions. We begin by establishing the following monotonicity property.

Lemma 1. *Consider a collection of segments* $\{[y_0, y_2), [y_1, y_3), \ldots, [y_{n-2}, y_n)\}$*, with* n *even. Suppose an individual at* y_k*,* $0<k<n$ *is considering joining the coalition of segments* $[y_0, y_n)$ *but is indifferent to doing so. Then all individuals at nodes* y_j*,* $j>k$*, prefer not to join the merged coalition of segments.*

Proof. Consider an individual at the next higher node, y_{k+1}. If he joins the large coalition he will face higher taxes relative to y_k but identical insurance from remaining in the smaller segment $[y_k, y_{k+2})$. The improvement in insurance from joining the coalition is worse than the improvement for the individual at the y_k node, however, because the lower boundary of the coalition of segments is farther away, while the upper boundary is closer: the probability that he will first exit at the upper boundary increases relative to the probability that he will exit from the lower boundary relative to the y_k node. Therefore he values the insurance of the coalition of segments less. Combined with the negative impact of the increased taxes, the individual will be made worse off by joining the coalition. Therefore the value of joining the coalition of segments shrinks monotonically as endowment y increases. □

Proposition 2. *A contract that is stable against two-segment coalitions is stable against* N*-segment coalitions,* $N>2$*.*

Proof. It is not immediately clear whether a blocking coalition of three segments could form when a two-segment coalition would fail. But we can rule this out as follows. Proposition 1 shows that above a certain critical size D^*, two-segment coalitions will not form, but below that size they will form. Therefore we can restrict our attention to whether the desire to form coalitions becomes stronger above this threshold if we allow three-segment coalitions, or more generally N-segment coalitions.

Suppose that a coalition with $3N-1$ segments is proposed. The smallest such example is $N=1$, in which case there are two segments in the coalition, which is our elemental coalition. The next possibility is $N=2$, in which case five segments propose a coalition. Without loss of generality suppose that the segments are S_1 (boundaries 0 and $2D$, centered at D), S_2, S_3, S_4, and S_5, with boundaries at 0, D, $2D$, $3D$, $4D$, $5D$, and $6D$. Individuals in the segments S_1-S_5 can enter the coalition when they hit any of the boundaries D–$5D$. Notice that the individuals at $4D$ are 2/3 of the distance from 0 to the upper boundary $6D$. From their perspective, the proposed coalition is no different from a proposed coalition that would join two segments [0, $4D$] and [$2D$, $6D$]. We demonstrated in Proposition 1 that these individuals would block the coalition if $2D>D^*$. Therefore such five-segment coalitions would be blocked.

Similar reasoning holds for larger N. For numbers between $3N-1$ and $3(N+1)-1$, we can be assured that there will be boundary points higher than the critical 2/3 distance between the lowest and highest boundary points of the proposed coalition. By the mono-tonicity property of Lemma 1, Those individuals will also block the coalitions. For example, a three-segment coalition consisting of the segments [0, $2D$], [D, $3D$], and [$2D$, $4D$] would have individuals entering at the boundary points D, $2D$, and $3D$. Since $3D$>2/3 $4D$, the individuals at $3D$ will block the coalition. Thus, we can restrict attention to blocking coalitions of size two.[5] □

Thus there is a critical size of segments that deters the formation of coalitions of segments. Blocking coalitions will form for small segment sizes, and small segment sizes are not an equilibrium. Exactly the same result occurs as ρ, the degree of impatience, shrinks. In this case the payoff from not deviating is simply a lottery of the two payoffs y_k and y_{k+1}, and the blocking coalition is the utility of the certainty-equivalent payoff, which is preferred. Increasing patience therefore increases the extent of cooperation and efficiency.

We illustrate our result with a numerical example, which we present in Fig. 1, for a value of $a=1.0$.

Fig. 1 illustrates that as the subjective rate of time discount parameter ρ increases, thus expressing increased impatience, the critical segment size decreases, resulting in

[5]We extend this reasoning further. Comparing the gain from an N-segment coalition to the gain from a smaller coalition, if N is large, then we can view the segments as approximating a Brownian process, and our thought experiment—one large coalition embedded in an infinite number of small segments-starts to look like our original contemplation of one segment embedded in an otherwise autarkic economy in Taub and Chade (2002). In that paper we showed that defection by high-endowment players was inevitable, and that result can be applied here: the high-endowment segments in the coalition won't want to join it.

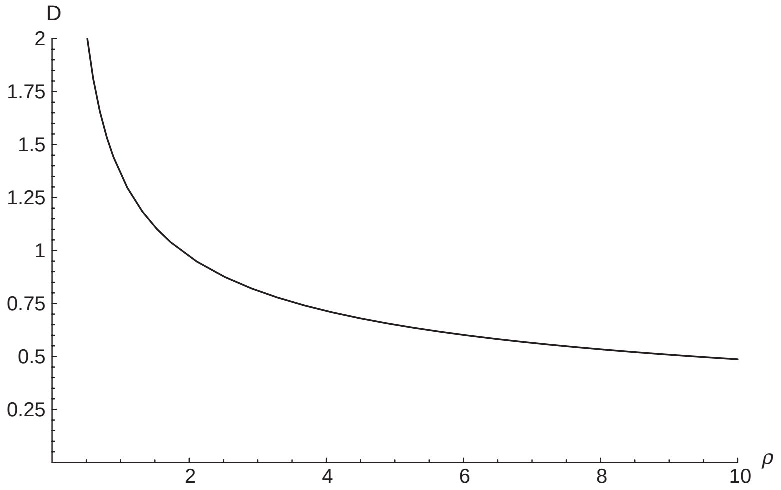

Fig. 1. Critical segment size as a function of subjective rate of time discount.

decreased insurance, and in the limit, risk-sharing collapses completely as the segment size shrinks to zero. As ρ approaches the minimum allowable value of $a^2/2=0.5$, the critical segment size tends to infinity—reflecting full ex ante risk sharing.

6. Discussion

The results demonstrate that there is a minimum segment size but that there is no incentive to increase segment size beyond this threshold level; there will be "crystallization" into agglomerations of individuals, and these agglomerations will be highly persistent. Our model is abstract, but we note its potential to model a concept that is central in economics: the extent of the market. An essential part of market specialization is the ability to diversify the risk inherent in specialized productive activity via insurance. The model here demonstrates that there are limits on the extent of such insurance imposed by incentives and ultimately by the value of time preference. Economies with low values of the rate of time preference, and a correspondingly high degree of patience, will be able to sustain a high degree of mutual insurance and consequently a high degree of specialization, and conversely for economies with high rates of time preference. Our model is a potential foundation for an empirical exploration of these ideas.

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References

Farrell, J., Scotchmer, S., 1988. Partnerships. Quarterly Journal of Economics 103 (2), 279–297.
Karatzas, I., Shreve, S., 1991. Brownian Motion and Stochastic Calculus. Springer-Verlag, New York.
Taub, Bart, Chade, Hector, 2002. Segmented risk-sharing in a continuous-time setting. Economic Theory 20 (4), 645–675.