Q3 $\quad \underbrace{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}}\ = x \qquad u(x,0) = \begin{cases} 0, & x<1 \\ x-1, & x\geq 1 \end{cases}$

$$\left.\begin{aligned} &① \text{ — } \frac{dx}{dt} = u \\ &② \text{ — } \frac{du}{dt} = x \end{aligned}\right\} \quad \frac{d(x+u)}{dt} = (x+u) \Rightarrow x(t)+u(t) = [x(0)+u(0)]e^t$$

plug into ①

$$\frac{dx}{dt} = -x + [x(0)+u(0)]e^t$$

solve

$$x(t) = x(0)e^{-t} + [x(0)+u(0)]\int_0^t e^{\hat{t}} e^{-(t-\hat{t})} d\hat{t}$$

$$\begin{cases} x(t) = x(0)e^{-t} + [x(0)+u(0)]\sinh(t) & \text{— ③} \\ \text{Also, } u(t) = -x(t) + [x(0)+u(0)]e^t & \text{— ④} \end{cases}$$

(i) when $x(0)<1 \Rightarrow u(0)=0 \xRightarrow{③} x(t) = x(0)[e^{-t}+\sinh(t)]$

$= x(0)\cosh(t)$

$x(0) = \dfrac{x(t)}{\cosh(t)}$

④ $u(t) = -x(t) + \dfrac{x(t)}{\cosh(t)}e^t$

$\boxed{u(t) = x(t)\tanh(t)}$

Since $x(0)<1$, $\boxed{x(t) < \cosh(t)}$

(ii) when $x(0)\geq 1$

$\Rightarrow u(0) = x(0)-1 \xRightarrow{③} x(t) = x(0)e^{-t} + (2x(0)-1)\sinh(t)$

$x(0) = [x(t)+\sinh(t)]e^{-t}$

④ $u(t) = -x(t) + 2[x(t)+\sinh(t)] - e^t$

$\boxed{u(t) = x(t) - e^{-t}}$

Since $x(0)\geq 1$

$[x(t)+\sinh(t)]e^{-t} \geq 1$

$\Rightarrow \boxed{x(t) \geq \cosh(t)}$

Full solution:

$$u(x,t) = \begin{cases} x\tanh(t) & \text{if } x<\cosh(t) \\ x - e^{-t} & \text{if } x\geq\cosh(t) \end{cases}$$

#

Steady solution:

As $t\to\infty$, both segments approaches x

$\Rightarrow u_s(x) = u(x,\infty) = \underline{\underline{x}}$ #

Q4 $\frac{\partial u}{\partial t} + t e^{u} \frac{\partial u}{\partial x} = e^{-u}$ $\quad u(x,0) = \begin{cases} 0, & x < 1 \\ \ln x, & x \geq 1 \end{cases}$

① — $\frac{dx}{dt} = t e^{u}$

② — $\frac{du}{dt} = e^{-u} \Rightarrow \frac{du}{e^{-u}} = dt \Rightarrow e^{u(t)} = t + e^{u(0)}$ — ③

plug into ①

$\frac{dx}{dt} = t^2 + t e^{u(0)} \Rightarrow x(t) = x(0) + \frac{t^3}{3} + \frac{t^2}{2} e^{u(0)}$ — ④

(i) when $x(0) < 1 \Rightarrow u(0) = 0$ ③ $\Rightarrow e^{u(t)} = t + 1 \Rightarrow \boxed{u(t) = \ln(t+1)}$

④ $x(t) = x(0) + \frac{t^3}{3} + \frac{t^2}{2}$

$\boxed{x(t) < 1 + \frac{t^2}{2} + \frac{t^3}{3}}$

(ii) when $x(0) \geq 1 \Rightarrow u(0) = \ln(x(0)) \Rightarrow e^{u(0)} = x(0)$

④ $x(t) = \left(1 + \frac{t^2}{2}\right) x(0) + \frac{t^3}{3}$

③ $e^{u(t)} = t + x(0)$

$u(t) = \ln(t + x(0))$

$x(0) = \frac{x(t) - \frac{t^3}{3}}{(1 + \frac{t^2}{2})}$

$\boxed{u(t) = \ln\left(t + \frac{x(t) - \frac{t^3}{3}}{(1 + \frac{t^2}{2})}\right)}$

$\boxed{x(t) \geq 1 + \frac{t^2}{2} + \frac{t^3}{3}}$

Full solution:

$$u(x,t) = \begin{cases} \ln(t+1) & \text{if } x < 1 + \frac{t^2}{2} + \frac{t^3}{3} \\ \ln\left[t + \frac{(x - \frac{t^3}{3})}{(1 + \frac{t^2}{2})}\right], & \text{if } x \geq 1 + \frac{t^2}{2} + \frac{t^3}{3} \end{cases}$$

#

Q5

$$\underbrace{\frac{\partial u}{\partial t} + (1+u)\frac{\partial u}{\partial x} = u}$$

$$u(x,0) = \begin{cases} 0 & , x<0 \\ x^2 & , 0 \le x \le 1 \\ 1 & , x>1 \end{cases}$$

$$\frac{dx}{dt} = 1+u \Rightarrow \frac{dx}{dt} = 1 + u(0)e^t \Rightarrow \underbrace{x(t) = x(0) + t + u(0)(e^t - 1)}_{③}$$

$$\frac{du}{dt} = u \Rightarrow \underbrace{u(t) = u(0)e^t}_{④}$$

(i) when $x(0)<0 \Rightarrow u(0)=0 \overset{③}{\Longrightarrow} x(t) = x(0)+t \Rightarrow x(0) = x(t) - t$

④ $\Rightarrow \boxed{u(t)=0}$

Since $x(0)<0$ $\Rightarrow \boxed{x(t)<t}$

(iii) when $x(0)>1 \Rightarrow u(0)=1 \overset{③}{\Longrightarrow} x(t) = x(0)+t+e^t-1$

④ $\Rightarrow \boxed{u(t)=e^t}$

since $x(0)>1$, $\Rightarrow \boxed{x(t) > t+e^{-t}}$

(ii) when $\underline{0 \le x(0) \le 1} \Rightarrow u(0) = [x(0)]^2$

③ $\Rightarrow \underline{x(t) = x(0) + t + [x(0)]^2(e^t-1)}$

solve

$$x(0) = \frac{-1 \pm \sqrt{1-4(e^t-1)(t-x)}}{2(e^t-1)}$$

$$0+t+0^2(e^t-1) \le x(t) \le 1+t+1^2(e^t-1)$$

$$\Rightarrow \boxed{t \le x(t) \le t+e^t}$$

since $x(0) \ge 0$, reject negative root

$$\boxed{x(0) = \frac{-1+\sqrt{1-4(e^t-1)(t-x)}}{2(e^t-1)}, \quad \text{and } u(t) = [x(0)]^2 e^t}$$

Full solution:

$$u(x,t) = \begin{cases} 0 & \text{if } x<t \\ \left[\dfrac{-1+\sqrt{1-4(e^t-1)(t-x)}}{2(e^t-1)}\right]^2 e^t, & \text{if } t \le x \le t+e^t \\ e^t, & \text{if } x > t+e^t \end{cases}$$

#

Q6 $\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0$ $\quad u(x,0) = 1$, $\quad u_t(x,0) = x$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u}_{w} = \underbrace{\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u}_{w}$$

$w(x,t) = u_t(x,t) - u_x(x,t)$

$w(x,0) = u_t(x,0) - u_x(x,0) = x$

$$\Rightarrow \begin{cases} \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial x} = w & \text{— ①} \\[2ex] \dfrac{\partial u}{\partial t} - \dfrac{\partial u}{\partial x} = w & \text{— ②} \end{cases}$$

Solving ① by MOC ~~withe~~ with the b.c., $w(x,0) = x$,

we obtain $w(x,t) = (x-t)e^t$

$\Rightarrow$ ② becomes:

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = (x-t)e^t \quad \text{with the b.c., } u(x,0) = 1.$$

~~$u(x(0),0)=1$~~

MOC again $\dfrac{dx}{dt} = -1 \Rightarrow x(t) = x(0) - t \Rightarrow x(0) = x(t) + t$

$\dfrac{du}{dt} = (x-t)e^t \Rightarrow \dfrac{du}{dt} = (x(0) - 2t)e^t$

$$\begin{aligned} u(t) &= u(0) + x(0)(e^t - 1) - 2[te^t - e^t + 1] \\ &= 1 + (x(t)+t)(e^t - 1) \\ &\quad - 2[te^t - e^t + 1] \\ &= x(t)(e^t - 1) + 2e^t - te^t - t - 1 \end{aligned}$$

Full solution:

$$u(x,t) = x(e^t - 1) + 2e^t - te^t - t - 1$$

#