Prob 1 Solution (Thanks to Kevin Antrosiglio)

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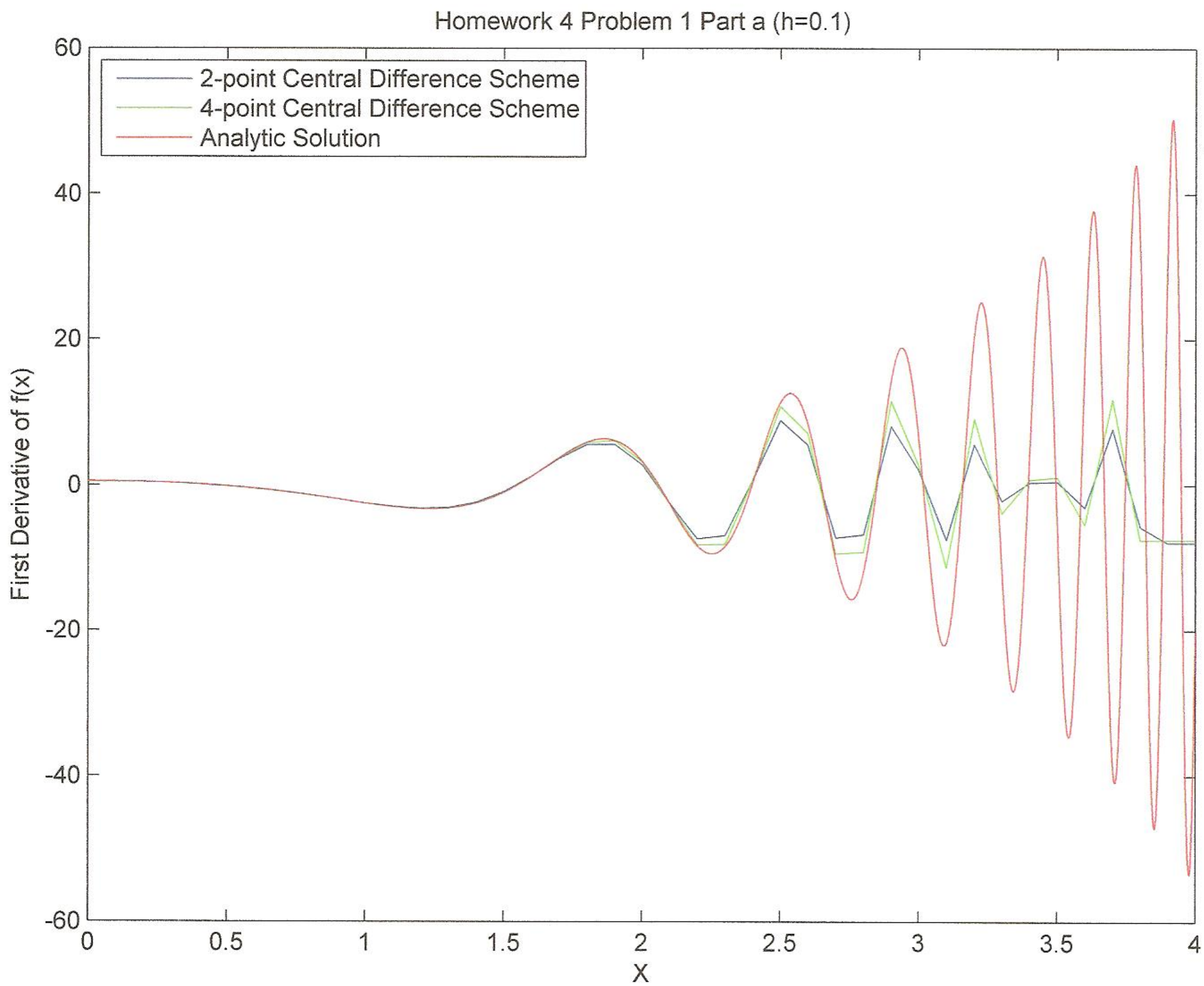
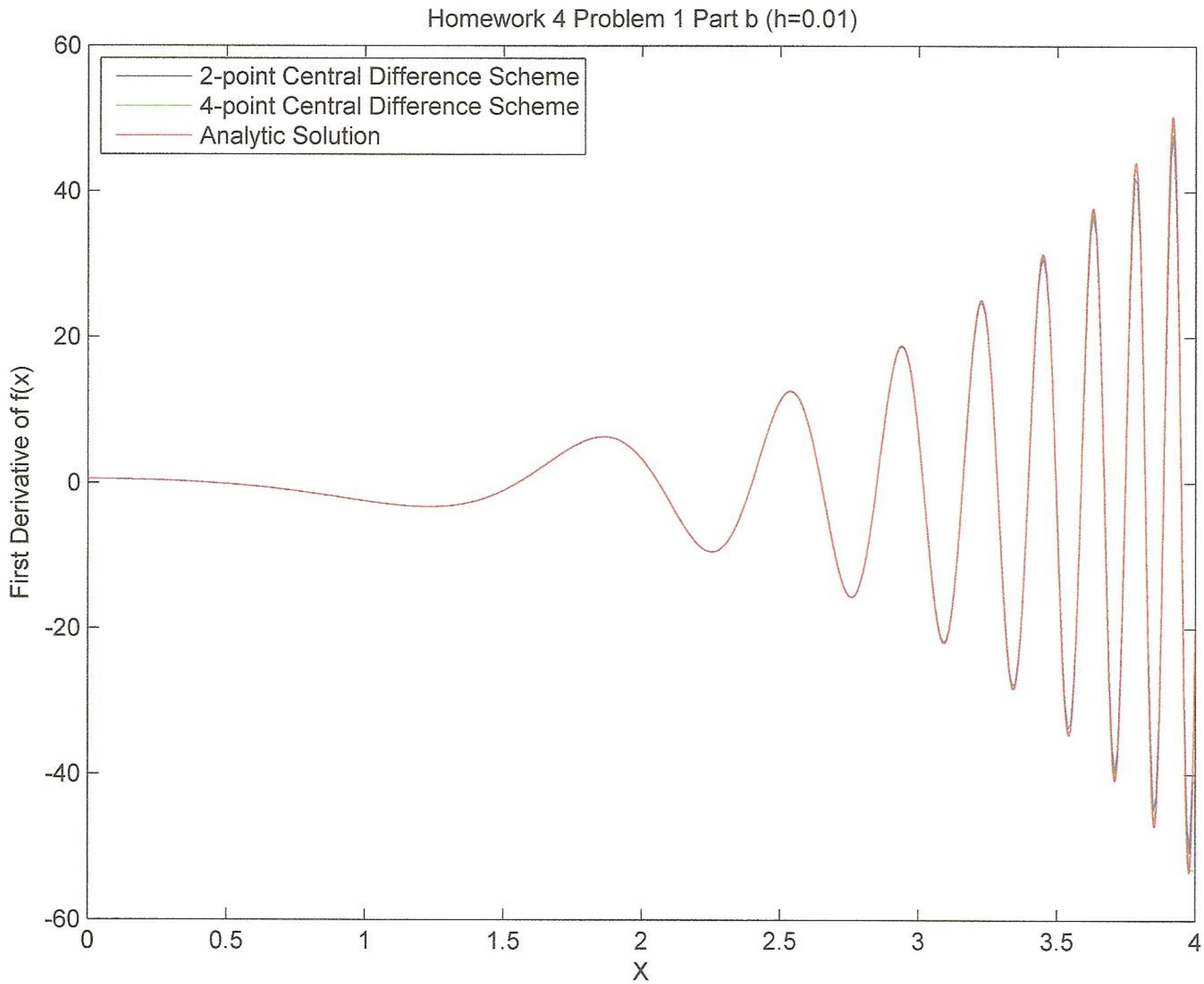
% HOMEWORK #4 PROBLEM #1
x1 = 0:0.1:4;
h1 = 0.1;
for z1 = 2:40
     central2(z1) = (sin(exp(x1(z1+1))) - sin(exp(x1(z1-1))))/(2*h1);
end
central2(1) = central2(2);
central2(41) = central2(40);
plot(x1,central2)
hold on
for z2 = 3:39
    central4(z2) = (sin(exp(x1(z2-2))) - 8*sin(exp(x1(z2-1))) + 8*sin(exp(x1(z2+1))) -↙
sin(exp(x1(z2+2))))/(12*h1);
end
central4(1) = central4(3);
central4(2) = central4(3);
central4(40) = central4(39);
central4(41) = central4(39);
plot(x1,central4,'g')
x = 0:0.0001:4;
fprime = exp(x).*cos(exp(x));
plot(x,fprime,'r')
legend('2-point Central Difference Scheme','4-point Central Difference Scheme','Analytic↙
Solution','Location','NorthWest')
xlabel('X')
ylabel('First Derivative of f(x)')
title('Homework 4 Problem 1 Part a (h=0.1)')
x2 = 0:0.01:4;
h2 = 0.01;
for z3 = 2:400
     central_2(z3) = (sin(exp(x2(z3+1))) - sin(exp(x2(z3-1))))/(2*h2);
end
central_2(1) = central_2(2);
central_2(401) = central_2(400);
figure
plot(x2,central_2)
hold on
for z4 = 3:399
    central_4(z4) = (sin(exp(x2(z4-2))) - 8*sin(exp(x2(z4-1))) + 8*sin(exp(x2(z4+1))) -↙
sin(exp(x2(z4+2))))/(12*h2);
end
central_4(1) = central_4(3);
central_4(2) = central_4(3);
central_4(400) = central_4(399);
central_4(401) = central_4(399);
plot(x2,central_4,'g')
plot(x,fprime,'r')
legend('2-point Central Difference Scheme','4-point Central Difference Scheme','Analytic↙
Solution','Location','NorthWest')
xlabel('X')
ylabel('First Derivative of f(x)')
```

Plots in next 2 pages


Homework 4 Problem 1 Part a (h=0.1)
2-point Central Difference Scheme
4-point Central Difference Scheme
Analytic Solution
First Derivative of f(x)
X
60
40
20
0
-20
-40
-60
0
0.5
1
1.5
2
2.5
3
3.5
4



Note: For h = 0.01 the numerical solutions are almost indistinguishable from the analytic solution.

Prob 2 Solution (Thanks to Olen Hatch)

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2} + \frac{f'''(x_i)h^3}{6} + Oh^4$$

$$f(x_{i+2}) = f(x_i) + f'(x_i)(3h) + \frac{f''(x_i)(3h)^2}{2} + \frac{f'''(x_i)(3h)^3}{6} + Oh^4$$

$$f(x_{i-1}) = f(x_i) + f'(x_i)(-2h) + \frac{f''(x_i)(-2h)^2}{2} + \frac{f'''(x_i)(-2h)^3}{6} + Oh^4$$

$$a\left(f(x_{i+1}) = f(x_i) + f'(x_i)h + \tfrac{1}{2}f''(x_i)h^2 + \tfrac{1}{6}f'''(x_i)h^3 + Oh^4 \right)$$

$$b\left(f(x_{i+2}) = f(x_i) + 3f'(x_i)h + \tfrac{9}{2}f''(x_i)h^2 + \tfrac{27}{6}f'''(x_i)h^3 + Oh^4 \right)$$

$$1\left(f(x_{i-1}) = f(x_i) - 2f'(x_i)h + 2f''(x_i)h^2 - \tfrac{8}{6}f'''(x_i)h^3 + Oh^4 \right)$$

$$a + 3b - 2 = 0$$
$$\tfrac{1}{6}a + \tfrac{27}{6}b - \tfrac{8}{6} = 0$$

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 1/6 & \frac{27}{6} & 8/6 \end{array}\right] \xrightarrow[\text{TI-83 Plus}]{\text{RREF}} \left[\begin{array}{ccc} 1 & 0 & 5/4 \\ 0 & 1 & 1/4 \end{array}\right]$$

$$a = 5/4 \quad b = \tfrac{1}{4}$$

$$\tfrac{5}{4}f(x_{i+1}) = \tfrac{5}{4}f(x_i) + \tfrac{5}{4}f'(x_i)h + \tfrac{5}{8}f''(x_i)h^2 + \tfrac{5}{24}f'''(x_i)h^3 + Oh^4$$

$$\tfrac{1}{4}f(x_{i+2}) = \tfrac{1}{4}f(x_i) + \tfrac{3}{4}f'(x_i)h + \tfrac{9}{8}f''(x_i)h^2 + \tfrac{27}{24}f'''(x_i)h^3 + Oh^4$$

$$f(x_{i-1}) = f(x_i) - 2f'(x_i)h + 2f''(x_i)h^2 - \tfrac{8}{6}f'''(x_i)h^3 + Oh^4$$

$$f(x_{i-1}) + \tfrac{5}{4}f(x_{i+1}) + \tfrac{1}{4}f(x_{i+2}) = \tfrac{5}{2}f(x_i) + \tfrac{15}{4}f''(x_i)h^2 + Oh^4$$

$$\frac{4}{15h^2}\left[f(x_{i-1}) - \tfrac{5}{2}f(x_i) + \tfrac{5}{4}f(x_{i+1}) + \tfrac{1}{4}f(x_{i+2}) = \frac{15h^2}{4}f''(x_i) + Oh^4 \right]$$

$$\boxed{f''(x_i) = \frac{4}{15h^2}f(x_{i-1}) - \frac{2}{3h^2}f(x_i) + \frac{1}{3h^2}f(x_{i+1}) + \frac{1}{15h^2}f(x_{i+2}) + Oh^2}$$

thus $\boxed{A = \frac{4}{15h^2}}$ $\boxed{B = \frac{-2}{3h^2}}$ $\boxed{C = \frac{1}{3h^2}}$ $\boxed{D = \frac{1}{15h^2}}$

Prob 3 Solution (Thanks to Olen Hatch)

$$a\left\{ f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2} + \frac{f'''(x_i)h^3}{6} + \frac{f^{iv}(x_i)h^4}{24} + \frac{f^{v}(x_i)h^5}{120} + O h^6 \right\}$$

$$b\left[f(x_{i+2}) = f(x_i) + f'(x_i)(2h) + \frac{f''(x_i)(2h)^2}{2} + \frac{f'''(x_i)(2h)^3}{6} + \frac{f^{iv}(x_i)(2h)^4}{24} + \frac{f^{v}(x_i)(2h)^5}{120} + O h^6 \right]$$

$$c\left[f(x_{i+3}) = f(x_i) + f'(x_i)(3h) + \frac{f''(x_i)(3h)^3}{2} + \frac{f'''(x_i)(3h)^3}{6} + \frac{f^{iv}(x_i)(3h)^4}{24} + \frac{f^{v}(x_i)(3h)^5}{120} + O h^6 \right]$$

$$d\left[f(x_{i-1}) = f(x_i) + f'(x_i)(-h) + \frac{f''(x_i)(-h)^2}{2} + \frac{f'''(x_i)(-h)^3}{6} + \frac{f^{iv}(x_i)(-h)^4}{24} + \frac{f^{v}(x_i)(-h)^5}{120} + O h^6 \right]$$

$$1\left\{ f(x_{i-2}) = f(x_i) + f'(x_i)(-2h) + \frac{f''(x_i)(-2h)^2}{2} + \frac{f'''(x_i)(-2h)^2}{6} + \frac{f^{iv}(x_i)(-2h)^4}{24} + \frac{f^{v}(x_i)(-2h)^5}{120} + O h^6 \right\}$$

$$\frac{a}{2} + \frac{4b}{2} + \frac{9c}{2} + \frac{d}{2} + \frac{4}{2} = 0$$

$$\frac{a}{6} + \frac{8b}{6} + \frac{27c}{6} - \frac{d}{6} - \frac{8}{6} = 0$$

$$\frac{a}{24} + \frac{16b}{24} + \frac{81c}{24} + \frac{d}{24} + \frac{16}{24} = 0$$

$$\frac{a}{120} + \frac{32b}{120} + \frac{243c}{120} - \frac{d}{120} - \frac{32}{120} = 0$$

$$\left[\begin{array}{cccc|c} 1/2 & 2 & 9/2 & 1/2 & -2 \\ 1/6 & 4/3 & 9/2 & -1/6 & 4/3 \\ 1/24 & 16/24 & 81/24 & 1/24 & -16/24 \\ 1/120 & 32/120 & 243/120 & -1/120 & 32/120 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 2/3 \\ 0 & 0 & 0 & 1 & -10 \end{array}\right]$$

$a = 20 \quad b = -5 \quad c = \frac{2}{3} \quad d = -10$

Continue to next page

Prob 3 continued

$$20\left[f(x_{i+1}) = f(x_i) + f'(x_i)h + \tfrac{1}{2}f''(x_i)h^2 + \tfrac{1}{6}f'''(x_i)h^3 + \tfrac{1}{24}f^{iv}(x_i)h^4 + \tfrac{1}{120}f^{v}(x_i)h^5 + Oh^6\right]$$

$$-5\left[f(x_{i+2}) = f(x_i) + 2f'(x_i)h + 2f''(x_i)h^2 + \tfrac{8}{6}f'''(x_i)h^3 + \tfrac{16}{24}f^{iv}(x_i)h^4 + \tfrac{32}{120}f^{v}(x_i)h^5 + Oh^6\right]$$

$$2/3\left[f(x_{i+3}) = f(x_i) + 3f'(x_i)h + \tfrac{9}{2}f''(x_i)h^2 + \tfrac{27}{6}f'''(x_i)h^3 + \tfrac{81}{24}f^{iv}(x_i)h^4 + \tfrac{243}{120}f^{v}(x_i)h^5 + Oh^6\right]$$

$$-10\left[f(x_{i-1}) = f(x_i) - f'(x_i)h + \tfrac{1}{2}f''(x_i)h^2 - \tfrac{1}{6}f'''(x_i)h^3 + \tfrac{1}{24}f^{iv}(x_i)h^4 - \tfrac{1}{120}f^{v}(x_i)h^5 + Oh^6\right]$$

$$1\left[f(x_{i-2}) = f(x_i) - 2f'(x_i)h + \tfrac{4}{2}f''(x_i)h^2 - \tfrac{8}{6}f'''(x_i)h^3 + \tfrac{16}{24}f^{iv}(x_i)h^4 - \tfrac{32}{120}f^{v}(x_i)h^5 + Oh^6\right]$$

$$20f(x_{i+1}) = 20f(x_i) + 20f'(x_i)h + 10f''(x_i)h^2 + \tfrac{20}{6}f'''(x_i)h^3 + \tfrac{20}{24}f^{iv}(x_i)h^4 + \tfrac{20}{120}f^{v}(x_i)h^5 + Oh^6$$

$$-5f(x_{i+2}) = -5f(x_i) - 10f'(x_i)h - 10f''(x_i)h^2 - \tfrac{40}{6}f'''(x_i)h^3 - \tfrac{80}{24}f^{iv}(x_i)h^4 - \tfrac{160}{120}f^{v}(x_i)h^5 + Oh^6$$

$$\tfrac{2}{3}f(x_{i+3}) = \tfrac{2}{3}f(x_i) + 2f'(x_i)h + 3f''(x_i)h^2 + 3f'''(x_i)h^3 + \tfrac{9}{4}f^{iv}(x_i)h^4 + \tfrac{27}{20}f^{v}(x_i)h^5 + Oh^6$$

$$-10f(x_{i-1}) = -10f(x_i) + 10f'(x_i)h - 5f''(x_i)h^2 + \tfrac{10}{6}f'''(x_i)h^3 - \tfrac{10}{24}f^{iv}(x_i)h^4 + \tfrac{10}{120}f^{v}(x_i)h^5 + Oh^6$$

$$f(x_{i-2}) = f(x_i) - 2f'(x_i)h + 2f''(x_i)h^2 - \tfrac{8}{6}f'''(x_i)h^3 + \tfrac{16}{24}f^{iv}(x_i)h^4 - \tfrac{32}{120}f^{v}(x_i)h^5 + Oh^6$$

$$f(x_{i-2}) - 10f(x_{i-1}) + 20f(x_{i+1}) - 5f(x_{i+2}) + \tfrac{2}{3}f(x_{i+3}) = \tfrac{20}{3}f(x_i) + 20f'(x_i)h + Oh^6$$

$$\frac{1}{20h}\left(f(x_{i-2}) - 10f(x_{i-1}) - \tfrac{20}{3}f(x_i) + 20f(x_{i+1}) - 5f(x_{i+2}) + \tfrac{2}{3}f(x_{i+3}) + Oh^6 = 20f'(x_i)h\right)$$

$$\boxed{f'(x_i) = \frac{1}{20h}f(x_{i-2}) - \frac{1}{2h}f(x_{i-1}) - \frac{1}{3h}f(x_i) + \frac{f(x_{i+1})}{h} - \frac{1}{4h}f(x_{i+2}) + \frac{1}{30h}f(x_{i+3}) + Oh^5}$$

So $\boxed{A = \frac{1}{20h}}$ $\boxed{B = -\frac{1}{2h}}$ $\boxed{C = -\frac{1}{3h}}$ $\boxed{D = \frac{1}{h}}$

$\boxed{E = -\frac{1}{4h}}$ $\boxed{F = \frac{1}{30h}}$

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