Prob 2(b)

sep. of var. on the PDE and b.c. (i), (ii)

$$\Rightarrow \begin{cases} \boxed{G'' = cG \quad G'(0)=0 \quad G'(100)=0} \longrightarrow \underline{c = 0,\ -\left(\frac{n\pi}{100}\right)^2,\ n=1,2,3,\ldots} \\ \underline{\ddot{H} = -cH} \end{cases}$$

$$G_0(x) = 1, \quad G_n(x) = \cos\left(\frac{n\pi x}{100}\right)$$

$$\ddot{H}_0 = 0 \Rightarrow H_0(y) = A_0 y + B_0$$

$n \neq 0$ $\quad \ddot{H}_n = -c_n H_n = \left(\frac{n\pi}{100}\right)^2 H_n \Rightarrow H_n(y) = A_n \cosh\left(\frac{n\pi y}{100}\right) + B_n \sinh\left(\frac{n\pi y}{100}\right)$

Observing b.c. (iii) and (iv), only $n=0$ and $n=100$ matter.

$\Rightarrow$ Full solution can be written as

$$u(x,y) = A_0 y + B_0 + \left[A_{100}\cosh(\pi y) + B_{100}\sinh(\pi y)\right]\cos(\pi x)$$

$$\rightarrow u_y(x,y) = A_0 + \left[\pi A_{100}\sinh(\pi y) + \pi B_{100}\cosh(\pi y)\right]\cos(\pi x)$$

<u>b.c. (iii)</u> $\rightarrow \quad A_0 + \pi B_{100}\cos(\pi x) = 3 \Rightarrow A_0 = 3,\ B_{100} = 0.$

<u>b.c. (iv)</u> $\rightarrow \quad A_0 + \pi A_{100}\sinh(\pi)\cos(\pi x) = 3 + \cos(\pi x)$

$$\Rightarrow A_0 = 3, \quad \pi A_{100}\sinh(\pi) = 1 \Rightarrow A_{100} = \frac{1}{\pi\sinh(\pi)}$$

Full solution is

$$u(x,y) = 3y + B_0 + \frac{\cosh(\pi y)\cos(\pi x)}{\pi\sinh(\pi)} \quad \#$$

In its final form, B_0 remains undetermined.
This means there are <u>infinite many</u> solutions.
This is consistent with the conclusion from Prob 2(a).

Prob 3

sep. of var. on the PDE and b.c. (iii), (iv) : $u \sim G(x)H(y)$

$\rightarrow \quad G\ddot{H} + x^2 G''H + 3x G'H + GH = 0 \quad , \quad \underbrace{H'(0)=0, \ H(\tfrac{\pi}{2})=0}$

$\rightarrow \quad \dfrac{\ddot{H}}{H} + \dfrac{x^2G'' + 3xG' + G}{G} = 0$

$$\rightarrow \begin{cases} \ddot{H} = cH \qquad H'(0)=0, \ H(\tfrac{\pi}{2})=0 \\ \underbrace{x^2G'' + 3xG' + G = -cG} \end{cases}$$

$\rightarrow \quad C_n = -n^2 \quad n = 1, 3, 5, 7, \ldots$

$H_n(y) = \cos(ny)$

$n = 1, 3, 5, 7, \ldots$

$$x^2 G_n'' + 3x G_n' + G_n = -C_n G_n = n^2 G_n$$

Observing b.c. (i) and (ii), only $n=1$ and $n=3$ matter.

$n=1$: $\quad x^2 G_1'' + 3x G_1' + G_1 = G_1 \Rightarrow x^2 G_1'' + 3x G_1' = 0$

Assume $G_1(x) \sim x^p \Rightarrow p(p-1) + 3p = 0 \Rightarrow p = 0, -2$

$G_1(x) = A_1 x^0 + B_1 x^{-2} = A_1 + B_1 x^{-2}$

$n=3$: $\quad x^2 G_3'' + 3x G_3' + G_3 = 9 G_3 \Rightarrow x^2 G_3'' + 3x G_3' - 8 G_3 = 0$

Assume $G_3(x) \sim x^p \Rightarrow p(p-1) + 3p - 8 = 0 \Rightarrow p = 2, -4$

$G_3(x) = A_3 x^2 + B_3 x^{-4}$

Full solution is $u(x,y) = [A_1 + B_1 x^{-2}]\cos(y) + [A_3 x^2 + B_3 x^{-4}]\cos(3y)$

<u>b.c. (i)</u> : $(A_1 + B_1)\cos(y) + (A_3 + B_3)\cos(3y) = 63\cos(3y)$

$\Rightarrow \quad A_1 + B_1 = 0 \qquad A_3 + B_3 = 63$

<u>b.c. (ii)</u> : $(A_1 + \tfrac{1}{4}B_1)\cos(y) + (4A_3 + \tfrac{1}{16}B_3)\cos(3y) = 3\cos(y)$

$\Rightarrow \quad A_1 + \tfrac{1}{4}B_1 = 3 \qquad 4A_3 + \tfrac{1}{16}B_3 = 0$

$\Downarrow \qquad\qquad \Downarrow$

$A_1 = 4, \ B_1 = -4 \qquad A_3 = -1, \ B_3 = 64$

Full solution: $\underline{u(x,y) = \left(4 - \dfrac{4}{x^2}\right)\cos(y) + \left(\dfrac{64}{x^4} - x^2\right)\cos(3y)}$ #

Prob 4

Sep. of var. on the PDE and b.c. (i) and (ii), $u \sim G(x)H(y)$

$$\rightarrow \quad G\ddot{H} + HG'' + 9\pi^2 GH = 0 \qquad G'(0)=0,\ G'(1)=0$$

$$\rightarrow \begin{cases} \boxed{G'' = cG \quad G'(0)=0,\ G'(1)=0} \rightarrow c = 0,\ -(n\pi)^2,\ n=1,2,3,\dots \\ \ddot{H} = (-c - 9\pi^2)H \end{cases}$$

$$G_0(x) = 1 \quad G_n(x) = \cos(n\pi x),$$

$$\ddot{H}_n = (-c_n - 9\pi^2)H_n$$

observing b.c. (iii) and (iv), only $n = 0$, 3, and 5 are relevant.

$$n=0: \quad \ddot{H}_0 = -9\pi^2 H_0 \Rightarrow H_0(y) = A_0\cos(3\pi y) + B_0\sin(3\pi y)$$

$$n=3: \quad \ddot{H}_3 = 0 \Rightarrow H_3(y) = A_3 y + B_3$$

$$n=5: \quad \ddot{H}_5 = 16\pi^2 H_5 \Rightarrow H_5(y) = A_5\cosh(4\pi y) + B_5\sinh(4\pi y)$$

Full solution: $u(x,y) = A_0\cos(3\pi y) + B_0\sin(3\pi y) + (A_3 y + B_3)\cos(3\pi x)$
$\qquad + [A_5\cosh(4\pi y) + B_5\sinh(4\pi y)]\cos(5\pi x)$

$$\rightarrow \quad u_y(x,y) = -3\pi A_0\sin(3\pi y) + 3\pi B_0\cos(3\pi y) + A_3\cos(3\pi x)$$
$$+ [4\pi A_5\sinh(4\pi y) + 4\pi B_5\cosh(4\pi y)]\cos(5\pi x)$$

b.c. (iii): $A_0 + B_3\cos(3\pi x) + A_5\cos(5\pi x) = 1 + \cos(5\pi x)$

$$\Rightarrow A_0 = 1,\ B_3 = 0,\ A_5 = 1$$

b.c. (iv): $3\pi B_0 \underbrace{\cos(3\pi)}_{-1} + A_3\cos(3\pi x)$
$\qquad + [4\pi\sinh(4\pi) + 4\pi B_5\cosh(4\pi)]\cos(5\pi x) = \cos(3\pi x)$

$$\Rightarrow B_0 = 0,\ A_3 = 1,\quad \sinh(4\pi) + B_5\cosh(4\pi) = 0$$
$$\Rightarrow B_5 = -\tanh(4\pi)$$

Full solution:

$$u(x,y) = \cos(3\pi y) + y\cos(3\pi x) + [\cosh(4\pi y) - \tanh(4\pi)\sinh(4\pi y)]\cos(5\pi x)$$

#