Introduction to Partial Differential Equation - I. Quick overview

To help explain the correspondence between a PDE and a real world phenomenon, we will use t to denote time and (x, y, z) to denote the 3 spatial coordinates

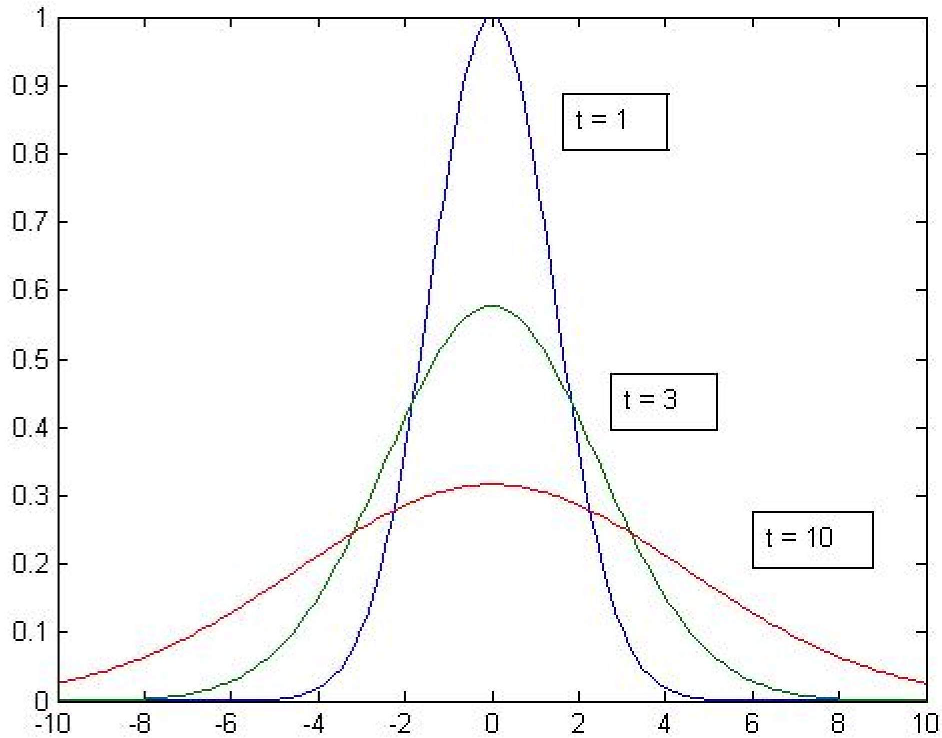
Some "classical" linear PDEs

Heat (or diffusion) equation: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, *describes the diffusion of temperature or the density of a chemical constituent from an initially concentrated distribution (e.g., a "hot spot" on a metal rod, or a speck of pollutant in the open air)*

A typical solution (when the initial distribution of u is a "spike"): $u(x,t) = \frac{1}{\sqrt{t}} \exp(-\frac{x^2}{4t})$

(Exercise: Verify that this solution does satisfy the original equation)

The figure in next page shows this solution at a few different times. As time increases, $u(x)$ becomes broader, its maximum decreases, but its "center of mass" does not move. These features characterize a "diffusion process".

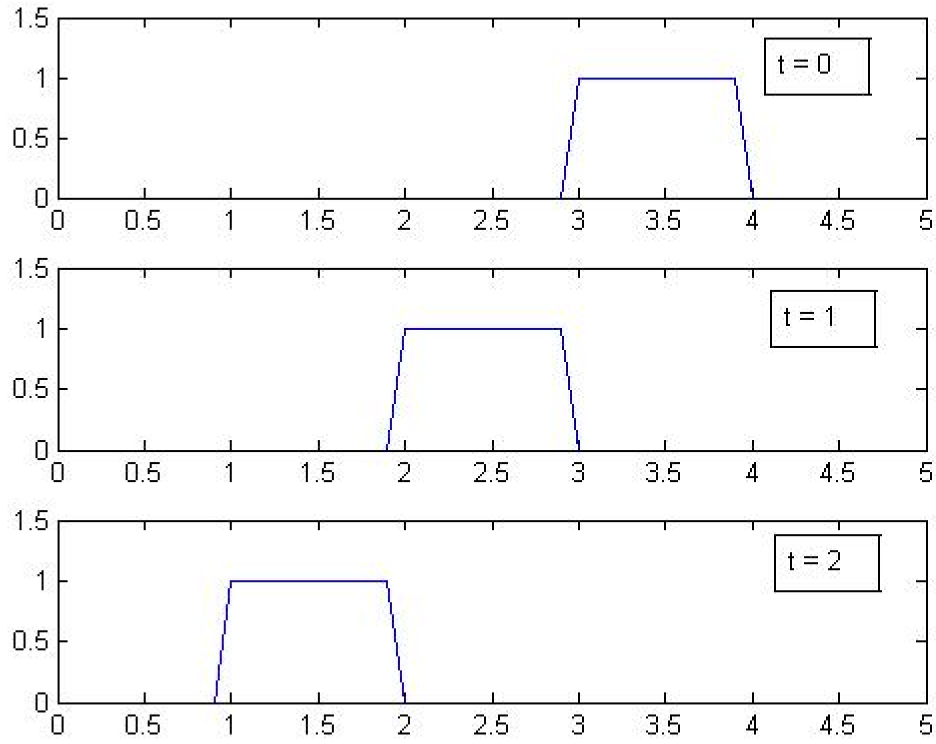


Solution of the heat equation at different times. The three curves are $u(x, 1)$, $u(x, 3)$, and $u(x,10)$

(How this solution is obtained is beyond the scope of this course - not to worry about the detail.)

Linear advection equation: $\dfrac{\partial u}{\partial t} = c\,\dfrac{\partial u}{\partial x}$ *, describes the constant movement of an initial distribution of u with a "speed" of* $-c$ *along the x-axis. The distribution moves while preserving its shape.*

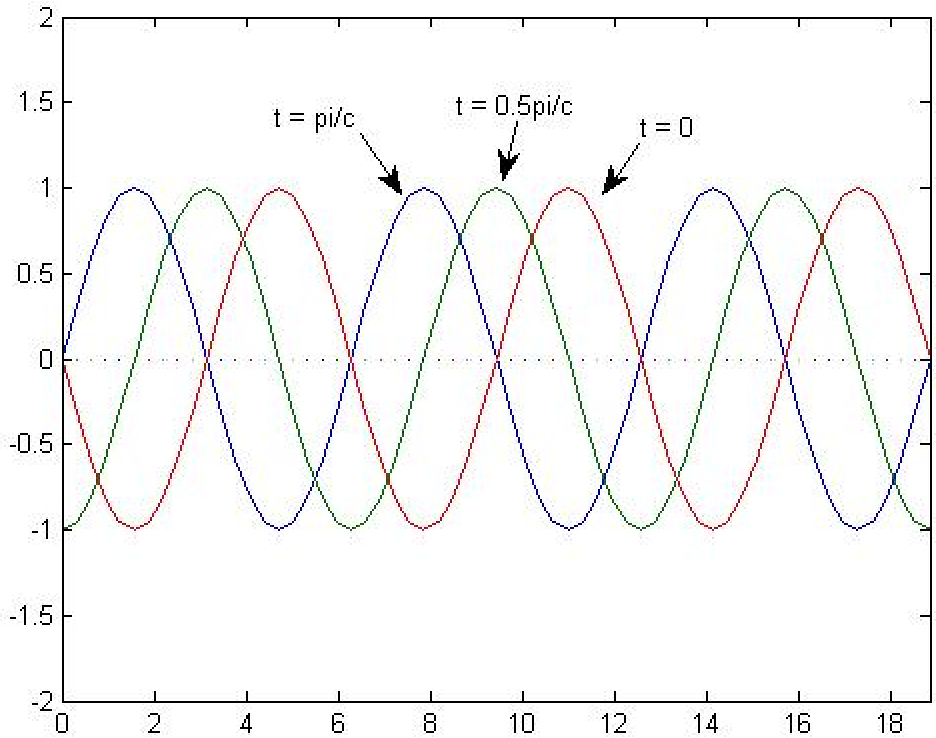
A typical solution: $u(x, t) = F(\xi)$, $\xi \equiv x + ct$; F can be any function that depends only on $x+ct$.
(Exercise: Verify that this is indeed a solution of the original equation.)
The following figure illustrate the behavior of the solution with c = 1. The initial condition, $u(x, t = 0)$, is a "top hat" structure. At later times, this structure moves to the left with a "speed" of $\delta x/\delta t = -1$ while preserving its shape. (The δx and δt here are the increments in space and time in the following diagrams.)



The 3 panels are $u(x, 0)$, $u(x, 1)$, and $u(x, 2)$

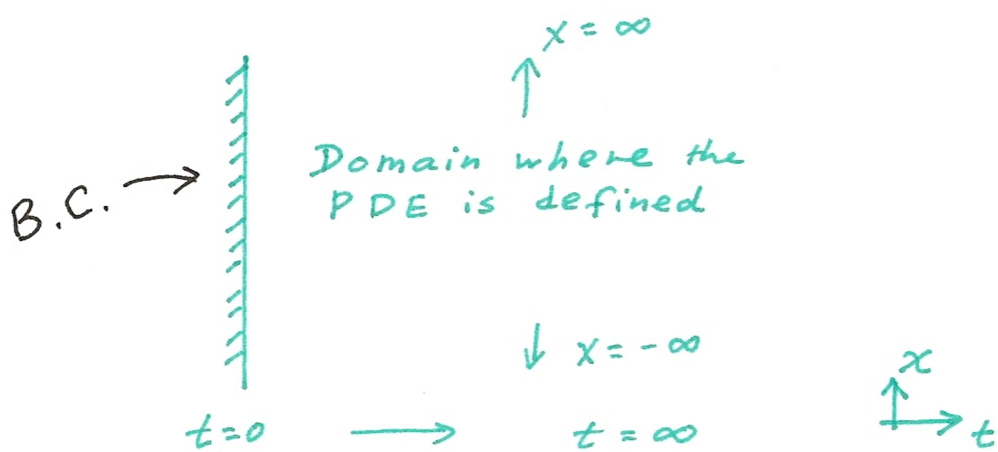
Linear wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, *describes wave motion*

For example, a simple traveling sinusoidal structure, $u(x, t) = sin(x + ct)$, as illustrated below, is a solution of the equation. (While at this level the solution is similar to that of the linear advection equation, more interesting behavior would emerge when we consider the superposition of different sinusoidal "modes", and when we introduce more interesting boundary conditions for the two equations. We will skip this detail.)



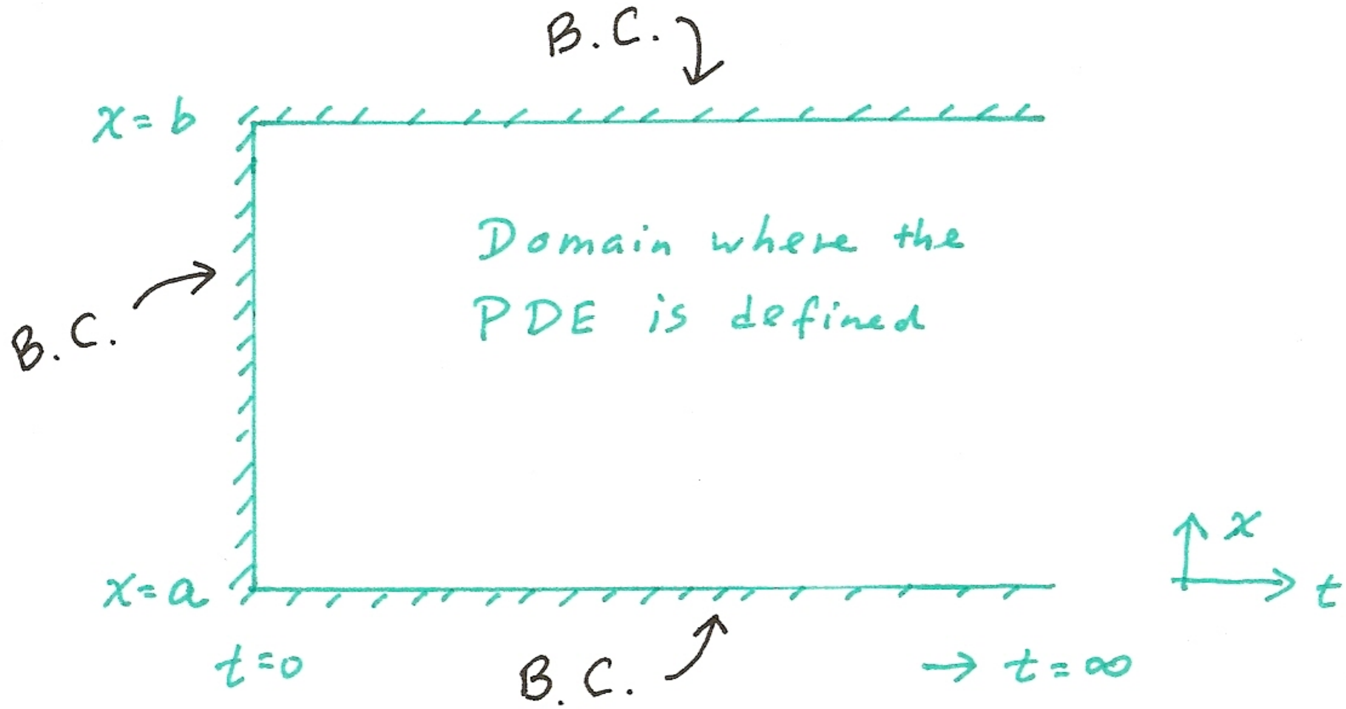
Boundary conditions (I)

In the three examples discussed above, we have not emphasized the role of boundary conditions. The simple solutions of the heat equation and linear advection equation are valid for the unbounded domain in x, $x \in (\infty, -\infty)$, and "semi-infinite" domain in t, $t \in [0, \infty)$, and under the boundary condition that u is well-behaved (e.g., decays to zero) as $x \to \infty$ and $x \to -\infty$. A PDE that is defined on the semi-infinite domain in time is often called an "evolution equation", which is to be solved with a given initial state, $u(x, 0) = G(x)$. The following diagram illustrate the domain for a PDE defined on $x \in (\infty, -\infty)$ and $t \in [0, \infty)$. Note that the prescribed initial state provides the "boundary condition" at the "wall", $t = 0$, of the domain.



Boundary conditions (II)

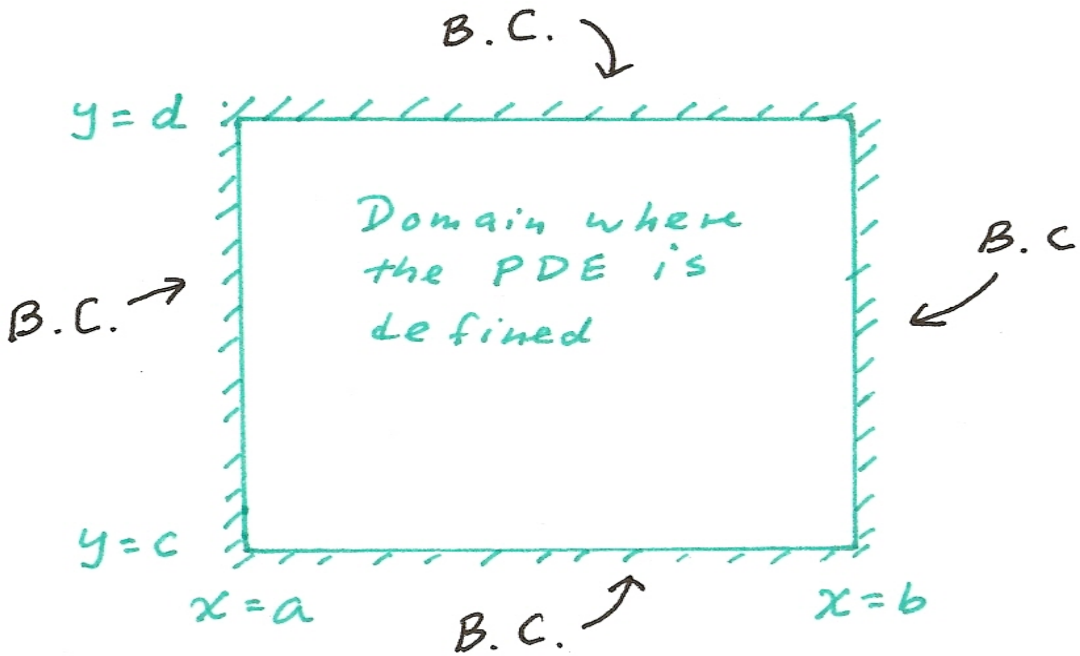
In real world applications, the heat equation is often defined on a finite interval in x, $x \in [a, b]$ (consider the problem of describing the temperature distribution on a finite metal rod), and a semi-infinite domain in t as before. The following diagram illustrates the domain for the PDE in this case. In addition to the boundary condition at $t = 0$, $u(x,0) = G(x)$, two more b.c. are needed at $x = a$ and $x = b$ for all t. They can be written as $u(a, t) = P(t)$ and $u(b, t) = Q(t)$. Note that $G(x)$ itself has to satisfy the boundary conditions of $G(a) = P(0)$ and $G(b) = Q(0)$. The prescription of $P(t)$, $Q(t)$ and $G(x)$ on the three "walls" of the domain is necessary for solving the PDE.



Boundary conditions (III), and Laplace equation

There are yet situations when the PDE may be defined on a closed domain. A famous example is the Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. (It belongs to the more general class of *elliptic equations*.)

The closed domain is illustrated in the following. In this case, boundary conditions need to be specified at all of the four walls.



Different types of PDEs often need to be matched with different types of boundary conditions in order for their solutions to exist and be unique.

Heat equation in two- and three-dimensions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \text{(2-D)}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \qquad \text{(3-D)}$$

The behavior of the solutions of these equations is similar to that of the 1-D heat equation. An initially concentrated distribution in u will spread in space and become more smooth as t increases.

For the 2-D case and for a closed domain with u specified on the "walls", the solution may reach "equilibrium" as $t \to \infty$. At this limit, u ceases to change further so $\partial u/\partial t \approx 0$. Then, the 2-D heat equation is reduced to the 2-D Laplace equation discussed before. In other words, the Laplace equation describes the equiribrium solution (or "steady state solution") of the 2-D heat diffusion problem.