Q2 $G'' = cG \qquad G'(0)=2 \quad G'(1)=2$

① ②

eigenfunction

$c>0$ general sol.: $G(x) = A\cosh(\sqrt{c}x) + B\sinh(\sqrt{c}x)$

$\rightarrow G'(x) = A\sqrt{c}\sinh(\sqrt{c}x) + B\sqrt{c}\cosh(\sqrt{c}x)$

b.c. ①: $B\sqrt{c} = 2 \Rightarrow \boxed{B = 2/\sqrt{c}}$ plug back

b.c. ②: $A\sqrt{c}\sinh(\sqrt{c}) + B\sqrt{c}\cosh(\sqrt{c}) = 2$

plug back

$\Rightarrow A\sqrt{c}\sinh(\sqrt{c}) + 2\cosh(\sqrt{c}) = 2$

$\Rightarrow \boxed{A = [2 - 2\cosh(\sqrt{c})]/[\sqrt{c}\sinh(\sqrt{c})]}$

All positive values of c are OK as eigenvalues.

$c=0$ general sol: $G(x) = Ax + B$

$\rightarrow G'(x) = A \qquad$ b.c. ①: $A=2$

b.c. ②: $A=2$

arbitrary

$c=0$ is an eigenvalue,

corresponding eigenfunction: $G(x) = 2x + B$

$c<0$ general sol.: $G(x) = A\cos(\sqrt{-c}\,x) + B\sin(\sqrt{-c}\,x)$

$\rightarrow G'(x) = -A\sqrt{-c}\sin(\sqrt{-c}\,x) + B\sqrt{-c}\cos(\sqrt{-c}\,x)$

b.c. ①: $B\sqrt{-c} = 2 \Rightarrow B = 2/\sqrt{-c}$

b.c. ②: $-A\sqrt{-c}\sin(\sqrt{-c}) + B\sqrt{-c}\cos(\sqrt{-c}) = 2$

$\Rightarrow -A\sqrt{-c}\sin(\sqrt{-c}) = 2[1 - \cos(\sqrt{-c})] \quad \text{———} (*)$

IF $\sin(\sqrt{-c}) \neq 0 \Rightarrow A = -\dfrac{2 - 2\cos(\sqrt{-c})}{\sqrt{-c}\sin(\sqrt{-c})}$

IF $\sin(\sqrt{-c}) = 0 \Rightarrow c = -(n\pi)^2,\ n = 1, 2, 3, \ldots$

FOR $n = 2, 4, 6, \ldots$, $\cos(\sqrt{-c}) = 1$

$\Rightarrow (*)$ is satisfied for <u>any</u> ~~non-zero~~ value of A

For $n = 1, 3, 5, \ldots$, $\cos(\sqrt{-c}) = -1$

$\Rightarrow (*)$ leads to <u>contradiction</u>, since <u>$0 = 4$</u>

Q2 continues

Summary: ~~Eigenf~~ Eigenvalues & eigenfunctions:

$c>0$ (All values OK): eigenfunction:

$$G_c(x) = \frac{2-2\cosh(\sqrt{c})}{\sqrt{c}\,\sinh(\sqrt{c})}\cdot\cosh(\sqrt{c}\,x) + \frac{2}{\sqrt{c}}\sinh(\sqrt{c}\,x)$$

$c=0$ (OK): eigenfunction: $G_0(x) = 2x + B$.

B is arbitrary.

$c<0$ (i) All values <u>except</u> $c = -(n\pi)^2$, $n = 1, 2, 3, \dots$

$$G_c(x) = -\frac{2-2\cos(\sqrt{-c})}{\sqrt{-c}\,\sin(\sqrt{-c})}\cos(\sqrt{-c}\,x) + \frac{2}{\sqrt{-c}}\sin(\sqrt{-c}\,x)$$

(ii) $c = -(n\pi)^2$, with $n = 2, 4, 6, 8, \dots$

$$G_c(x) = A\cos(\sqrt{-c}\,x) + \frac{2}{\sqrt{-c}}\sin(\sqrt{-c}\,x)$$

where A can be any arbitrary number

(iii) $c = -(n\pi)^2$, with $n = 1, 3, 5, 7, \dots$

<u>AVOID</u>, they are NOT eigenvalues

Q4 Using Fourier series:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx}$$

plug into PDE

$$\dot{C}_n = in^5 C_n + (-in^3) C_n$$

$$-n^2\left(\frac{1}{1+t}\right) C_n + \left(\frac{4}{1+t}\right) C_n$$

$n=0$: $\dot{C}_0 = \frac{4}{1+t} C_0$

$\Rightarrow C_0(t) = C_0(0)(1+t)^4$
$= (1+t)^4$

$n=1$: $\dot{C}_1(t) = \frac{3}{1+t} C_1$

$\Rightarrow C_1(t) = C_1(0)(1+t)^3$
$= \frac{1}{2i}(1+t)^3$

$n=2$: $\dot{C}_2(t) = i24\, C_2$

$\Rightarrow C_2(t) = C_2(0) e^{i24t}$
$= \frac{1}{2} e^{i24t}$

$$u(x,0) = \sum_{n=-\infty}^{\infty} C_n(0) e^{inx}$$

From the b.c. at $t=0$, by visual inspection:

$C_0(0) = 1$

$C_1(0) = \frac{1}{2i}$ $\left(C_{-1}(0) = \frac{-1}{2i}\right)$

$C_2(0) = \frac{1}{2}$ $\left(C_{-2}(0) = \frac{1}{2}\right)$

$\Rightarrow$ All other $C_n(0) = 0$

$\Rightarrow$ only $n=0,1,2$ need to be considered

Full Solution:

$$u(x,t) = C_0(t) + \left\{\boxed{C_1(t)e^{ix} + C_2(t)e^{i2x}}\right\} + \{\text{c.c. of } \boxed{}\}$$

$$\Downarrow$$

$$u(x,t) = C_0(t) + 2\,\mathrm{Re}\left\{C_1(t)e^{ix} + C_2(t)e^{i2x}\right\}$$

$C_1(t)e^{ix} + C_2(t)e^{i2x}$

$= \frac{-i}{2}(\cos(x) + i\sin(x)) \cdot (1+t)^3$

$+ \frac{1}{2} e^{i(2x+24t)}$

$\Rightarrow \mathrm{Re}[C_1(t)e^{ix} + C_2(t)e^{i2x}]$

$= \frac{1}{2}\sin(x)(1+t)^3 + \frac{1}{2}\cos(2x+24t)$

$$u(x,t) = (1+t)^4 + \sin(x)(1+t)^3 + \cos(2x+24t)$$

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Q5 Using Fourier series:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx}$$

plug into PDE

$$\ddot{C}_n = -n^2 C_n + 2in\dot{C}_n + 3\dot{C}_n + 4C_n$$

$$u(x,0) = \sum_{n=-\infty}^{\infty} C_n(0) e^{inx}$$

From b.c. (i), by visual inspection, $C_2(0) = \frac{1}{2i}$. All other $C_n(0) = 0$

From b.c. (ii), by visual inspection, $\dot{C}_0(0) = 10$, All other $\dot{C}_n(0) = 0$

$\Rightarrow$ only need to consider $n = 0, 2$

n=0: $\ddot{C}_0 - 3\dot{C}_0 - 4C_0 = 0$

Let $C_0 \sim e^{\alpha t} \Rightarrow \alpha^2 - 3\alpha - 4 = 0$

$\alpha = 4, -1$

$$C_0(t) = A_0 e^{4t} + B_0 e^{-t}$$

$$\Rightarrow \dot{C}_0(t) = 4A_0 e^{4t} - B_0 e^{-t}$$

given $C_0(0) = 0$, $\dot{C}_0(0) = 10 \Rightarrow \left.\begin{aligned} A_0 + B_0 &= 0 \\ 4A_0 - B_0 &= 10 \end{aligned}\right\} \Rightarrow \begin{aligned} A_0 &= 2 \\ B_0 &= -2 \end{aligned}$

$$\Rightarrow C_0(t) = 2e^{4t} - 2e^{-t}$$

n=2: $\ddot{C}_2 = (i2+3)\dot{C}_2 \Rightarrow$ Let's define $D \equiv \dot{C}_2$

$\Rightarrow \dot{D} = (i2+3)D$

$\Rightarrow D(t) = D(0) e^{(i2+3)t}$

$= \dot{C}_2(0) e^{(i2+3)t} = 0$

since $\dot{C}_2(0) = 0$

$\dot{C}_2 = 0 \Leftarrow$

$\Rightarrow C_2(t) = C_2(0) = \frac{1}{2i}$

Full solution:

$$u(x,t) = C_0(t) + \boxed{C_2(t) e^{i2x}} + \text{c.c. of } \boxed{}$$

$$= C_0(t) + 2\,\mathrm{Re}\{C_2(t) e^{i2x}\}$$

$$= C_0(t) + \sin(2x)$$

$$= 2e^{4t} - 2e^{-t} + \sin(2x)$$

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$C_2(t) e^{i2x}$

$= \frac{-i}{2}(\cos(2x) + i\sin(2x))$

$= \frac{1}{2}\sin(2x) +$ imaginary part

$\mathrm{Re}\{C_2(t) e^{i2x}\} = \frac{1}{2}\sin(2x)$