Introduction to Partial Differential Equation (IV): Further examples for Laplace Equation

Example 1 (irregular domain): Solve Laplace equation within a T-shaped domain as shown below. For the simplest case (assume $\Delta x = \Delta y$ and with minimum resolution), we have

```
      b1    b2    b3    b4    b5
      +-----+-----+-----+-----+
      |     |     |     |     |
      |     |     |     |     |
  b16 +-----u1----u2----u3----+ b6
      |     |     |     |     |
      |     |     |     |     |
  b15 +----b14----u4----b8----+ b7
            |     |     |
            |     |     |
        b13 +-----u5----+ b9
            |     |     |
            |     |     |
        b12 +-----+-----+ b10
                 b11
```

where u*N* are unknowns and b*N* are known values of u(x,y) given by the boundary conditions. Using 2nd order central difference in x and y, the discretized version of Laplace equation becomes

```
     -4 u1 +   u2                        = -(b2+b14+b16)
        u1 - 4 u2 +    u3  +   u4        = - b3
               u2 + -4 u3                = -(b4+b6+b8)
               u2          - 4 u4 +   u5 = -(b8+b14)
                               u4 - 4 u5 = -(b9+b11+b13)
```

which can be readily solved by standard matrix manipulations (e.g., Gauss elimination).

Example 2 (unequal Δx and Δy): Solve Laplace equation within a tall rectangular domain with the grid shown below and with the setting of $\Delta y = 2\ \Delta x$. Use 2nd order central difference scheme for both x and y.

```
          b10   b9   b8   b7
           +----+----+----+
           |    |    |    |
           |    |    |    |
           |    |    |    |
       b11 +----u3---u4---+ b6
           |    |    |    |
           |    |    |    |
           |    |    |    |
       b12 +----u1---u2---+ b5
           |    |    |    |
           |    |    |    |
           |    |    |    |
           +----+----+----+
           b1   b2   b3   b4
```

Recall that the finite difference formula for Laplace equation is (using the convention that u(i,j) is the value of u at the i-th grid point in x and j-th grid point in y)

$$\frac{u(i+1,j)-2u(i,j)+u(i-1,j)}{(\Delta x)^2} + \frac{u(i,j+1)-2u(i,j)+u(i,j-1)}{(\Delta y)^2} = 0$$

With $\Delta y = 2\Delta x$, it becomes

$$4u(i+1,j)-8u(i,j)+4u(i-1,j)+u(i,j+1)-2u(i,j)+u(i,j-1) = 0 ,$$

or

$$-10u(i,j)+4u(i+1,j)+4u(i-1,j)+u(i,j+1)+u(i,j-1) = 0.$$

Using this formula and the grid system shown above, we have

$$\begin{aligned}
-10\,u_1 + 4\,u_2 + u_3 \phantom{{}+u_4} &= -(4\,b_{12} + b_2) \\
4\,u_1 - 10\,u_2 + u_4 &= -(b_3 + 4\,b_5) \\
u_1 - 10\,u_3 + 4\,u_4 &= -(b_9 + 4\,b_{11}) \\
u_2 + 4\,u_3 - 10\,u_4 &= -(4\,b_6 + b_8) \quad ,
\end{aligned}$$

which can be readily solved with typical matrix manipulation. Beware that the unequal weight for x and y has to be applied to the boundary values, too. For example, in the right hand side of the equation for the first row, we have a factor of 4 in front of b_{12} but a factor of only 1 in front of b_2.