MAE 384 Numerical Methods for Engineers

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(Huei rhymes with "way")

Tu/Th 12:00-1:15 PM SCOB 252

Textbook: *Numerical Methods for Engineers and Scientists: An Introduction with Applications Using Matlab,* Gilat and Subramaniam, Required

Office hours: Tuesday 3:00-5:00 PM, Wednesday 2:00-3:00 PM, or by appointment

Course website:

http://www.public.asu.edu/~hhuang38/MAE384.html

Posting of homework + solutions

Update of course/exam schedule

Supplementary course material & slides

- Please review:

(1) Calculus and ordinary differential equation
(2) Linear algebra
(3) Matlab

Self study:

Ch. 2, Ch. 4 (sec. 4.1-4.4), and Appendix A of G&S textbook

Review of calculus & differential equation will be very useful for the second half of the course - More difficult; Separates A's from B's

Quick exercise: Work through Example 3-2 in G&S textbook and make sure every step in that example is transparent to you. If not, consult Calculus textbooks.

Course outline

Part I Basic numerical methods (Ch. 1, 3-9 of Galit & Subramaniam)

- Overview and numerical errors
- Nonlinear equations
- System of linear equations (matrix equation, eigenvalue problem)
- Curve fitting and interpolation
- Numerical differentiation and integration
- Ordinary differential equation
  - Initial value problem
  - Boundary value problem

Part II Introduction to Partial Differential Equation (Lecture note)

- Analytic solution of PDE
- Numerical solution of PDE

Grade: 50% Homework (5-6 assignments)
20% Mid-term (One exam)
30% Final

See course website for projected time line for HW & exams

Exams are open book, open note. Only basic calculators are allowed.

Matlab

• Tutorials on Matlab will be given during the first 5-6 weeks of the semester, then on a learn-while-needed basis through the semester

• This is a course on Numerical Methods, not Computer Programming

BUT ...

• Computer programming is extremely useful for performing lengthy, and often repetitive, calculations that are characteristic of almost all numerical methods

See further remarks in Syllabus.

Materials related to Matlab will be <u>excluded</u> from the exams.
They will however be very important for homework.

Exams : One midterm + final

Laptops, programmable calculators, hand-held devices with programming functions will not be allowed

Examples of calculators that are permitted for exams:

Typical cost ~ $10 (as low as $5 at back-to-school specials)

A quick overview

Slides will be posted at http://www.public.asu.edu/~hhuang38/MAE384.html

(Don't worry about the details. They will be explained in later lectures.)

Why study numerical methods?

Most (> 99.9%) of real world problems in science/engineering are complicated enough that they can only be solved *numerically*

Analytic (exact, closed-form) solutions - the kind of "math" you have been learning all the way - are in fact very rare. Nevertheless, these gems are the "core truth" that helps us understand the structure of the mathematical problems and test the integrity of our numerical schemes.

Analytic vs. numerical solutions - example

Quadratic equation $$a x^2 + b x + c = 0$$

analytic solutions : $$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

It works for any set of values of (a,b,c) -- powerful

It tells us that real solutions exist only when $b^2 - 4ac \geqslant 0$

- The properties of the solutions are transparent

Numerical solution:

-- Can only deal with a given set of (a,b,c) at a time
-- Solution is approximate; Error estimate is needed

We want the computer to do repeated search for the solutions, but not blindly.

A clear set of rules ("steps" that can be repeated over and over again) has to be designed based on sound mathematical reasoning. Often, a good "numerical scheme" would guarantee a solution at a desirable level of accuracy.

Numerical solution - quadratic equations

-- Always useful to "visualize" the problem before solving it

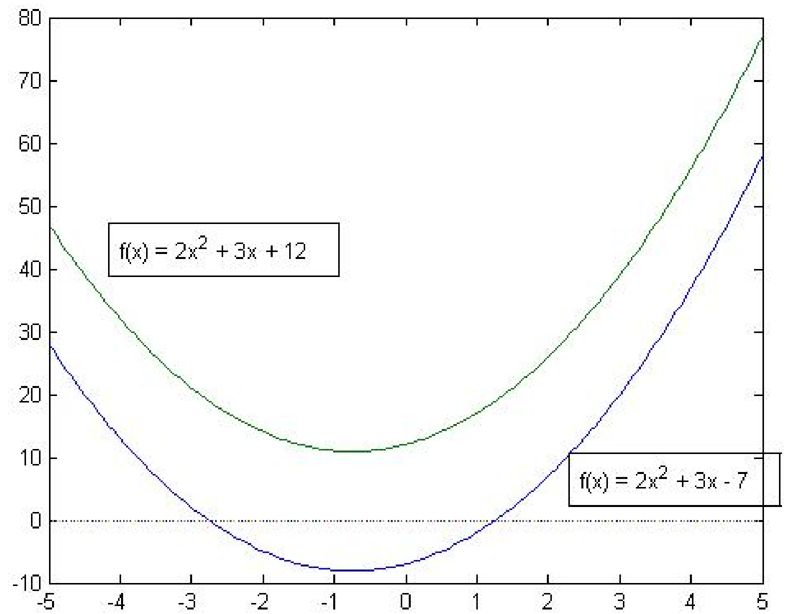
- $f(x) = ax^2 + bx + c$ is a parabola
- The real solutions of $ax^2 + bx + c = 0$ are the intersections of the parabola and the zero line (x-axis)

Examples

Green: $2x^2 + 3x + 12 = 0$ has no real solutions; No intersection

Blue: $2x^2 + 3x - 7 = 0$ has two real solutions; Two intersections

---> Let's try to find a solution to this equation

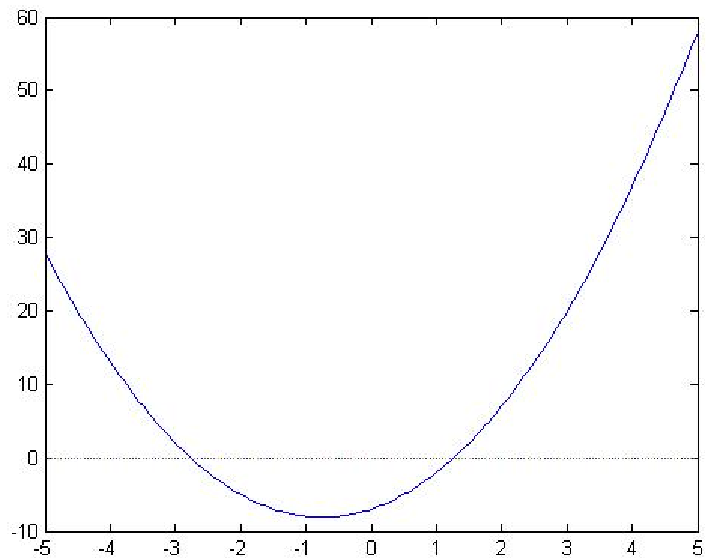


Numerical solution:

- $f(x) = ax^2 + bx + c$ is a parabola
- The real solutions of $ax^2 + bx + c = 0$ are the intersections of the parabola and the zero line (x-axis)

- If X is a solution, and $x1$ and $x2$ are two points immediately to its left and right on the x-axis, $f(x1)$ and $f(x2)$ will have opposite signs

In other words, if $f(x1)$ and $f(x2)$ have opposite signs, a solution falls within the interval ($x1$, $x2$). The goal of our numerical scheme is to systematically refine this interval.



A numerical scheme (Bisection method)

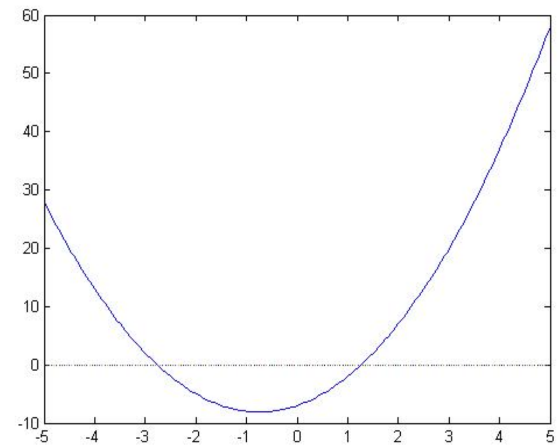
1. Pick two points, x1 and x2, such that f(x1) and f(x2) have opposite signs (this has to be done manually)

2. "Bisect" the interval, (x1, x2), into (x1, xm) and (xm, x2), where xm is the mid-point of the original interval. Keep the half interval for which f(x) retains opposite signs at the two end points.

3. Repeat step 2 until the refined interval is short enough (depending on how accurate you want the solution to be). The mid-point of this interval is our numerical solution. The interval itself is the "error bar".

The "Repeat step 2" is where a computer's brute force can provide great help, but the numerical scheme is the soul of the procedure.

Example: Solve $f(x) = 0 \quad f(x) \equiv 2\,x^2 + 3\,x - 7$



Analytic solutions: $x = (-3 \pm \sqrt{65})/4$,

$\Rightarrow x = $ **$-$2.76556443...** *and* **1.26556443...**

Numerical solution:

(for demonstration, just the positive solution)

Observe that $f(0) = -7 < 0$ and $f(2) = 7 > 0$ => Pick (0,2) as initial interval that contains a solution

bisect (0,2)	$f(1) = -2 < 0$	=> pick (1, 2) as the refined interval
bisect (1,2)	$f(1.5) = 2 > 0$	=> pick (1, 1.5)
bisect (1, 1.5)	$f(1.25) = -0.125 < 0$	=> pick (1.25, 1.5)
bisect (1.25, 1.5)	$f(1.375) = 0.3828125 > 0$	=> pick (1.25, 1.375)
	... ...	

After 9 iterations, the interval is refined into (1.2578125, 1.265625)

If we stop here, the numerical solution is **x = 1.26171875± 0.00390625**

After 20 iterations, it is improved to **x = 1.26556587 ± 0.00000191**

The procedure for the numerical solution looks cumbersome, nothing of the elegance of the analytic solution. It is also inefficient compared to the analytic solution. So, why bother?

- As the mathematical problem becomes more complicated, analytic solutions become exceedingly hard to come by, eventually nonexistent.

- One numerical scheme can be adapted to solve a wide range of problems. Often, the level of complexity of the numerical scheme does not increase dramatically with the complexity of the mathematical problem.

Cubic equation $a\,x^3 + b\,x^2 + c\,x + d = 0$

Analytic solution

Solution of Cubic Equations

3.8.2 Given $z^3+a_2z^2+a_1z+a_0=0$, let

$$q=\frac{1}{3}\,a_1-\frac{1}{9}\,a_2^2;\ r=\frac{1}{6}\,(a_1a_2-3a_0)-\frac{1}{27}\,a_2^3.$$

If $q^3+r^2>0$, one real root and a pair of complex conjugate roots,

$q^3+r^2=0$, all roots real and at least two are equal,

$q^3+r^2<0$, all roots real (irreducible case).

Let

$$s_1=[r+(q^3+r^2)^{\frac{1}{2}}]^{\frac{1}{3}},\ s_2=[r-(q^3+r^2)^{\frac{1}{2}}]^{\frac{1}{3}}$$

then

$$z_1=(s_1+s_2)-\frac{a_2}{3}$$

$$z_2=-\frac{1}{2}\,(s_1+s_2)-\frac{a_2}{3}+\frac{i\sqrt{3}}{2}\,(s_1-s_2)$$

$$z_3=-\frac{1}{2}\,(s_1+s_2)-\frac{a_2}{3}-\frac{i\sqrt{3}}{2}\,(s_1-s_2).$$

If z_1, z_2, z_3 are the roots of the cubic equation

$$z_1+z_2+z_3=-a_2$$

$$z_1z_2+z_1z_3+z_2z_3=a_1$$

$$z_1z_2z_3=-a_0$$

(From Abramowitz & Stegun's *Handbook*)

Numerical solution

Bisection method works.

No need to upgrade.

Quartic equation $a\,x^4 + b\,x^3 + c\,x^2 + d\,x + e = 0$

Analytic solution

Solution of Quartic Equations

3.8.3 Given $z^4+a_3z^3+a_2z^2+a_1z+a_0=0$, find the real root u_1 of the cubic equation

$$u^3-a_2u^2+(a_1a_3-4a_0)u-(a_1^2+a_0a_3^2-4a_0a_2)=0$$

and determine the four roots of the quartic as solutions of the two quadratic equations

$$v^2+\left[\frac{a_3}{2}\mp\left(\frac{a_3^2}{4}+u_1-a_2\right)^{\frac{1}{2}}\right]v+\frac{u_1}{2}\mp\left[\left(\frac{u_1}{2}\right)^2-a_0\right]^{\frac{1}{2}}=0$$

If all roots of the cubic equation are real, use the value of u_1 which gives real coefficients in the *quadratic equation and select signs so that if

$$z^4+a_3z^3+a_2z^2+a_1z+a_0=(z^2+p_1z+q_1)(z^2+p_2z+q_2),$$

then

$$p_1+p_2=a_3,\ p_1p_2+q_1+q_2=a_2,\ p_1q_2+p_2q_1=a_1,\ q_1q_2=a_0.$$

If z_1, z_2, z_3, z_4 are the roots,

$$\Sigma z_i=-a_3,\ \Sigma z_iz_jz_k=-a_1,$$

$$\Sigma z_iz_j=a_2,\ z_1z_2z_3z_4=a_0.$$

(From Abramowitz & Stegun's *Handbook*)

Numerical solution

Again, bisection method works.

No need to change.

Quintic equation $a\,x^5 + b\,x^4 + c\,x^3 + d\,x^2 + \mathrm{e}\,x + f = 0$, and beyond

Analytic solution

Does not exist

(except for a few isolated special cases)

Numerical solution

Bisection method works as usual.

Bisection and other numerical methods become the only options.

Types of mathematical problems that we will learn to solve numerically

Nonlinear equations

$$x^7 + 3x^3 + 5x + 9 = 0 \qquad\qquad x + \sin x + 0.1 = 0$$

Ordinary differential equations

Initial value problem

$$\frac{du}{dx} - u = \sin x + 1 \quad , u(0) = 0$$

Boundary value problem

$$\frac{d^2u}{dx^2} + 2x\frac{du}{dx} + 5u - \cos(3x) = 0 \quad , u(0) = 1.5 \quad , u(\pi) = 0$$

Partial differential equations

(Will be introduced in Part 2)

System of linear equations

$$\begin{pmatrix} 3 & -2 & 5 & 3.5 \\ 1.5 & 4 & 7 & -4 \\ 2 & 5 & 0.5 & 6 \\ -1.5 & 9 & 7 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 3 \\ -2 \\ 7 \end{pmatrix}$$

Matrix eigenvalue problem

$$\begin{pmatrix} 3 & -2 & 5 & 3.5 \\ 1.5 & 4 & 7 & -4 \\ 2 & 5 & 0.5 & 6 \\ -1.5 & 9 & 7 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Other useful techniques:

Numerical differentiation and integration; Curve fitting and interpolation

Numerical errors

- Everything is digitized and finite in a computer (or calculator)

  For example, $\frac{1}{3} = 0.33333333333333333333333\ldots$ (exact)

  For a calculator with 9-digit capacity, "1 ÷ 3" = 0.333333333

  The error is 0.0000000003333333...

  This is an example of **round-off error**

Another example (round-off error)

$A = 0.34982$, exact
$B = 0.29817$, exact
$A \times B = 0.1043058294$, exact

For a prototypical "five digit machine", even though A and B individually are exact, $A \times B$ is not
=> $A \times B = 0.10430$, error = 0.0000058294 (which we throw away)

In this manner, round-off errors will accumulate at every step of the calculation

- In addition to **round-off errors**, we will also encounter **truncation errors** in numerical procedures.

- The "digits" in the above examples are 10-based. In a real computer, the representation of a number is 2-based (i.e., in **binary**).

  We will discuss these issues in later lectures.

Numerical solutions are approximate, not exact.

Error estimation is always important.