Q1

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos(n\pi x)$$

Building blocks determined from b.c. (i), (ii)

$$Q(x,t) = \underbrace{q_0(t) + \sum_{n=1}^{\infty} q_n(t)\cos(n\pi x)}$$

where

$$Q(x,t) = \underbrace{t e^{-\pi^2 t}\cos(\pi x) + 100 e^{-t}}$$

visual inspection:

$$q_0(t) = 100 e^{-t}, \quad q_1(t) = t e^{-\pi^2 t}, \quad q_n(t) = 0 \text{ otherwise}$$

From b.c. (iii), $a_1(0) = 1$ and $a_n(0) = 0$ for $n \neq 1$.

$\Rightarrow$ Only $n = 0, 1$ need to be processed.

$n=0$: $\cancel{\frac{da_0}{dt}}$ $\frac{da_0}{dt} = q_0 = 100 e^{-t} \Rightarrow a_0(t) = \cancelto{0}{a_0(0)} + 100(1 - e^{-t})$

$n=1$: $\frac{da_1}{dt} = -\pi^2 a_1 + q_1 = -\pi^2 a_1 + t e^{-\pi^2 t}$

$$\Rightarrow a_1(t) = a_1(0) e^{-\pi^2 t} + \int_0^t q_1(\hat{t})\, e^{-\pi^2 (t - \hat{t})}\, d\hat{t}$$

$$= e^{-\pi^2 t} + e^{-\pi^2 t}\int_0^t \hat{t}\, d\hat{t} = \left(1 + \frac{t^2}{2}\right) e^{-\pi^2 t}$$

Full solution:

$$u(x,t) = a_0(t) + a_1(t)\cos(\pi x)$$

$$= 100(1 - e^{-t}) + \left(1 + \frac{t^2}{2}\right) e^{-\pi^2 t}\cos(\pi x)$$

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Q2

Steady solution $u_s(x)$ satisfies

$$u_s'' + 4u_s = 0 \ , \ u_s'(0)=0 \ \text{(I)}, \ u_s(\pi)=1 \ \text{(II)}$$

$$u_s(x) = A\cos(2x) + B\sin(2x)$$

$$u_s'(x) = -2A\sin(2x) + 2B\cos(2x)$$

b.c. (I) $\rightarrow$ $0 = u_s'(0) = 2B \Rightarrow \underline{B=0}$

b.c (II) $\rightarrow$ $1 = u_s(\pi) = A\cos(2\pi) + 0 = A \Rightarrow \underline{A=1}$

$\Rightarrow$ $u_s(x) = \cos(2x)$.

Let $\hat{u}(x,t) = u(x,t) - u_s(x)$

$$\Rightarrow \frac{\partial \hat{u}}{\partial t} = \frac{\partial^2 \hat{u}}{\partial x^2} + 4\hat{u}$$

(i) $\hat{u}_x(0,t)=0$, (ii) $\hat{u}(\pi,t)=0$, (iii) $\hat{u}(x,0) = \cos(0.5x)$

sep. of var. $\hat{u} \sim G(x)H(t)$

$$\Rightarrow G\dot{H} = HG'' + 4GH \Rightarrow \frac{\dot{H}}{H} - 4 = \frac{G''}{G} = c \qquad G'(0)=0, \ G(\pi)=0$$

$$c = -\left(\tfrac{1}{2}\right)^2, -\left(\tfrac{3}{2}\right)^2, \ldots \qquad \cancel{c = \left(\tfrac{0}{2}\right)^2,}$$

$$G_n(x) = \cos\left(\frac{nx}{2}\right), \ n = 1, 3, 5, \ldots$$

$$\rightarrow \frac{\dot{H}_n}{H_n} = c_n + 4 = 4 - \left(\frac{n}{2}\right)^2, \ n=1,3,5,\ldots$$

From b.c. (iii), only $n=1$ matters

$$\cancel{H_n} \ H_1(t) = e^{[4-(\frac{1}{2})^2]t} = e^{\frac{15}{4}t} \qquad G_1(x) = \cos\left(\frac{x}{2}\right)$$

$$\hat{u}(x,t) = a_1\cos\left(\frac{x}{2}\right) \cancel{e^{4-\frac{1}{2}}} e^{\frac{15}{4}t} \rightarrow a_1 = 1$$

$$= \cos\left(\frac{x}{2}\right) e^{\frac{15}{4}t}$$

Full solution: $u(x,t) = \cos(2x) + \cos\left(\frac{x}{2}\right) e^{\frac{15}{4}t}$ #

Q3

Steady solution satisfies

$$u_s'' + 0{,}25\,u_s + 2020 = 0 \qquad \overset{(I)}{u_s'(0) = 0} \quad \overset{(II)}{u_s'(\pi) = -1}$$

let $\hat{u}_s = u_s + 8080 \longrightarrow \hat{u}_s' = u_s'$

$$\Rightarrow \quad \hat{u}_s'' = -0{,}25\,\hat{u}_s'' \Rightarrow \hat{u}_s(x) = A\cos\left(\tfrac{x}{2}\right) + B\sin\left(\tfrac{x}{2}\right)$$

$$\hat{u}_s'(x) = -\tfrac{A}{2}\sin\left(\tfrac{x}{2}\right) + \tfrac{B}{2}\cos\left(\tfrac{x}{2}\right)$$

b.c. (I) $\rightarrow 0 = \hat{u}_s'(0) = \frac{B}{2} \Rightarrow B = 0$

b.c. (II) $\rightarrow -1 = \hat{u}_s'(\pi) = -\frac{A}{2}\sin\left(\frac{\pi}{2}\right) + 0 = -\frac{A}{2} \Rightarrow \underline{A = 2}$

$$\Rightarrow \quad \hat{u}_s(x) = 2\cos\left(\tfrac{x}{2}\right) \Rightarrow u_s(x) = 2\cos\left(\tfrac{x}{2}\right) - 8080.$$

Let $\tilde{u}(x,t) = u(x,t) - u_s(x)$

$$\Rightarrow \boxed{\begin{array}{l} \dfrac{\partial \tilde{u}}{\partial t} = \dfrac{\partial^2 \tilde{u}}{\partial x^2} + 0{,}25\,\tilde{u}, \\ \text{(i) } \tilde{u}_x(0,t) = 0 \quad \text{(ii) } \tilde{u}_x(\pi,t) = 0 \end{array}} \quad \text{(iii) } \tilde{u}(x,0) = 8080$$

Sep. of var. $\tilde{u} \sim G(x)H(t)$

$$G\dot{H} = G''H + 0{,}25\,GH \Rightarrow \frac{\dot{H}}{H} - 0{,}25 = \boxed{\frac{G''}{G} = c \quad \begin{array}{l} G'(0) = 0 \\ G'(\pi) = 0 \end{array}}$$

$$\frac{\dot{H}_0}{H_0} - 0{,}25 = 0 \Leftarrow \left[\begin{array}{l} c = 0, -n^2. \; n = 1, 2, 3, \dots \\ \text{From b.c.(iii), only } \underline{c = 0} \\ \qquad G_0(x) = 1 \quad \text{matters.} \end{array}\right.$$

$$\Rightarrow H_0(t) = H_0(0)\,e^{\frac{t}{4}}$$

$$\tilde{u}(x,t) = a_0\,e^{\frac{t}{4}} \Rightarrow a_0 = 8080$$

$$= 8080\,e^{t/4}$$

Full solution: $u(x,t) = 8080\left(e^{\frac{t}{4}} - 1\right) + 2\cos\left(\frac{x}{2}\right)$

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$\underline{Q4}$ Use F.S. : $u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx}$

$\Rightarrow \quad \ddot{C}_n = -n^2 C_n + 4C_n + q_n(t)$

$\underline{n=0}$: $\ddot{C}_0 = 4C_0 + 8$

Let $\hat{C}_0 = C_0 + 2 \Rightarrow \ddot{\hat{C}}_0 = 4\hat{C}_0$

$\hat{C}_0(t) = A_0 \cosh(2t) + B_0 \sinh(2t)$

$\Rightarrow \quad C_0(t) = A_0 \cosh(2t) + B_0 \sinh(2t) - 2$

$\dot{C}_0(t) = 2A_0 \sinh(2t) + 2B_0 \cosh(2t)$

$C_0(0) = 1 \Rightarrow A_0 - 2 = 1 \Rightarrow A_0 = 3$

$\dot{C}_0(0) = 0 \Rightarrow 0 = 2B_0 \Rightarrow B_0 = 0$

$C_0(t) = 3\cosh(2t) - 2$

$\underline{n=1}$: $\ddot{C}_1 = 3C_1 \quad C_1(t) = A_1 \cosh(\sqrt{3}t) + B_1 \sinh(\sqrt{3}t)$

$\dot{C}_1(t) = \sqrt{3}A_1 \sinh(\sqrt{3}t) + \sqrt{3}B_1 \cosh(\sqrt{3}t)$

$C_1(0) = 0 \Rightarrow A_1 = 0, \quad \dot{C}_1(0) = \frac{1}{2} \Rightarrow B_1 = \frac{1}{2\sqrt{3}}$

$C_1(t) = \frac{1}{2\sqrt{3}} \sinh(\sqrt{3}t)$

$\underline{n=2}$ $\quad \ddot{C}_2 = q_2(t) = \frac{t}{2i}$

$\dot{C}_2(t) = \cancel{\dot{C}_2(0)}^{\,0} + \frac{1}{2i}\frac{t^2}{2}$

$C_2(t) = \cancel{C_2(0)}^{\,0} + \frac{1}{2i}\frac{t^3}{6} = \left(\frac{-i}{2}\right)\frac{t^3}{6}$

$Q(x,t) = \sum_{n=-\infty}^{\infty} q_n(t) e^{inx}$

$Q(x,t) = 8 + t\sin(2x)$

$= 8 + \frac{t}{2i} e^{i2x} + \frac{-t}{2i} e^{-i2x}$

$\Downarrow$ visual inspection:

$q_0(t) = 8$

$q_2(t) = \frac{t}{2i}, \quad q_{-2}(t) = \frac{-t}{2i}$

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b.c. (i):

$u(x,0) = 1$

$\Rightarrow \quad C_0(0) = 1,$

all other $C_n(0) = 0$

b.c. (ii):

$u_t(x,0) = \cos(x)$

$= \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}$

$\Rightarrow \quad \dot{C}_1(0) = \frac{1}{2}, \quad \dot{C}_{-1}(0) = \frac{1}{2}$

all other $\dot{C}_n(0) = 0$

From the above, only $n = 0, 1, 2$ need to be solved

⇒ Full solution:

$$u(x,t) = C_0(t) + \left\{ \boxed{C_1(t)e^{ix} + C_2(t)e^{i2x}} + \text{c.c. of } \square \right\}$$

$$= \left[3\cosh(2t) - 2\right] + \boxed{\frac{1}{2\sqrt{3}}\sinh(\sqrt{3}t)\,e^{ix} + \left(\frac{-i}{2}\right)\frac{t^3}{6}e^{i2x}}$$

$$+ \text{ c.c. of } \square .$$

$$= \left[3\cosh(2t) - 2\right] + 2\cdot Re\left\{ \frac{1}{2\sqrt{3}}\sinh(\sqrt{3}t)\,e^{ix} + \left(\frac{-i}{2}\right)\frac{t^3}{6}e^{i2x} \right\}$$

$$= \left[3\cosh(2t) - 2\right] + \frac{1}{\sqrt{3}}\sinh(\sqrt{3}t)\cos(x)$$

$$+ \frac{t^3}{6}\cancel{\cos}\sin(2x)$$

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