MAE 384 Fall 2009 Homework #6

1. A boundary value problem is given as

$$\frac{d^2u}{dx^2} + 5\frac{du}{dx} + 4u = 0 \quad , \; u(0) = 0 \; , \; u(1) = e^{-3} - 1 \; .$$

Discretize this system by using the *three-point central difference* scheme (Fifth formula under "Second Derivative" in Table 6-1) for the second derivative, u'', and the *two-point forward difference* scheme (First formula under "First Derivative" in Table 6-1 in bottom of p. 247) for the first derivative, u'. Then, solve the problem numerically for the two cases with (i) $\Delta x = 0.25$, and (ii) $\Delta x = 0.1$. Plot the numerical solutions against the analytic solution, $u(x) = e^{1-4x} - e^{1-x}$. **(4 points)**

2. Find the general solution of the partial differential equation,

$$\frac{\partial^2 u}{\partial x \partial y} - 2u = 0 \quad ,$$

using the method of separation of variables. (**2 points)**

3. The following PDE,

$$\frac{\partial u}{\partial t} = 2\,\frac{\partial^2 u}{\partial x^2} \quad ,$$

is defined on the semi-infinite domain with $x \in (-\infty, \infty)$, $t \in [0, \infty)$.

(a) Discretize the equation by using the *two-point forward difference* scheme for $\partial u/\partial t$ and the *three-point central difference* scheme for $\partial^2 u/\partial x^2$. Write the resulted finite difference formula for the PDE into the form,

$$u_{i,j+1} = P\,u_{i-1,j} + Q\,u_{i,j} + R\,u_{i+1,j} \quad ,$$

where $u_{i,j} \equiv u(i\Delta x, j\Delta t)$. What are the P, Q, and R as a function of $(\Delta x, \Delta t)$?

(b) Choosing $\Delta x = 0.5$, $\Delta t = 0.1$, and the boundary condition (given at $t = 0$),

$$u_{5,0} = 1, \text{ and } u_{i,0} = 0 \text{ for all } i \neq 5 \; ,$$

determine the values of all non-zero $u_{i,j}$ at $j = 1$ and 2, i.e., at $t = 0.1$ and 0.2.
(**3 points)**