MAE384 Fall 2011 Homework #2

1. Using the *Fixed-point iteration method*, find all of the solutions for the equation,

$$x\,e^{x} - x^{4} + 1 = 0$$

within the interval of $-3 < x < 3$. Must show your choice of the specific functional form and the corresponding initial guess for the iteration for each solution. [**3 points]**

2. A system of linear equations is given as

$$\begin{aligned} 6x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 5x_2 + 2x_3 &= 13 \\ x_1 + x_2 + 6x_3 &= 6\ . \end{aligned} \qquad \text{Eq. (1)}$$

(a) Solve the system by *Gauss elimination*. Show your procedure.
(b) Solve the system by the *Jacobi iterative method*, using (x_1, x_2, x_3) = (0,0,0) as the initial guess.
(c) Solve the system by the *Gauss-Seidel iterative method*, using (x_2, x_3) = (0,0) as the initial guess.
In (b) and (c), the outcome of the iterative process will be considered satisfactory if the "numerical error" defined in Part (d) is reduced to $E < 0.01$.

(d) In (b) and (c), if the exact solution of the linear system is not known, one way to measure the success of the iterative process is by plugging the numerical solution back to Eq. (1) and check the discrepancy between the l.f.s. and r.h.s. More precisely, Eq. (1) can be written as

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\ ,$$

where

$$\boldsymbol{A} \equiv \begin{pmatrix} 6 & 1 & 2 \\ 2 & 5 & 2 \\ 1 & 1 & 6 \end{pmatrix}\ ,\quad \boldsymbol{x} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\ , \text{ and } \quad \boldsymbol{b} \equiv \begin{pmatrix} 9 \\ 13 \\ 6 \end{pmatrix}\ .$$

The "residual" vector that quantifies the discrepancy can be defined as

$$\boldsymbol{r} \equiv \boldsymbol{A}\,\boldsymbol{x_S} - \boldsymbol{b}\ ,$$

where $\boldsymbol{x_S}$ is the numerical solution. Then, the final measure of the "numerical error" can be defined as the length of the $\boldsymbol{r}$ vector. For this problem, we adopt the *Euclidean 2-norm* (Eq. (4.72) in textbook),

$$E \equiv \|\boldsymbol{r}\| \equiv \sqrt{r_1^{\,2} + r_2^{\,2} + r_3^{\,2}}\ ,$$

where r_1, r_2, and r_3 are the 3 components of the $\boldsymbol{r}$ vector. Using this definition, evaluate E after every iteration in (b) and (c) and plot them as a function of the number of iterations in order to compare the efficiency of the Jacobi and Gauss-Seidel method. Comment on the result.

(e) Using the *Euclidean norm* for matrix (Eq. 4.76 in textbook), evaluate the *condition number* for the system in Eq. (1). Is this system ill-conditioned?
[5 points]