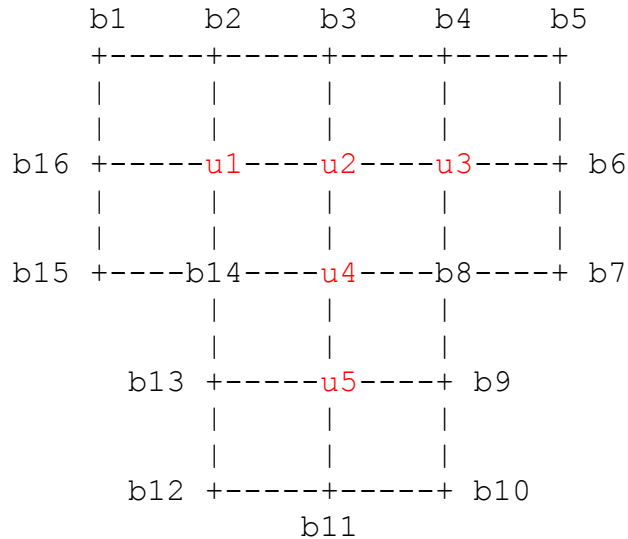


Introduction to Partial Differential Equation (IV): Further examples for Laplace Equation

Example 1 (irregular domain): Solve Laplace equation within a T-shaped domain as shown below. For the simplest case (assume $\Delta x = \Delta y$ and with minimum resolution), we have

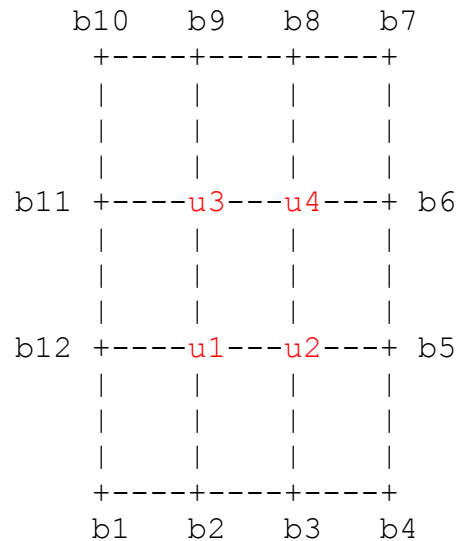


where u_N are unknowns and b_N are known values of $u(x,y)$ given by the boundary conditions. Using 2nd order central difference in x and y , the discretized version of Laplace equation becomes

$$\begin{aligned}
 -4 u_1 + u_2 &= -(b_2+b_{14}+b_{16}) \\
 u_1 - 4 u_2 + u_3 + u_4 &= -b_3 \\
 u_2 + -4 u_3 &= -(b_4+b_6+b_8) \\
 u_2 - 4 u_4 + u_5 &= -(b_8+b_{14}) \\
 u_4 - 4 u_5 &= -(b_9+b_{11}+b_{13})
 \end{aligned}$$

which can be readily solved by standard matrix manipulations (e.g., Gauss elimination).

Example 2 (unequal Δx and Δy): Solve Laplace equation within a tall rectangular domain with the grid shown below and with the setting of $\Delta y = 2 \Delta x$. Use 2nd order central difference scheme for both x and y.



Recall that the finite difference formula for Laplace equation is (using the convention that $u(i,j)$ is the value of u at the i -th grid point in x and j -th grid point in y)

$$\frac{u(i+1,j) - 2u(i,j) + u(i-1,j)}{(\Delta x)^2} + \frac{u(i,j+1) - 2u(i,j) + u(i,j-1)}{(\Delta y)^2} = 0$$

With $\Delta y = 2\Delta x$, it becomes

$$4u(i+1,j) - 8u(i,j) + 4u(i-1,j) + u(i,j+1) - 2u(i,j) + u(i,j-1) = 0 ,$$

or

$$-10u(i,j) + 4u(i+1,j) + 4u(i-1,j) + u(i,j+1) + u(i,j-1) = 0 .$$

Using this formula and the grid system shown above, we have

$$\begin{aligned} -10 u_1 + 4 u_2 + u_3 &= -(4 b_{12} + b_2) \\ 4 u_1 - 10 u_2 + u_4 &= -(b_3 + 4 b_5) \\ u_1 - 10 u_3 + 4 u_4 &= -(b_9 + 4 b_{11}) \\ u_2 + 4 u_3 - 10 u_4 &= -(4 b_6 + b_8) \end{aligned} ,$$

which can be readily solved with typical matrix manipulation. Beware that the unequal weight for x and y has to be applied to the boundary values, too. For example, in the right hand side of the equation for the first row, we have a factor of 4 in front of b_{12} but a factor of only 1 in front of b_2 .