MAE384 Fall 2009 HW6 Discussion

Prob 1 Solution   Thanks to Nayeli Gomez

(The reference solution incorporated the end points into the matrices so they become 5 x 5 and 11 x 11 for $\Delta x = 0.25$ and $\Delta x = 0.1$, respectively. Since u(x) are given at the end points, the matrices could be reduced to 3 x 3 and 9 x 9, i.e., for the interior points only. This is a very minor detail. - HPH)

① $\frac{d^2u}{dx^2} + 5\frac{du}{dx} + 4u = 0 \;;\; u(0) = 0,\; u(1) = e^{-3} - 1$

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{(\Delta x)^2} + \frac{5u_{i+1} - 5u_i}{\Delta x} + 4u_i = 0$$

(i) $\Delta x = 0.25;\; u_0 = 0,\; u_4 = e^{-3} - 1$

$$u_{i-1} - 2u_i + u_{i+1} + \tfrac{5}{4}u_{i+1} - \tfrac{5}{4}u_i + \tfrac{1}{4}u_i = 0$$

$$u_{i-1} - 3u_i + \tfrac{9}{4}u_{i+1} = 0$$

$$u_0 - 3u_1 + \tfrac{9}{4}u_2 = 0$$
$$u_1 - 3u_2 + \tfrac{9}{4}u_3 = 0$$
$$u_2 - 3u_3 + \tfrac{9}{4}u_4 = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & \frac{9}{4} & 0 & 0 \\ 0 & 1 & -3 & \frac{9}{4} & 0 \\ 0 & 0 & 1 & -3 & \frac{9}{4} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e^{-3}-1 \end{bmatrix}$$

... continued on matlab...

(ii) $\Delta x = 0.1;\; u_0 = 0,\; u_{10} = e^{-3} - 1$

$$u_{i-1} - 2u_i + u_{i+1} + \tfrac{1}{2}u_{i+1} - \tfrac{1}{2}u_i + \tfrac{1}{25}u_i = 0$$

$$u_{i-1} - \tfrac{123}{50}u_i + \tfrac{3}{2}u_{i+1} = 0$$

$$u_0 - \tfrac{123}{50}u_1 + \tfrac{3}{2}u_2 = 0$$
$$u_1 - \tfrac{123}{50}u_2 + \tfrac{3}{2}u_3 = 0$$
$$u_2 - \tfrac{123}{50}u_3 + \tfrac{3}{2}u_4 = 0$$
$$u_3 - \tfrac{123}{50}u_4 + \tfrac{3}{2}u_5 = 0$$
$$u_4 - \tfrac{123}{50}u_5 + \tfrac{3}{2}u_6 = 0$$
$$u_5 - \tfrac{123}{50}u_6 + \tfrac{3}{2}u_7 = 0$$
$$u_6 - \tfrac{123}{50}u_7 + \tfrac{3}{2}u_8 = 0$$
$$u_7 - \tfrac{123}{50}u_8 + \tfrac{3}{2}u_9 = 0$$
$$u_8 - \tfrac{123}{50}u_9 + \tfrac{3}{2}u_{10} = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\frac{123}{50} & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{123}{50} & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{123}{50} & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{123}{50} & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{123}{50} & \frac{3}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{123}{50} & \frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{123}{50} & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{123}{50} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{123}{50} & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ e^{-3}-1 \end{bmatrix}$$

... continued on matlab...

Matlab code and plot are in next two pages.

(Prob 1 Solution continued)
Matlab code

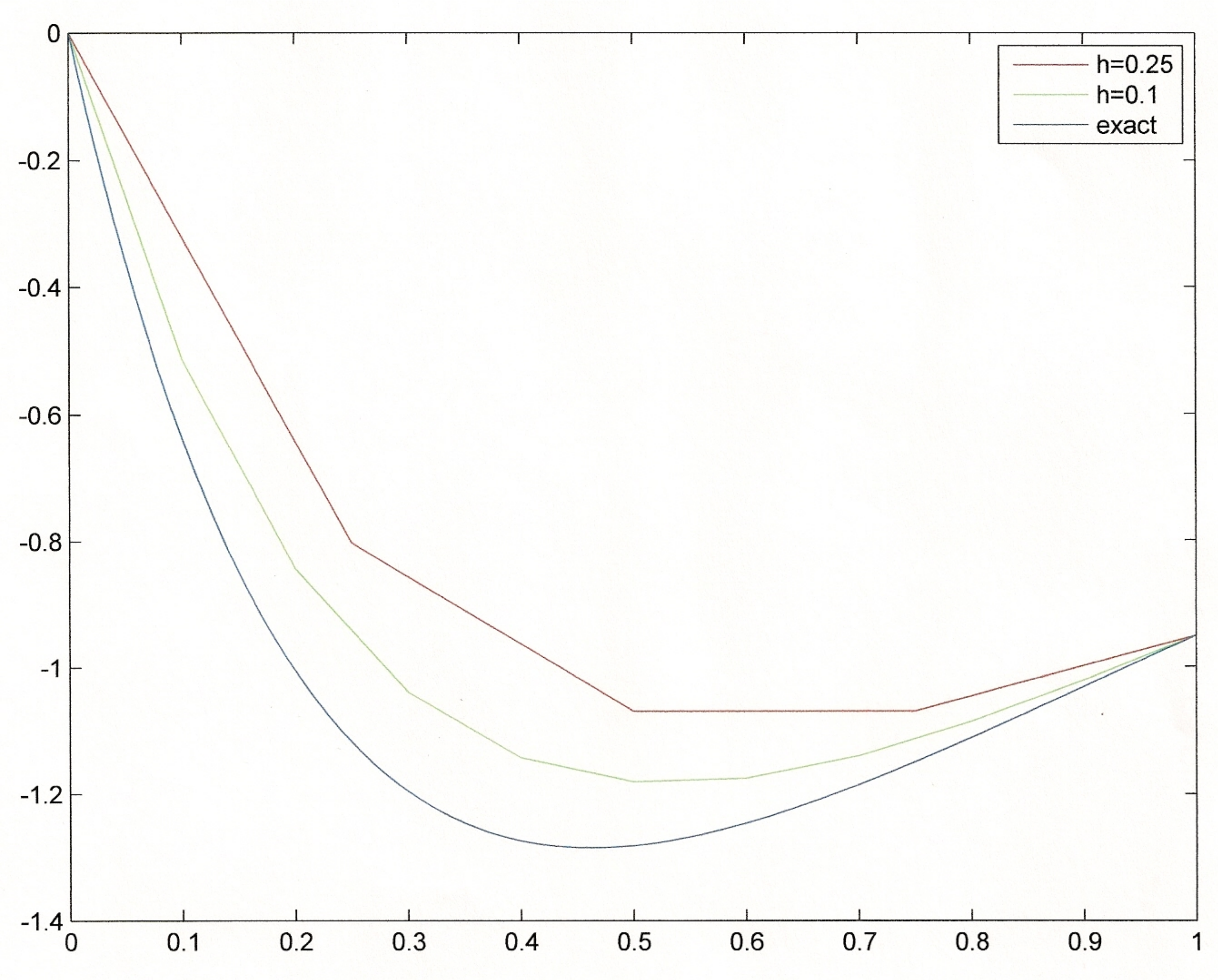
```
%%Problem 1%%
%%(i)%%
A=[1 0 0 0 0; 1 -3 9/4 0 0; 0 1 -3 9/4 0; 0 0 1 -3 9/4; 0 0 0 0 1];
B=[0; 0; 0; 0; exp(-3)-1];
U1=inv(A)*B
U1 =

         0
   -0.8017
   -1.0690
   -1.0690
   -0.9502
%%(ii)%%
C=[1 0 0 0 0 0 0 0 0 0 0; 1 -123/50 3/2 0 0 0 0 0 0 0 0;...
    0 1 -123/50 3/2 0 0 0 0 0 0 0; 0 0 1 -123/50 3/2 0 0 0 0 0 0;...
    0 0 0 1 -123/50 3/2 0 0 0 0 0; 0 0 0 0 1 -123/50 3/2 0 0 0 0;...
    0 0 0 0 0 1 -123/50 3/2 0 0 0; 0 0 0 0 0 0 1 -123/50 3/2 0 0;...
    0 0 0 0 0 0 0 1 -123/50 3/2 0; 0 0 0 0 0 0 0 0 1 -123/50 3/2;...
    0 0 0 0 0 0 0 0 0 0 1];
D=[0; 0; 0; 0; 0; 0; 0; 0; 0; 0; exp(-3)-1];
U2=inv(C)*D
2 =

         0
   -0.5134
   -0.8420
   -1.0386
   -1.1420
   -1.1804
   -1.1746
   -1.1394
   -1.0855
   -1.0207
   -0.9502

%%plot%%
x1=[0:0.25:1];x2=[0:0.1:1];
x=[0:0.01:1];y=exp(1-4*x)-exp(1-x);
plot(x1,U1,'r',x2,U2,'g',x,y,'b')
```

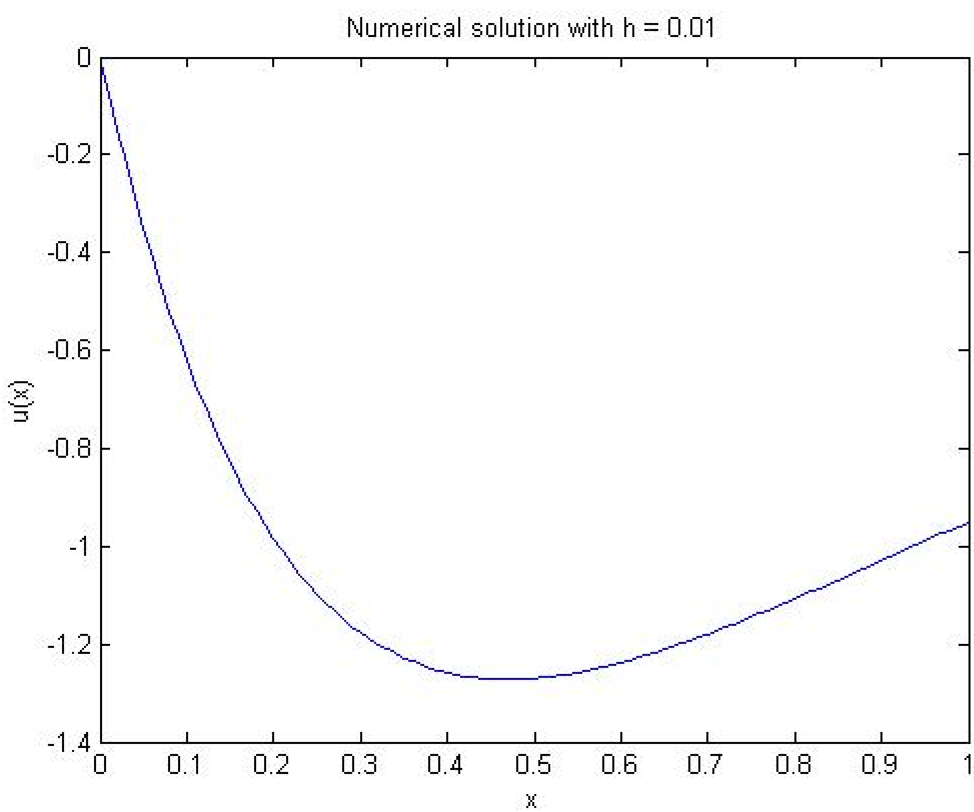
(Prob 1 Solution continued)
Plot of numerical and analytic solutions



Prob 1   Another example of Matlab code (prepared by HPH)

This code works for any given number of grid points for the interval, [0,1]. It is based on the formulation of the reference solution in page 1. The parameter "N" is the number of intervals; N = 4 and N = 10 correspond to $\Delta x = 0.25$ and $\Delta x = 0.1$, respectively. The example below is for $\Delta x = 0.01$. In this case, the numerical solution (see plot) is almost identical to the analytic solution.

```
N = 100;
h = 1/N;
x = [0:h:1];
a = zeros(N+1,N+1);
b = zeros(N+1);
a(1,1) = 1;
a(N+1,N+1) = 1;
for i = 2:N
  a(i,i-1) = 1;
  a(i,i)   = 4*h^2-5*h-2;
  a(i,i+1) = 1+5*h;
end
b(1) = 0;
b(N+1) = exp(-3)-1;
u = a\b;
plot(x,u)
```



Prob 2, Solution   Thanks to Cyle Teal

2) $\dfrac{\partial^2 u}{\partial x \partial y} - 2u = 0 \qquad u(x,y) = G(x)H(y)$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \qquad \frac{\partial u}{\partial y} = G(x)\frac{dH}{dy}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{dG}{dx}\cdot\frac{dH}{dy}$$

$$\Rightarrow \frac{dG}{dx}\frac{dH}{dy} = 2GH \qquad \frac{1}{G}\frac{dG}{dx} = \frac{2H}{\frac{dH}{dy}}$$

$$\frac{1}{G}\frac{dG}{dx} = C \qquad \frac{2H}{\frac{dH}{dy}} = C$$

$$\int \frac{1}{G}\,dG = \int C\,dx \qquad \Rightarrow \int \frac{2}{C}\,dy = \int \frac{1}{H}\,dH$$

$$\Rightarrow \ln G = Cx \qquad \Rightarrow \ln H = \frac{2y}{C}$$

$$\Rightarrow \underline{G(x) = K_1 e^{Cx}} \qquad \Rightarrow \underline{H(y) = K_2 e^{\frac{2y}{C}}}$$

$$u(x,y) = G(x)H(y) \Rightarrow \boxed{u(x,y) = K\cdot\exp\left(Cx + \frac{2y}{C}\right)}$$

Prob 3 Solution Thanks to Cyle Teal

3) $$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2}$$

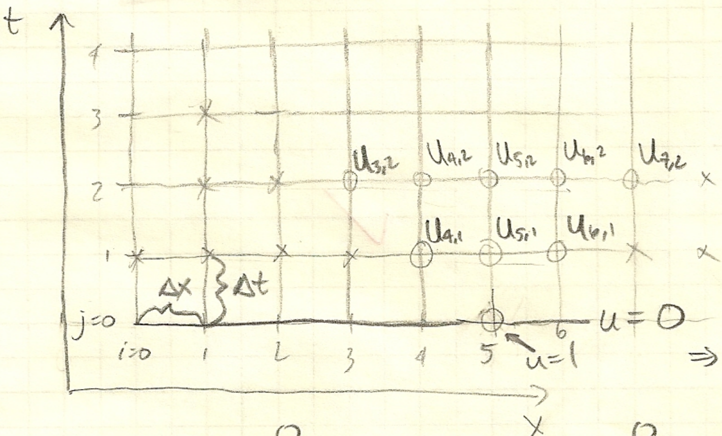
(a) $$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})}{(\Delta x)^2}$$

$$u_{i,j+1} = \frac{2\Delta t}{(\Delta x)^2}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) + u_{i,j}$$

$$\Rightarrow u_{i,j+1} = \frac{2\Delta t}{(\Delta x)^2}u_{i-1,j} + \left(\frac{-4\Delta t}{(\Delta x)^2} + 1\right)u_{i,j} + \frac{2\Delta t}{(\Delta x)^2}u_{i+1,j}$$

$$\Rightarrow \boxed{P = \frac{2\Delta t}{(\Delta x)^2} \qquad Q = 1 - \frac{4\Delta t}{(\Delta x)^2} \qquad R = \frac{2\Delta t}{(\Delta x)^2}}$$

(b)



$\Delta t = 0.1$
$\Delta x = .5$

$$\Rightarrow u_{i,j+1} = .8u_{i-1,j} - .6u_{i,j} + .8u_{i+1,j}$$

$$u_{4,1} = \frac{2(0.1)}{(.5)^2}\cancel{u_{3,0}}^{\,0} + \left(1 - \frac{4(.1)}{(.5)^2}\right)\cancel{u_{4,0}}^{\,0} + \frac{2(.1)}{(.5)^2}u_{5,0}$$

$$\Rightarrow u_{4,1} = \frac{2(.1)}{(.5)^2}(1) \Rightarrow \boxed{u_{4,1} = .8}$$

$$u_{5,1} = \left(1 - \frac{4(.1)}{(.5)^2}\right)u_{5,0} \Rightarrow \boxed{u_{5,1} = -.6}$$

$$u_{6,1} = \frac{2(.1)}{(.5)^2}u_{5,0} \Rightarrow \boxed{u_{6,1} = .8}$$

(Continued)

(Prob 3 Solution continued)

$$u_{3,2} = .8\,\cancelto{0}{u_{2,1}} - .6\,\cancelto{0}{u_{3,1}} + .8\,u_{4,1}$$

$$\Rightarrow u_{3,2} = .8(.8) \Rightarrow \boxed{u_{3,2} = .64}$$

$$u_{4,2} = .8\,\cancelto{0}{u_{3,1}} - .6\,u_{4,1} + .8\,u_{5,1}$$

$$\Rightarrow u_{4,2} = -.6(.8) + .8(-.6) \Rightarrow \boxed{u_{4,2} = -.96}$$

$$u_{5,2} = .8u_{4,1} - .6\,u_{5,1} + .8(u_{6,1})$$

$$u_{5,2} = .8(.8) - .6(-.6) + .8(.8)$$

$$\Rightarrow \boxed{u_{5,2} = 1.64}$$

$$u_{6,2} = .8\,u_{5,1} - .6\,u_{6,1} + .8\,\cancelto{0}{u_{7,1}}$$

$$u_{6,2} = .8(-.6) - .6(.8) \Rightarrow \boxed{u_{6,2} = -.96}$$

$$u_{7,2} = .8u_{6,1} - .6\,\cancelto{0}{u_{7,1}} + .8\,\cancelto{0}{u_{8,1}}$$

$$u_{7,2} = .8(.8) \Rightarrow \boxed{u_{7,2} = .64}$$