

## MAE384 Fall 2009 Homework #2 Discussion

### Prob 1.

There are many possible choices of "g(x)" that work for (a) or (b). None will work for both. Therefore, we need to use two different g(x)'s to complete the solutions. The first pair of valid choices are

$$g_1(x) = x + \sin(x) + \cos(x) - 0.2, \text{ which works for (a) (convergence to the positive solution)}$$

$$g_2(x) = x - \sin(x) - \cos(x) + 0.2, \text{ which works for (b) (convergence to the negative solution)}$$

Another pair of valid choices are

$$g_3(x) = \cos^{-1}(0.2 - \sin(x)), \text{ which works for (a)}$$

$$g_4(x) = \sin^{-1}(0.2 - \cos(x)), \text{ which works for (b).}$$

One can also use  $g_1(x)$  for (a) and  $g_4(x)$  for (b), and so on. There are many other forms of  $g(x)$  that will work. See attached sample solution for an interesting choice of  $g(x)$  for (b).

To test whether a  $g(x)$  will work, one needs to evaluate  $|g'(x)|$  in the vicinity of the desired solution. For this purpose, it is useful to plot  $|g'(x)|$  along with the original  $f(x)$  ( $= \sin(x) + \cos(x) - 0.2$  for our problem). Figure 1 in next page shows  $|g'(x)|$  vs.  $f(x)$  for all of the 4 choices of  $g(x)$  listed above. Note that

$$dg_3/dx = \cos(x)/[1 - (0.2 - \sin(x))^2]^{1/2},$$

and

$$dg_4/dx = \sin(x)/[1 - (0.2 - \cos(x))^2]^{1/2}.$$

It can be immediately verified that all of the  $g(x)$ 's that work for a desired solution satisfy  $|g'(x)| < 1$  in the neighborhood of that solution.

### Prob. 2

See sample solution. (Two versions: One by hand, one using Matlab.)

Figure 1 for discussion of Prob. 1 (prepared by HPH)

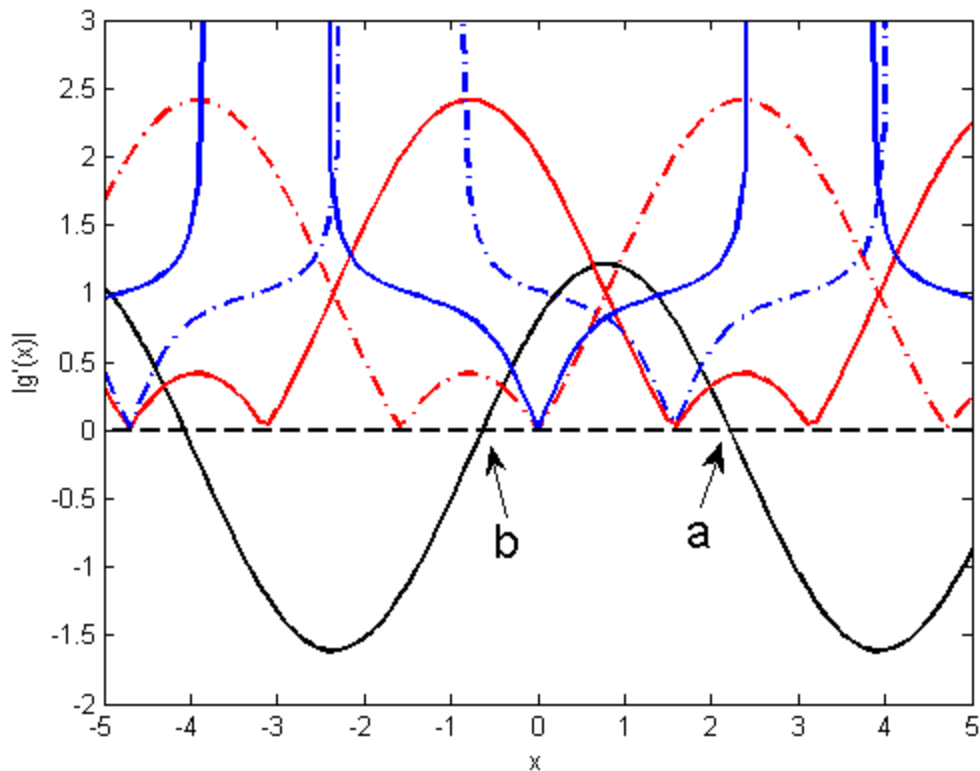


Fig. 1 Black solid:  $f(x)$  Red solid:  $|g_1'(x)|$  Red dashed:  $|g_2'(x)|$   
Blue solid:  $|g_3'(x)|$  Blue dashed:  $|g_4'(x)|$

For the record, the following is the Matlab program that does the plotting for Fig. 1. (A little trick is needed to plot the inverse trigonometric functions because they are undefined on certain intervals. As a compromise, their values are set to 999 there. Then, the domain of the plot is set to  $-2 < y < 3$  so we don't see the effect of those ugly values.)

```
x = [-5:0.1:5];
f = sin(x)+cos(x)-0.2;
gp1 = abs(1+cos(x)-sin(x));
gp2 = abs(1-cos(x)+sin(x));
for k = 1:101
    if (cos(x(k)) > -0.8)
        gp3(k) = abs(sin(x(k))/sqrt(1-(0.2-cos(x(k)))^2));
    else
        gp3(k) = 999.;
    end
    if (sin(x(k)) > -0.8)
        gp4(k) = abs(cos(x(k))/sqrt(1-(0.2-sin(x(k)))^2));
    else
        gp4(k) = 999.;
    end
    z(k) = 0;
end
plot(x,f,'k-',x,gp1,'r-',x,gp2,'r-.',x,gp3,'b-',x,gp4,'b-.',...
     x,z,'k--','LineWidth',2)
axis([-5 5 -2 3])
```

Solution, HW2 Prob 1 Thanks to Ben Warner

$$\sin(x) + \cos(x) - 0.2 = 0$$

from graph, 1<sup>st</sup> positive solution  $\approx 2$   
 1<sup>st</sup> negative solution  $\approx -0.5$

Fixed Point Iterative Method

$$x_{i+1} = g(x_i)$$

$$|g'(x)| < 1$$

a)  $x = \sin(x) + \cos(x) - 0.2 + x$  positive solution, interval (2,3) ✓

$$g(x) = \sin(x) + \cos(x) + x - 0.2$$

$$g'(x) = \cos(x) - \sin(x) + 1$$

$$|\cos(x) - \sin(x) + 1| < 1 \text{ over the interval } (2,3) \text{ (suitable } g(x) \text{ since Lipschitz cont.)}$$

$$g(x) = \sin(x) + \cos(x) + x - 0.2$$

$$\text{let } x_1 = 2$$

$$x_2 = g(x_1) = \sin(2) + \cos(2) + 2 - 0.2 = 2.29315059\dots$$

$$x_3 = g(x_2) = 2.18224908\dots$$

$$x_4 = g(x_3) = 2.2270639\dots$$

$$x_5 = g(x_4) = 2.20919817\dots$$

$$x_6 = g(x_5) = 2.2163345\dots$$

Solution after 5 iterations: 2.2163345072061

b) Try:  $x = \sin(x) + \cos(x) + x - 0.2$  interval (-1,0)

$$g'(x) = \cos(x) - \sin(x) + 1$$

$$g'(-1) = 2.3818$$

$$g'(0) = 2$$

} not suitable

Try:  $\cos(x) = 0.2 - \sin(x)$

$$1 = \frac{0.2 - \sin(x)}{\cos(x)}$$

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$$1 = \frac{0.2}{\cos(x)} - \tan(x)$$

$$x + 1 = \frac{0.2}{\cos(x)} - \tan(x) + x$$

$$x = \frac{0.2}{\cos(x)} - \tan(x) + x - 1 = g(x)$$

$$g'(x) = \left| \frac{-\sin(x)(\sin(x) - 0.2)}{\cos^2(x)} \right| < 1 \quad \text{on the interval } (-0.4, -0.7)$$

let  $g(x) = \frac{0.2}{\cos(x)} - \tan(x) + x - 1$

let  $x_1 = -0.4$

$$x_2 = g(x_1) = \frac{0.2}{\cos(-.4)} - \tan(-.4) + (-.4) - 1 = -0.7601 \dots$$

$$x_3 = g(x_2) = -0.5335$$

$$x_4 = g(x_3) = -0.7106$$

$$x_5 = g(x_4) = -0.5862$$

$$x_6 = g(x_5) = -0.6821$$

$$x_7 = g(x_6) = -0.6123$$

$$x_8 = g(x_7) = -0.6655$$

$$x_9 = g(x_8) = -0.6263$$

$$x_{10} = g(x_9) = -0.6560 \quad \checkmark$$

$$x_{11} = g(x_{10}) = -0.6339$$

$$x_{12} = g(x_{11}) = -0.6506$$

$$x_{13} = g(x_{12}) = -0.6381$$

Solution, HW2 Prob 2 (by hand) Thanks to Sean Lucas

$$2, \quad 6x_1 + 2x_2 + 3x_3 = 2$$

$$2x_1 + 8x_2 + x_3 = 8$$

$$2x_1 + x_2 + 6x_3 = -4$$

Gauss-Elimination

$$(a) \begin{pmatrix} 6 & 2 & 3 & 2 \\ 2 & 8 & 1 & 8 \\ 2 & 1 & 6 & -4 \end{pmatrix} \xrightarrow{\substack{-3R_2+R_1 \\ -R_3+R_2 \\ R_3 \leftrightarrow R_1}} \begin{pmatrix} 2 & 1 & 6 & -4 \\ 0 & 7 & -5 & 12 \\ 0 & -1 & -15 & 14 \end{pmatrix}$$

$$\xrightarrow{\substack{7R_3+R_2 \\ R_3 \leftrightarrow R_2}} \begin{pmatrix} 2 & 1 & 6 & -4 \\ 0 & -1 & -15 & 14 \\ 0 & 0 & -110 & 110 \end{pmatrix} \xrightarrow{-110R_3} \begin{pmatrix} 2 & 1 & 6 & -4 \\ 0 & -1 & -15 & 14 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{\substack{15R_3+R_2 \\ -1R_2}} \begin{pmatrix} 2 & 1 & 6 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{\substack{-6R_3+R_1 \\ -1R_2+R_1 \\ \frac{1}{2}R_1}} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\Rightarrow \bar{x}_T = \begin{pmatrix} \frac{1}{2} \\ 1 \\ -1 \end{pmatrix}$$

(b) Gauss-Seidel Iterative Method:

$$\begin{pmatrix} 6 & 2 & 3 \\ 2 & 8 & 1 \\ 2 & 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & 1 & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{6} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 1 \\ -\frac{2}{3} \end{pmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{6} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 1 \\ -\frac{2}{3} \end{pmatrix}$$

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$$\Rightarrow \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{6} & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 1 \\ -\frac{2}{3} \end{pmatrix} \Rightarrow \bar{X} = \begin{pmatrix} \frac{1}{3} \\ 1 \\ -\frac{2}{3} \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{6} & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$\Rightarrow X_1 = \frac{1}{3} + \frac{X_2}{3} - \frac{X_3}{2} \quad \text{initial guess } X_2=0 \quad X_3=0$$

$$X_2 = 1 - \frac{X_1}{4} - \frac{X_3}{8}$$

$$X_3 = \frac{-2}{3} - \frac{X_1}{3} - \frac{X_2}{6}$$

$$X_1^{(1)} = \frac{1}{3} - \frac{0}{3} - \frac{0}{2} = \frac{1}{3} \quad \Rightarrow \quad X_1^{(2)} = \frac{1}{3} - \frac{1}{3} \left( \frac{11}{12} \right) - \frac{1}{2} \left( \frac{-67}{72} \right) = \frac{71}{144}$$

$$X_2^{(1)} = 1 - \left( \frac{1}{3} \right) \frac{1}{4} - \frac{0}{8} = \frac{11}{12} \quad X_2^{(2)} = 1 - \frac{1}{4} \left( \frac{71}{144} \right) - \frac{1}{8} \left( \frac{-67}{72} \right) = \frac{143}{144}$$

$$X_3^{(1)} = \frac{-2}{3} - \frac{1}{3} \left( \frac{1}{3} \right) - \frac{11}{12} \left( \frac{1}{6} \right) = \frac{-67}{72} \quad X_3^{(2)} = \frac{-2}{3} - \frac{1}{3} \left( \frac{71}{144} \right) - \frac{1}{6} \left( \frac{143}{144} \right) = \frac{-287}{288}$$

$$X_1^{(3)} = \frac{1}{3} - \frac{1}{3} \left( \frac{143}{144} \right) - \frac{1}{2} \left( \frac{-287}{288} \right) = \frac{865}{1728}$$

$$X_2^{(3)} = 1 - \frac{1}{4} \left( \frac{865}{1728} \right) - \frac{1}{8} \left( \frac{-287}{288} \right) = \frac{6908}{6912}$$

$$X_3^{(3)} = \frac{-2}{3} - \frac{1}{3} \left( \frac{865}{1728} \right) - \frac{1}{6} \left( \frac{6908}{6912} \right) = \frac{-41476}{41472}$$

$$\Rightarrow \text{updated 3 times: } X_1 = \frac{865}{1728}$$

$$X_2 = \frac{6908}{6912}$$

$$X_3 = \frac{-41476}{41472}$$

$$c) E = \frac{\|\bar{X}_T - \bar{X}_N\|}{\|\bar{X}_T\|} \quad \|\bar{X}\| = \sqrt{\sum_{i=1}^n x_i^2}$$

$$\|\bar{X}_T\| = \sqrt{\left(\frac{1}{2}\right)^2 + (1)^2 + (-1)^2} = \sqrt{2\frac{1}{4}} = 1.5$$

$$\|\bar{X}_T - \bar{X}_{N1}\| = \sqrt{\left(\frac{1}{2} - \frac{1}{3}\right)^2 + \left(1 - \frac{11}{12}\right)^2 + \left(-1 + \frac{67}{72}\right)^2} = \sqrt{\frac{205}{5184}} = .1989$$

$$\Rightarrow E_1 = \frac{.1989}{1.5} = .1326$$

$$\|\bar{X}_T - \bar{X}_{N2}\| = \sqrt{\left(\frac{1}{2} - \frac{71}{144}\right)^2 + \left(1 - \frac{143}{144}\right)^2 + \left(-1 + \frac{287}{288}\right)^2} = .01042$$

$$\Rightarrow E_2 = \frac{.01042}{1.5} = .00694$$

$$\|\bar{X}_T - \bar{X}_{N3}\| = \sqrt{\left(\frac{1}{2} - \frac{865}{1728}\right)^2 + \left(1 - \frac{6908}{6912}\right)^2 + \left(-1 + \frac{41476}{41472}\right)^2} = .0008241$$

$$\Rightarrow E_3 = \frac{.0008241}{1.5} = 0.0005494$$

## Solution, HW2 Prob 2 (Matlab code, prepared by HPH)

The following program is for both 2(b) and 2(c). From 2(a), we have obtained the exact solution as  $(x_1, x_2, x_3) = (0.5, 1, -1)$ . They are given in the program as  $(x1true, x2true, x3true)$ . The 3rd line is for calculating the norm of the true solution. The first three lines within the loop are the formula for the iteration. The Gauss-Seidel method is perfect for programming: As we write the formula for  $x_1, x_2$ , and  $x_3$  in sequence, we automatically adopt the latest updates in all of the formula. The 4th line inside the loop calculates the true relative error for the outcome of each iteration.

```
x2 = 0; x3 = 0;
x1true = 0.5; x2true = 1; x3true = -1;
base = sqrt(x1true^2+x2true^2+x3true^2);
for k = 1:8
    x1 = 1/3 - x2/3 - x3/2;
    x2 = 1 - x1/4 - x3/8;
    x3 = -2/3 - x1/3 - x2/6;
    tre = sqrt((x1-x1true)^2+(x2-x2true)^2+(x3-x3true)^2)/base;
    fprintf('Iteration No.%2i yields (x1,x2,x3) = (%7.4f, %7.4f, %7.4f) \n',...
        k,x1,x2,x3)
    fprintf('                True relative error = %8.5f \n',tre)
end
```

### Outcome:

```
Iteration No. 1 yields (x1,x2,x3) = ( 0.3333, 0.9167, -0.9306)
    True relative error = 0.13257
Iteration No. 2 yields (x1,x2,x3) = ( 0.4931, 0.9931, -0.9965)
    True relative error = 0.00694
Iteration No. 3 yields (x1,x2,x3) = ( 0.5006, 0.9994, -1.0001)
    True relative error = 0.00055
Iteration No. 4 yields (x1,x2,x3) = ( 0.5002, 1.0000, -1.0001)
    True relative error = 0.00017
Iteration No. 5 yields (x1,x2,x3) = ( 0.5001, 1.0000, -1.0000)
    True relative error = 0.00004
Iteration No. 6 yields (x1,x2,x3) = ( 0.5000, 1.0000, -1.0000)
    True relative error = 0.00001
Iteration No. 7 yields (x1,x2,x3) = ( 0.5000, 1.0000, -1.0000)
    True relative error = 0.00000
Iteration No. 8 yields (x1,x2,x3) = ( 0.5000, 1.0000, -1.0000)
    True relative error = 0.00000
```