Prob 2

Sep. of var. : $(1+t)G\dot{H} = G''H + \pi^2 GH$

$u \sim G(x)H(t)$

$$\Rightarrow (1+t)\frac{\dot{H}}{H} - \pi^2 = \boxed{\frac{G''}{G} = c}$$

b.c. (i) & (ii) $\Rightarrow$ $\boxed{\begin{array}{l} G(0)=0 \\ G(1)=0 \end{array}}$

$\boxed{c_n = -(n\pi)^2}$, $n = 1, 2, 3, \ldots$

$G_n(x) = \sin(n\pi x)$, $n = 1, 2, 3, \ldots$

$$(1+t)\frac{\dot{H}_n}{H_n} - \pi^2 = -(n\pi)^2$$

$$\Rightarrow \quad \frac{\dot{H}_n}{H_n} = (1-n^2)\pi^2\left(\frac{1}{1+t}\right)$$

when $n=1$, $\dot{H}_1 = 0 \Rightarrow H_1(t) = A_1 \leftarrow$ OK to set to 1

otherwise, $H_n(t) = \underbrace{H_n(0)}_{\text{OK to set to 1}}(1+t)^{(1-n^2)\pi^2}$

Observing b.c. (iii), only $n=1,2$ are relevant.

The full solution is

$$u(x,t) = a_1 G_1(x)H_1(t) + a_2 G_2(x)H_2(t)$$
$$= a_1 \sin(\pi x) + a_2 \sin(2\pi x)(1+t)^{-3\pi^2} \quad \text{— } (*)$$

Setting t to 0 and comparing $(*)$ with b.c. (iii), we determine $a_1 = 2$, $a_2 = 1$.

$$\Rightarrow \quad u(x,t) = 2\sin(\pi x) + \sin(2\pi x)(1+t)^{-3\pi^2}$$

✱

The steady solution can be obtained by pushing $t \to \infty$ in the full solution. As $t \to \infty$, $(1+t)^{-3\pi^2} \to 0$.

Thus, the steady solution is

$$u_s(x) = 2\sin(\pi x).$$

✱

Prob 3

sep. of var. : $G\dot{H} = e^{-t} G'' H + G H \cos(t)$ ↳ b.c. (i) & (ii) $\boxed{G'(0)=0 \quad G'(1)=0}$

$u \sim G(x)H(t)$

$$\Rightarrow \underbrace{e^t \left[\frac{\dot{H}}{H} - \cos(t) \right]} = \boxed{\frac{G''}{G}} = c$$

$$\boxed{c = 0, \{-(n\pi)^2, n = 1, 2, 3, \dots\}}$$

$G_0(x) = 1 \qquad G_n(x) = \cos(n\pi x), \quad n = 1, 2, 3, \dots$

when $c = 0$

$$e^t \left[\frac{\dot{H}_0}{H_0} - \cos(t) \right] = 0$$

$$\Rightarrow \frac{\dot{H}_0}{H_0} = \cos(t), \quad \Rightarrow H_0(t) = H_0(0)\, e^{\int_0^t \cos(t)\,dt} = \underbrace{H_0(0)}_{\text{OK to set to 1}} e^{\sin(t)}$$

otherwise,

$$\frac{\dot{H}_n}{H_n} = \cos(t) - (n\pi)^2 e^{-t}$$

$$\Rightarrow H_n(t) = H_n(0)\, e^{\int_0^t [\cos(t) - (n\pi)^2 e^{-t}]\,dt}$$

$$= \underbrace{H_n(0)}_{\text{OK to set to 1.}} e^{\sin(t) - (n\pi)^2(1 - e^{-t})}$$

Observing b.c. (iii), only $n = 0, 1$ are relevant

Full solution is

$$u(x,t) = a_0 G_0(x) H_0(t) + a_1 G_1(x) H_1(t)$$

$$= a_0 e^{\sin(t)} + a_1 \cos(\pi x)\, e^{\sin(t) - \pi^2(1 - e^{-t})} \qquad \text{—— } (*)$$

setting t to 0 and comparing $(*)$ to b.c. (iii), we obtain $a_0 = 1$ and $a_1 = 1$.

Full solution :

$$u(x,t) = e^{\sin(t)} + \cos(\pi x)\, e^{\sin(t) - \pi^2(1 - e^{-t})}$$

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