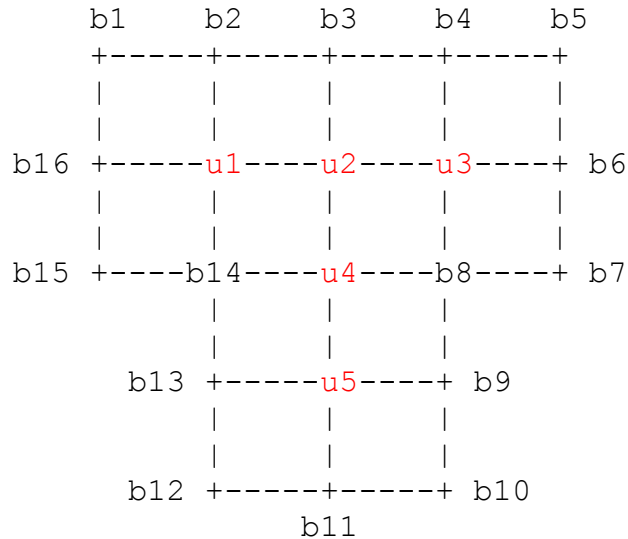


## Introduction to Partial Differential Equation (IV): Further examples for Laplace Equation

**Example 1** (irregular domain): Solve Laplace equation within a T-shaped domain as shown below. For the simplest case (assume  $\Delta x = \Delta y$  and with minimum resolution), we have

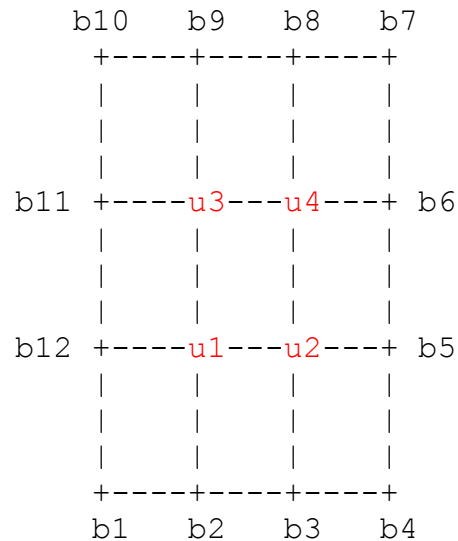


where  $u_N$  are unknowns and  $b_N$  are known values of  $u(x,y)$  given by the boundary conditions. Using 2nd order central difference in  $x$  and  $y$ , the discretized version of Laplace equation becomes

$$\begin{aligned}
 -4 u_1 + u_2 &= -(b_2+b_{14}+b_{16}) \\
 u_1 - 4 u_2 + u_3 + u_4 &= -b_3 \\
 u_2 + -4 u_3 &= -(b_4+b_6+b_8) \\
 u_2 - 4 u_4 + u_5 &= -(b_8+b_{14}) \\
 u_4 - 4 u_5 &= -(b_9+b_{11}+b_{13})
 \end{aligned}$$

which can be readily solved by standard matrix manipulations (e.g., Gauss elimination).

**Example 2** (unequal  $\Delta x$  and  $\Delta y$ ): Solve Laplace equation within a tall rectangular domain with the grid shown below and with the setting of  $\Delta y = 2 \Delta x$ . Use 2nd order central difference scheme for both x and y.



Recall that the finite difference formula for Laplace equation is (using the convention that  $u(i,j)$  is the value of  $u$  at the  $i$ -th grid point in  $x$  and  $j$ -th grid point in  $y$ )

$$\frac{u(i+1,j)-2u(i,j)+u(i-1,j)}{(\Delta x)^2} + \frac{u(i,j+1)-2u(i,j)+u(i,j-1)}{(\Delta y)^2} = 0$$

With  $\Delta y = 2\Delta x$ , it becomes

$$4u(i+1,j)-8u(i,j)+4u(i-1,j)+u(i,j+1)-2u(i,j)+u(i,j-1) = 0 ,$$

or

$$-10u(i,j)+4u(i+1,j)+4u(i-1,j)+u(i,j+1)+u(i,j-1) = 0.$$

Using this formula and the grid system shown above, we have

$$\begin{aligned} -10 u_1 + 4 u_2 + u_3 &= -(4 b_{12} + b_2) \\ 4 u_1 - 10 u_2 + u_4 &= -(b_3 + 4 b_5) \\ u_1 - 10 u_3 + 4 u_4 &= -(b_9 + 4 b_{11}) \\ u_2 + 4 u_3 - 10 u_4 &= -(4 b_6 + b_8) \end{aligned} ,$$

which can be readily solved with typical matrix manipulation. Beware that the unequal weight for  $x$  and  $y$  has to be applied to the boundary values, too. For example, in the right hand side of the equation for the first row, we have a factor of 4 in front of  $b_{12}$  but a factor of only 1 in front of  $b_2$ .