MAE384 Fall 2012 HW2 solutions

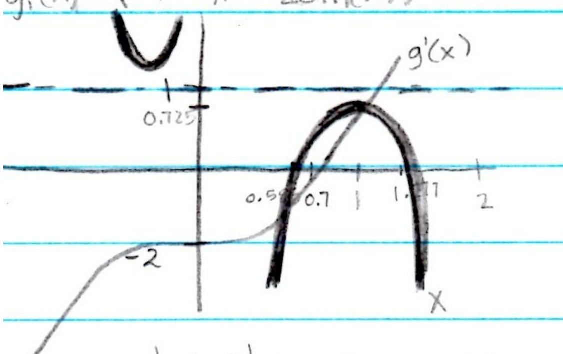
**Prob 1** The solutions are x = 0.5355 and x = 1.2899. There are many valid choices of "g(x)". See further discussion in p.2. An example of solution by hand (thanks to Shelby Rode):

1. $1-x^3+2\sin x-\frac{1}{x}=0$

$1-x^3+2\sin(x)=\frac{1}{x}$

$x=\dfrac{1}{(1-x^3+2\sin(x))}$

$g_1'(x)=\dfrac{3-x^2-2\cos(x)}{(-1+x^3-2\sin(x))^2}$




– since $|g'(x)|$ is less than one, g(x) is a valid selection

$x=g(x)$

$x={}^1/_{1-x^3+2\sin(x)}$

$x_0=0.7$

$x_1=g(x_0)={}^1/_{1-0.7^3+2\sin(0.7)}=0.514$

$x_2=g(x_1)={}^1/_{1-(.514)^3+2\sin(.514)}=0.541$

$x_3=g(x_2)={}^1/_{1-(.541)^3+2\sin(.541)}=0.534$

$x_4=g(x_3)={}^1/_{1-(.534)^3+2\sin(.534)}=0.536$

$x_5=g(x_4)={}^1/_{1-(.536)^3+2\sin(.536)}=0.536$

$\boxed{x_1=0.536}$

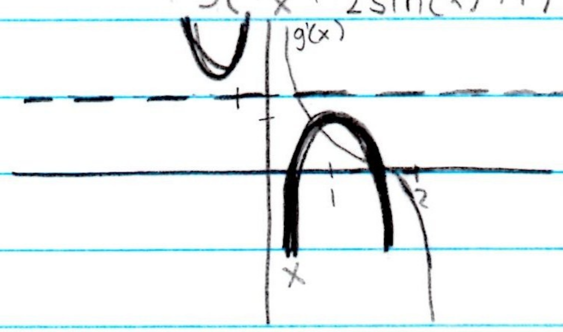
$1-x^3+2\sin x-\frac{1}{x}=0$

$x^3=1+2\sin(x)-\frac{1}{x}$

$x=\sqrt[3]{1+2\sin(x)-\frac{1}{x}}$

$\boxed{g_2(x)=\left(1+2\sin(x)-\frac{1}{x}\right)^{\frac{1}{3}}}$

$g_2'(x)=\dfrac{\frac{1}{x^2}+2\cos(x)}{3\left(-\frac{1}{x}+2\sin(x)+1\right)^{\frac{2}{3}}}$




– since $|g'(x)|$ is less than one, g(x) is a valid selection

$x=g(x)$

$x=\left(1+2\sin(x)-\frac{1}{x}\right)^{\frac{1}{3}}$

$x_0=1.5$

$x_1=\left(1+2\sin(1.5)-\frac{1}{1.5}\right)^{\frac{1}{3}}=1.325$

$x_2=\left(1+2\sin(1.325)-\frac{1}{1.325}\right)^{\frac{1}{3}}=1.298$

$x_3=\left(1+2\sin(1.298)-\frac{1}{1.298}\right)^{1/3}=1.292$

$x_4=\left(1+2\sin(1.292)-\frac{1}{1.292}\right)^{1/3}=1.29$

$x_5=\left(1+2\sin(1.29)-\frac{1}{1.29}\right)^{1/3}=1.29$

$\boxed{x_2=1.29}$

**Prob 1** Further discussion (by HPH)

There are many possible choices of g(x) that will lead to either the first or the second solution. It is useful to test if |g'(x)| < 1 over an interval that contains the target solution. (This can be visualized by superimposing the plots of f(x) and g'(x).) If so, an initial guess in that interval will converge to the corresponding solution. In the following, we list some of the good choices of g(x) for our problem.

For solution #1, x = 0.5355

| g(x) | Valid range of initial guess $x_1$ (we only tested the range of $0 < x_1 < 2$) | Remarks |
|---|---|---|
| $\dfrac{1}{1-x^3+2\sin(x)}$ | $0 < x_1 < 2$ | Most popular choice used by the majority of the class |
| $(x^{13}-2x^{10}\sin(x)+x^9)^{1/10}$ | $0 < x_1 < 1.25$ | Contributed by Misharr Rutnagar |
| $\dfrac{x^4+1}{1+2\sin(x)}$ | $0 < x_1 < 1.25$ | Contributed by Nolan Cheshire, Cody Peterson |
| $\dfrac{x}{x-x^4+2x\sin(x)}$ | $0 < x_1 < 2$ | Contributed by Aaron Zehe |

For solution #2, x = 1.2899

| g(x) | Valid range of initial guess $x_1$ (we only tested the range of $0 < x_1 < 2$) | Remarks |
|---|---|---|
| $\left(1+2\sin(x)-\dfrac{1}{x}\right)^{1/3}$ | $0.55 < x_1 < 2$ | Most popular choice used by the majority of the class |
| $(x+2x\sin(x)-1)^{1/4}$ | $0.55 < x_1 < 2$ | Contributed by Spencer McDonald |
| $\dfrac{2\sin(x)+1}{\left(x^2+\dfrac{1}{x^2}\right)}$ | $0.55 < x_1 < 2$ | Contributed by David Gonzalez, Anthony White |
| $(x^5-x^8+2x^5\sin(x))^{1/4}$ | $0.55 < x_1 < 1.4$ | Contributed by Trevor Keegan |

**Prob 2(a)** Solution by hand (Thanks to Shelby Rode)

2. $4x_1 + 2x_2 + x_3 = 10$

$3x_1 + 5x_2 + 0.5x_3 = 14$

$x + 0.5x_2 + 3x_3 = 8$

a)

$$\left[\begin{array}{ccc|c} 4 & 2 & 1 & 10 \\ 3 & 5 & 0.5 & 14 \\ 1 & 0.5 & 3 & 8 \end{array}\right] \xrightarrow{\frac{(1)}{4}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{4} & \frac{5}{2} \\ 3 & 5 & \frac{1}{2} & 14 \\ 1 & \frac{1}{2} & 3 & 8 \end{array}\right] \xrightarrow{(2)+[-3(1)]} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{4} & \frac{5}{2} \\ 0 & \frac{7}{2} & \frac{-1}{4} & \frac{13}{2} \\ 1 & \frac{1}{2} & 3 & 8 \end{array}\right]$$

$$\xrightarrow{3+[-1(1)]} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{4} & \frac{5}{2} \\ 0 & \frac{7}{2} & \frac{-1}{4} & \frac{13}{2} \\ 0 & 0 & \frac{11}{4} & \frac{11}{2} \end{array}\right] \xrightarrow{\frac{(2)}{7/2}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{4} & \frac{5}{2} \\ 0 & 1 & -\frac{1}{14} & \frac{13}{7} \\ 0 & 0 & \frac{11}{4} & \frac{11}{2} \end{array}\right] \xrightarrow[\frac{(3)}{11/4}]{} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{4} & \frac{5}{2} \\ 0 & 1 & -\frac{1}{14} & \frac{13}{7} \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{(3)+\frac{1}{14}(3)} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{4} & \frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{(1)+[-\frac{1}{4}(3)]} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{(1)+[-\frac{1}{2}(2)]} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \longrightarrow \text{exact solution}$$

**Prob 2(b)-2(e)** Solution by hand + plot (Thanks to Deion Schmidt)

① using $(x_1, x_2, x_3) = (0,0,0)$ [initially]

b) Solve the system by Jacobi Iterative.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{10}{4} \\ \frac{14}{5} \\ \frac{8}{3} \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{3}{5} & 0 & \frac{1}{10} \\ \frac{1}{3} & \frac{1}{6} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Jacobi Iterative

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{①} \begin{pmatrix} \frac{10}{4} \\ \frac{14}{5} \\ \frac{8}{3} \end{pmatrix} \xrightarrow{②} \begin{pmatrix} \frac{13}{30} \\ \frac{31}{30} \\ \frac{41}{30} \end{pmatrix} \xrightarrow{③} \begin{pmatrix} \frac{197}{120} \\ \frac{721}{300} \\ \frac{47}{20} \end{pmatrix}$$

$$\xrightarrow{④} \begin{pmatrix} \frac{853}{1200} \\ \frac{79}{50} \\ \frac{1547}{900} \end{pmatrix} \xrightarrow{⑤} \boxed{\begin{pmatrix} 1.28028 \\ 2.20161 \\ 2.16639 \end{pmatrix}}$$

$$\boxed{x_1 = 1.28028, \quad x_2 = 2.20161, \quad x_3 = 2.16639}$$

c) Solve the system by Gauss-Seidel.

Gauss-Seidel Iterative

$$\begin{pmatrix} - \\ 0 \\ 0 \end{pmatrix} \xrightarrow{①} \begin{array}{l} \frac{10}{4} - \frac{1}{2}(0) - \frac{1}{4}(0) \rightarrow x_1 = \frac{10}{4} \\ \frac{14}{5} - \frac{3}{5}(\frac{10}{4}) - \frac{1}{10}(0) \rightarrow x_2 = \frac{13}{10} \\ \frac{8}{3} - \frac{1}{3}(\frac{10}{4}) - \frac{1}{6}(\frac{13}{10}) \rightarrow x_3 = \frac{97}{60} \end{array}$$

Prob 2(b)-2(e) continued

$$\begin{pmatrix} \frac{10}{4} \\ \frac{13}{10} \\ \frac{97}{60} \end{pmatrix} \xrightarrow{②} \begin{array}{l} \frac{10}{4} - \frac{1}{2}\left(\frac{13}{10}\right) - \frac{1}{4}\left(\frac{97}{60}\right) \rightarrow x_1 = \frac{347}{240} \\ \frac{14}{5} - \frac{3}{5}\left(\frac{347}{240}\right) - \frac{1}{10}\left(\frac{97}{60}\right) \rightarrow x_2 = \frac{85}{48} \\ \frac{8}{3} - \frac{1}{3}\left(\frac{347}{240}\right) - \frac{1}{6}\left(\frac{85}{48}\right) \rightarrow x_3 = \frac{907}{480} \end{array}$$

$$\begin{pmatrix} \frac{347}{240} \\ \frac{85}{48} \\ \frac{907}{480} \end{pmatrix} \xrightarrow{③} \begin{array}{l} \frac{10}{4} - \frac{1}{2}\left(\frac{85}{48}\right) - \frac{1}{4}\left(\frac{907}{480}\right) \rightarrow x_1 = \frac{731}{640} \\ \frac{14}{5} - \frac{3}{5}\left(\frac{731}{640}\right) - \frac{1}{10}\left(\frac{907}{480}\right) \rightarrow x_2 = \frac{18487}{9600} \\ \frac{8}{3} - \frac{1}{3}\left(\frac{731}{640}\right) - \frac{1}{6}\left(\frac{18487}{9600}\right) \rightarrow x_3 = 1.965 \end{array}$$

$$\begin{pmatrix} \frac{731}{640} \\ \frac{18487}{9600} \\ 1.965 \end{pmatrix} \xrightarrow{④} \begin{array}{l} \frac{10}{4} - \frac{1}{2}\left(\frac{18487}{9600}\right) - \frac{1}{4}(1.96498) \rightarrow x_1 = 1.04589 \\ \frac{14}{5} - \frac{3}{5}(1.04589) - \frac{1}{10}(1.96498) \rightarrow x_2 = 1.97597 \\ \frac{8}{3} - \frac{1}{3}(1.04589) - \frac{1}{6}(1.97597) \rightarrow x_3 = 1.98871 \end{array}$$

$$\begin{pmatrix} 1.04589 \\ 1.97597 \\ 1.98871 \end{pmatrix} \xrightarrow{⑤} \begin{array}{l} \frac{10}{4} - \frac{1}{2}(1.97597) - \frac{1}{4}(1.98871) \rightarrow x_1 = 1.01484 \\ \frac{14}{5} - \frac{3}{5}(1.01484) - \frac{1}{10}(1.98871) \rightarrow x_2 = 1.99223 \\ \frac{8}{3} - \frac{1}{3}(1.01484) - \frac{1}{6}(1.99223) \rightarrow x_3 = 1.99635 \end{array}$$

$$\boxed{x_1 = 1.01484, \quad x_2 = 1.99223, \quad x_3 = 1.99635}$$

d) From outcome of (b) and (c), calculate and plot the numerical error as a function of # of iterations. [Euclidean 2-norm]

$$E = \|x_N - x_S\| \equiv \sqrt{(x_{N,1} - x_{S,1})^2 + (x_{N,2} - x_{S,2})^2 + (x_{N,3} - x_{S,3})^2}$$

$$x_N \equiv (x_{N,1}, x_{N,2}, x_{N,3})$$
$$x_S \equiv (x_{S,1}, x_{S,2}, x_{S,3})$$

Prob 2(b)-2(e) continued

## Jacobi Iterative Error

$$E_1 \rightarrow \sqrt{\left(\tfrac{10}{4}-1\right)^2+\left(\tfrac{14}{5}-2\right)^2+\left(\tfrac{8}{3}-2\right)^2} = 1.82605$$

$$E_2 \rightarrow \sqrt{\left(\tfrac{13}{30}-1\right)^2+\left(\tfrac{31}{30}-2\right)^2+\left(\tfrac{41}{30}-2\right)^2} = 1.28712$$

$$E_3 \rightarrow \sqrt{\left(\tfrac{197}{120}-1\right)^2+\left(\tfrac{721}{300}-2\right)^2+\left(\tfrac{47}{20}-2\right)^2} = 0.83481$$

$$E_4 \rightarrow \sqrt{\left(\tfrac{853}{1200}-1\right)^2+\left(\tfrac{79}{50}-2\right)^2+\left(\tfrac{1547}{900}-2\right)^2} = 0.58227$$

$$E_5 \rightarrow \sqrt{(1.28-1)^2+(2.20-2)^2+(2.17-2)^2} = 0.38326$$

MATLAB Plot is attached *

## Gauss-Seidel Iterative Error

$$E_1 \rightarrow \sqrt{\left(\tfrac{10}{4}-1\right)^2+\left(\tfrac{13}{10}-2\right)^2+\left(\tfrac{97}{60}-2\right)^2} = 1.69910$$

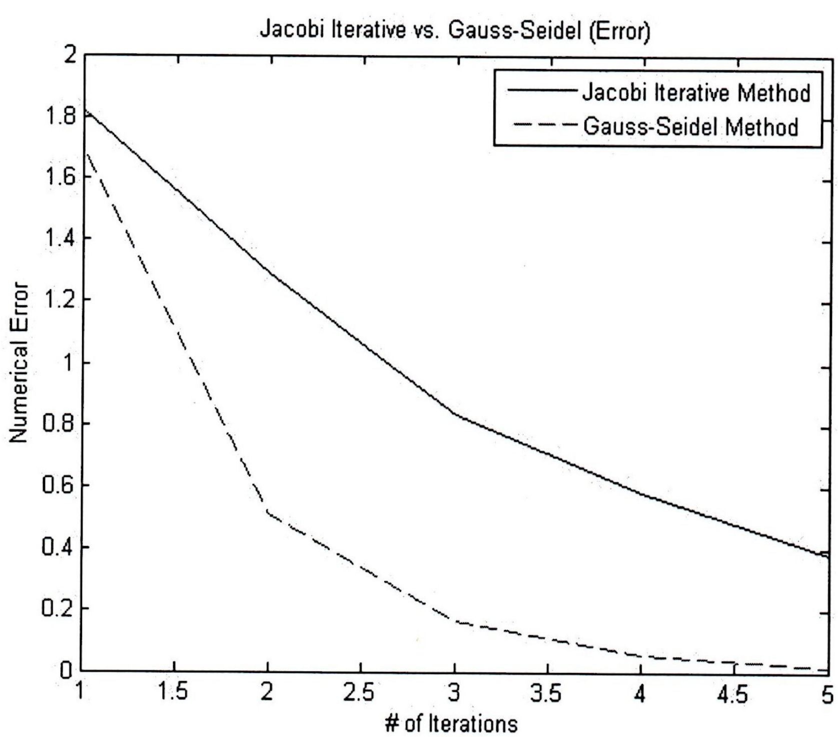
$$E_2 \rightarrow \sqrt{\left(\tfrac{347}{240}-1\right)^2+\left(\tfrac{85}{48}-2\right)^2+\left(\tfrac{907}{480}-2\right)^2} = 0.51330$$

$$E_3 \rightarrow \sqrt{\left(\tfrac{731}{640}-1\right)^2+\left(\tfrac{18487}{9600}-2\right)^2+(1.96-2)^2} = 0.16419$$

$$E_4 \rightarrow \sqrt{(1.05-1)^2+(1.98-2)^2+(1.99-2)^2} = 0.05302$$

$$E_5 \rightarrow \sqrt{(1.01-1)^2+(1.99-2)^2+(1.99-2)^2} = 0.01714$$

MATLAB Plot is attached *



Prob 2(b)-2(e) continued

Solution for 2(e) (slightly abbreviated)

$$[A^{-1}] = \begin{bmatrix} \frac{59}{154} & -\frac{1}{7} & -\frac{8}{77} \\ -\frac{17}{77} & \frac{2}{7} & \frac{2}{77} \\ -\frac{1}{11} & 0 & \frac{4}{11} \end{bmatrix} \quad *$$

condition number $\longrightarrow$ $\|[A]\| \; \|[A^{-1}]\|$

$$\text{mag} \begin{bmatrix} 4 & 2 & 1 \\ 3 & 5 & \frac{1}{2} \\ 1 & \frac{1}{2} & 3 \end{bmatrix} \times \text{mag} \begin{bmatrix} \frac{59}{154} & -\frac{1}{7} & -\frac{8}{77} \\ -\frac{17}{77} & \frac{2}{7} & \frac{2}{77} \\ -\frac{1}{11} & 0 & \frac{4}{11} \end{bmatrix}$$

$$\|[A]\| \longrightarrow \sqrt{(4)^2 + (2)^2 + (1)^2 + (3)^2 + (5)^2 + (\tfrac{1}{2})^2 + (1)^2 + (\tfrac{1}{2})^2 + (3)^2}$$

so $\|[A]\| = \sqrt{65.5} = 8.0932$

$$\|[A^{-1}]\| \longrightarrow \sqrt{(\tfrac{59}{154})^2 + (-\tfrac{1}{7})^2 + (-\tfrac{8}{77})^2 + (-\tfrac{17}{77})^2 + (\tfrac{2}{7})^2 + (\tfrac{2}{77})^2 + (-\tfrac{1}{11})^2 + (0)^2 + (\tfrac{4}{11})^2}$$

so $\|[A^{-1}]\| = \sqrt{0.4495} = 0.6705$

condition number $= (8.0932)(0.6705)$

$$\boxed{\text{condition number} = 5.43}$$

This system is $\boxed{\text{not ill-conditioned}}$ because the condition value is not much larger than one, therefore the system is well-conditioned instead of ill-conditioned because 5.43 is not that much greater than one.

**Prob 2(b)-2(e)** Solution by Matlab (Thanks to Daniel Miskin)

```
sol = [1;2;2]; % Exact solution from Gauss-Jordan Elimination (by hand)
A = [4,2,1;3,5,.5;1,.5,3];

% following the form x_i = d + [B]x_i-1
%% b)
xi = [0;0;0];
d = [5/2;14/5;8/3];
B = [0,-1/2,-1/4;-3/5,0,-1/10;-1/3,-1/6,0];
for i = 1:5
   xs = d+B*xi;
   xi = xs;
   xsteps(:,i) = xs;
end
xs

%% c)
x2 = 0;
x3 = 0;
for i = 1:5
x1 = d(1)+B(1,2)*x2+B(1,3)*x3;
x2 = d(2)+B(2,1)*x1+B(2,3)*x3;
x3 = d(3)+B(3,1)*x1+B(3,2)*x2;
xg_ssteps(:,i) = [x1;x2;x3];
end
xg_ssteps;
xg_ssteps(:,5)


%% d)
for i = 1:5
Ejacobi = sqrt((xsteps(1,i)-sol(1))^2+(xsteps(2,i)-sol(2))^2+(xsteps(3,i)-
sol(3))^2);
Eg_s = sqrt((xg_ssteps(1,i)-sol(1))^2+(xg_ssteps(2,i)-
sol(2))^2+(xg_ssteps(3,i)-sol(3))^2);
Ej(i) = Ejacobi;          % Jacobian error at each step
Eg(i) = Eg_s;             % Gauss-Seidel error at each step
end
i = 1:5;
plot(i,Ej,':ko',i,Eg,'-kx'); legend('Jacobi','Gauss-Seidel');
title('Error per Iteration');xlabel('iteration');ylabel('Error')



%% e)
norm1 = 0;
norm2 = 0;
Ai = inv(A);
for m = 1:3;
    for n = 1:3;
        norm1 = norm1+A(m,n)^2;
        norm2 = norm2+Ai(m,n)^2;
    end
end
norm1 = sqrt(norm1);          % Euclidean Norms
norm2 = sqrt(norm2);
C = norm1*norm2               % Condition number = ||A||*||inv(A)||
if C <= 10
    disp('System is not ill-conditioned.')
elseif C >= 100
    disp('system is ill-conditioned.')
else
   disp('system is marginally conditioned')
end
```

**Prob 2(e)** Further discussion (by HPH)

The condition number based on Euclidean norm is C = 5.426. Some of you have noticed that Matlab has a function to calculate the norm of a matrix or even the condition number associated with it. For example, one can obtain the condition number with the command,

```
C = norm(A)*norm(inv(A)) ,
```

or simply,

```
C = cond(A).
```

However, the default setting of those functions used a different kind of norm which is not Euclidean. Those who used the Matlab functions without any modification will obtain an answer that's slightly off. In order to force Matlab to adopt the Euclidean norm, an option of 'fro' needs to be specified when calling the functions:

```
C = norm(A, 'fro')*norm(inv(A), 'fro') ,
```

or

```
C = cond(A, 'fro').
```

Either of the above two commends will return the correct answer of C = 5.426. (Thanks to Darrell Jordan for pointing this out.) Note that 'fro' stands for 'Frobenius'. The Euclidean norm in our textbook is sometimes called the Frobenius norm in mathematical literature.

Also, note that C = 5.426 is small enough that the system is not ill-conditioned.