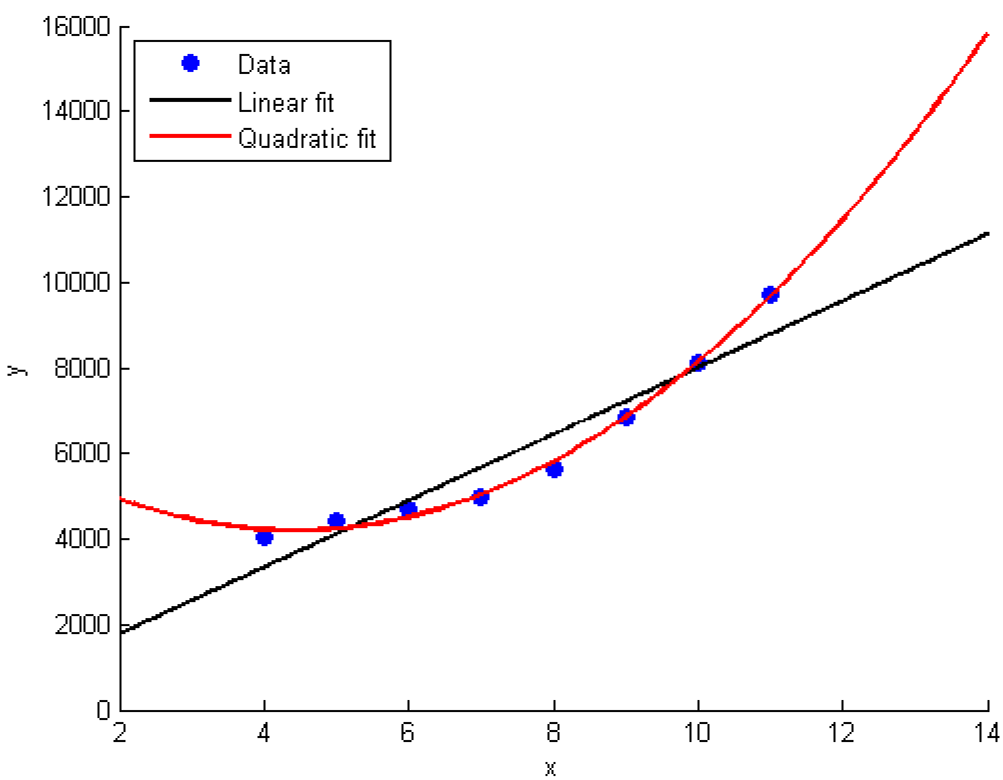
MAE384 Fall 2012 HW3 Solutions

Prob 1 (a) and (b), All-in-one Matlab code (prepared by HPH)

```
clear; clc
x = [4 5 6 7 8 9 10 11];
y = [4062 4404 4686 4969 5659 6840 8128 9716];
P4 = 0;P3 = 0; P2 = 0; P1 = 0; Q3 = 0; Q2 = 0; Q1 = 0;
for k = 1:8
    P4 = P4+x(k)^4; P3 = P3+x(k)^3; P2 = P2+x(k)^2; P1 = P1+x(k);
    Q3 = Q3+(x(k)^2)*y(k); Q2 = Q2+x(k)*y(k); Q1 = Q1+y(k);
end
M2 = [P4 P3 P2; P3 P2 P1; P2 P1 8] ; v2 = [Q3; Q2; Q1];
w2 = inv(M2)*v2; a = w2(1); b = w2(2); c = w2(3);
M1 = [P2 P1; P1 8]; v1 = [Q2; Q1];
w1 = inv(M1)*v1; p = w1(1); q = w1(2);
xp = [2:0.1:14];
yp1 = p*xp+q; yp2 = a*xp.^2+b*xp+c;
hold on
plot(x,y,'o','MarkerFaceColor','b')
plot(xp,yp1,'-k',xp,yp2,'-r','LineWidth',2)
xlabel('x');ylabel('y')
legend('Data','Linear fit','Quadratic fit','Location','NorthWest')
proj1 = p*14+q
proj2 = a*(14^2)+b*14+c
```



Linear fit: y = 11115 at x = 14

Quadratic fit: y = 15797 at x = 14

Problem 1 (a) and (b), solution by hand (Thanks to Christina Hays)

X	y
4	4062
5	4404
6	4686
7	4969
8	5659
9	6840
10	8128
11	9716

a. Perform linear least-squares regression.

$$y = ax + b$$

$$\begin{pmatrix} \sum_{i=1}^{8} x_i^2 & \sum_{i=1}^{8} x_i \\ \sum_{i=1}^{8} x_i & N \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{8} x_i y_i \\ \sum_{i=1}^{8} y_i \end{pmatrix}$$

$$\sum_{i=1}^{8} x_i^2 = 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = 492$$

$$\sum_{i=1}^{8} x_i = 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 = 60$$

$$\sum_{i=1}^{8} x_i y_i = 4(4062) + 5(4404) + 6(4686) + 7(4969) + 8(5659) + 9(6840) + 10(8128) + 11(9716) = 396155$$

$$\sum_{i=1}^{8} y_i = 4062 + 4404 + 4686 + 4969 + 5659 + 6840 + 8128 + 9716 = 48464$$

$$\begin{bmatrix} 492 & 60 \\ 60 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 396155 \\ 48464 \end{bmatrix}$$

$$492a + 60b = 396155$$
$$60a + 8b = 48464 \cdot 7.5$$

$$492a + 60b = 396155$$
$$\underline{450a + 60b = 363480}$$
$$42a = 32675$$
$$a = 777.976$$

$$60(777.976) + 8b = 48464$$
$$8b = 1785.429$$
$$b = 223.179$$

$$\boxed{y = 777.976x + 223.179}$$

Continued to next page

b. Perform quadratic least-squares regression

$$y = px^2 + qx + r$$

$$\begin{pmatrix} \sum_{i=1}^{N} x_i^4 & \sum_{i=1}^{N} x_i^3 & \sum_{i=1}^{N} x_i^2 \\ \sum_{i=1}^{N} x_i^3 & \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i & N \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{N} x_i^2 y_i \\ \sum_{i=1}^{N} x_i y_i \\ \sum_{i=1}^{N} y_i \end{pmatrix}$$

$$\sum_{i=1}^{N} x_i^4 = 4^4 + 5^4 + 6^4 + 7^4 + 8^4 + 9^4 + 10^4 + 11^4 = 39876$$

$$\sum_{i=1}^{N} x_i^3 = 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3 + 11^3 = 4320$$

$$\sum_{i=1}^{N} x_i^2 = 492$$

$$\sum_{i=1}^{N} x_i = 60$$

$$\sum_{i=1}^{N} x_i^2 y_i = 4^2(4062) + 5^2(4404) + 6^2(4686) + 7^2(4969) + 8^2(5659) + 9^2(6840) + 10^2(8128) + 11^2(9716) = 3491921$$

$$\sum_{i=1}^{N} x_i y_i = 396155 \qquad \sum_{i=1}^{N} y_i = 48464$$

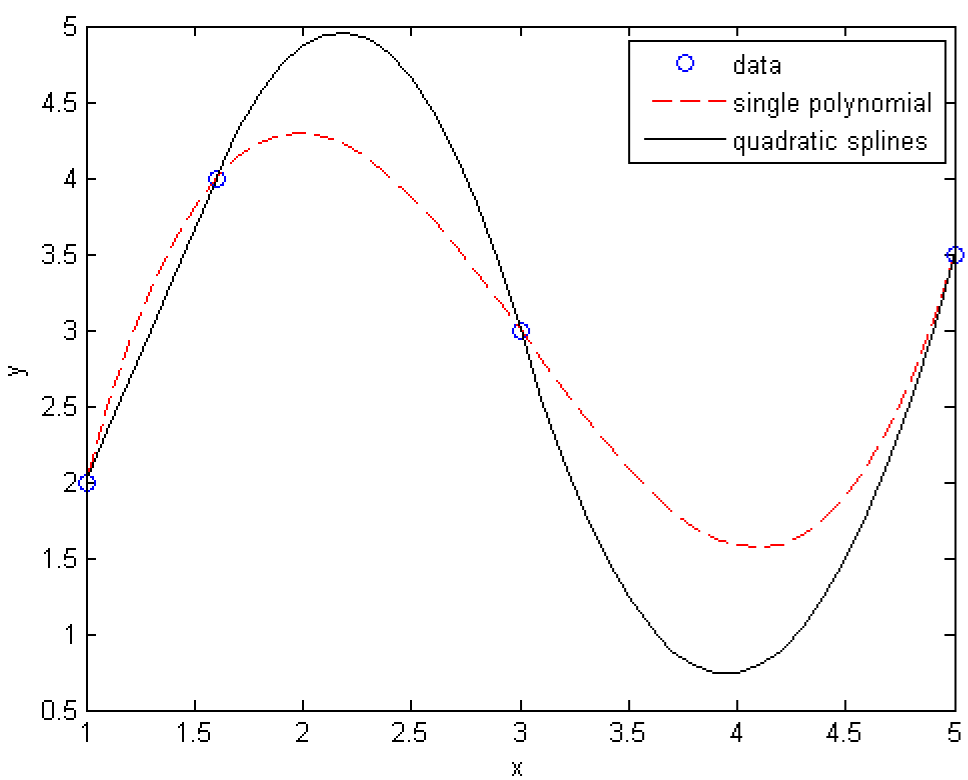
$$\begin{bmatrix} 39876 & 4320 & 492 \\ 4320 & 492 & 60 \\ 492 & 60 & 8 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 3491921 \\ 396155 \\ 48464 \end{bmatrix}$$

$$p = 126.5 \quad q = -1120.2 \quad r = 6677.1$$

$$\boxed{y = 126.5x^2 - 1120.2x + 6677.1}$$

Prob 2 (a) and (b) All-in-one Matlab code (prepared by HPH)

```
clear ; clc
x = [1 1.6 3 5]; y = [2 4 3 3.5];
A = [x(1)^2 x(1) 1 0 0 0 0 0 0; ...
     x(2)^2 x(2) 1 0 0 0 0 0 0; ...
     0 0 0 x(2)^2 x(2) 1 0 0 0; ...
     0 0 0 x(3)^2 x(3) 1 0 0 0; ...
     0 0 0 0 0 0 x(3)^2 x(3) 1; ...
     0 0 0 0 0 0 x(4)^2 x(4) 1; ...
     2*x(2) 1 0 -2*x(2) -1 0 0 0 0; ...
     0 0 0 2*x(3) 1 0 -2*x(3) -1 0; ...
     1 0 0 0 0 0 0 0 0];
 b = [y(1);y(2);y(2);y(3);y(3);y(4);0;0;0];
 v = inv(A)*b;
 a1 = v(1); b1 = v(2); c1 = v(3);
 a2 = v(4); b2 = v(5); c2 = v(6);
 a3 = v(7); b3 = v(8); c3 = v(9);
 xp1 = [x(1):0.1:x(2)] ; xp2 = [x(2):0.1:x(3)] ; xp3 = [x(3):0.1:x(4)];
 yp1 = a1*xp1.^2+b1*xp1+c1;
 yp2 = a2*xp2.^2+b2*xp2+c2;
 yp3 = a3*xp3.^2+b3*xp3+c3;
 M = [x(1)^3 x(1)^2 x(1) 1; ...
      x(2)^3 x(2)^2 x(2) 1; ...
      x(3)^3 x(3)^2 x(3) 1; ...
      x(4)^3 x(4)^2 x(4) 1];
 p = [y(1);y(2);y(3);y(4)];
 w = inv(M)*p;
 d1 = w(1); d2 = w(2); d3 = w(3); d4 = w(4);
 xp = [x(1):0.1:x(4)];
 yp = d1*(xp.^3)+d2*(xp.^2)+d3*xp+d4;
 plot(x,y,'o',xp,yp,'--r',xp1,yp1,'-k',xp2,yp2,'-k',xp3,yp3,'-k')
 xlabel('x');ylabel('y');legend('data','single polynomial','quadratic splines')
```



Prob 2(a) Derivation of the matrix system (Thanks to Julia Dyer)

a) Quadratic splines

4 points, 3 splines (1,2,3)

$a_1 = 0$

$i=1 \quad f_1(x) = a_1 x_1^2 + b_1 x_1 + c_1 = b_1 + c_1 = 2 \qquad (1)$

$f_1(x) = a_1 x_2^2 + b_1 x_2 + c_1 = 1.6b_1 + c_1 = 4 \qquad (2)$

$i=2 \quad f_2(x) = a_2 x_2^2 + b_2 x_2 + c_2 = (1.6)^2 a_2 + 1.6b_2 + c_2 = 4 \qquad (3)$

$f_2(x) = a_2 x_3^2 + b_2 x_3 + c_2 = (3)^2 a_2 + 3b_2 + c_2 = 3 \qquad (4)$

$i=3 \quad f_3(x) = a_3 x_3^2 + b_3 x_3 + c_3 = (3)^2 a_3 + 3b_3 + c_3 = 3 \qquad (5)$

$f_3(x) = a_3 x_4^2 + b_3 x_4 + c_3 = (5)^2 a_3 + 5b_3 + c_3 = 3.5 \qquad (6)$

interior knots:

$i=2 \quad 2\cancel{a_1 x_2}^{0} + b_1 = 2a_2 x_2 + b_2 \rightarrow 3.2a_2 - b_1 + b_2 = 0 \qquad (7)$

$i=3 \quad 2a_2 x_3 + b_2 = 2a_3 x_3 + b_3 \rightarrow 6a_2 - 6a_3 + b_2 - b_3 = 0 \qquad (8)$

$$\text{rref}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.6^2 & 1.6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 9 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 25 & 5 & 1 \\ -1 & 0 & 3.2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & -6 & -1 & 0 \end{bmatrix} = \begin{matrix} b_1 = 3.33 \\ c_1 = -1.33 \\ a_2 = -2.89 \\ b_2 = 12.585 \\ c_2 = -8.735 \\ a_3 = 2.505 \\ b_3 = -19.797 \\ c_3 = 39.839 \end{matrix}$$

answers

$f_1(x) = 3.33x - 1.33$ for $1 \le x \le 1.6$

$f_2(x) = -2.89x^2 + 12.585x - 8.735$ for $1.6 \le x \le 3$

$f_3(x) = 2.505x^2 - 19.797x + 39.84$ for $3 \le x \le 5$

Prob 2(a) Complete solution by hand (Thanks to Christina Hays)

Note: As discussed in class, the "big matrix" derived in the preceding page is quite sparse (with many zeros) and the individual equations are only loosely coupled. As such, it is actually possible to directly solve the whole system by hand without invoking the matrix inversion (or even the construction of the matrix in the first place).

X	y
1	2
1.6	4
3	3
5	3.5

$y_1 = a_1 x_1^2 + b_1 x_1 + c_1 \qquad y_2 = a_1 x_2^2 + b_1 x_2 + c_1$

$y_2 = a_2 x_2^2 + b_2 x_2 + c_2 \qquad y_3 = a_2 x_3^2 + b_2 x_3 + c_2$

$y_3 = a_3 x_3^2 + b_3 x_3 + c_3 \qquad y_4 = a_3 x_4^2 + b_3 x_4 + c_3$

$y_2' = y_2' \qquad y_3' = y_3'$

$2a_1 x_2 + b_1 = 2a_2 x_2 + b_2 \qquad 2a_2 x_3 + b_2 = 2a_3 x_3 + b_3$

$y_1'' = 0 = 2a_1 \qquad 2 = 0 + b_1(1) + c_1$

$4 = 0 + b_1(1.6) + c_1$

$b_1 = 3.33 \quad c_1 = -1.33$

$0 + 3.33 = 2a_2(1.6) + b_2$

$4 = a_2(1.6)^2 + b_2(1.6) + c_2$

$3 = a_2(3)^2 + b_2(3) + c_2$

$a_2 = -2.89 \quad b_2 = 12.57 \quad c_2 = -8.72$

$2(-2.89)(3) + 12.57 = 2a_3(3) + b_3$

$3 = a_3(3)^2 + b_3(3) + c_3$

$3.5 = a_3(5)^2 + b_3(5) + c_3$

$a_3 = 2.51 \quad b_3 = -19.83 \quad c_3 = 39.9$

$a_1 = 0$

$b_1 = 3.33$

$c_1 = -1.33$

$a_2 = -2.89$

$b_2 = 12.57$

$c_2 = -8.72$

$a_3 = 2.51$

$b_3 = -19.83$

$c_3 = 39.9$

quadratic splines:

1. $y = 3.33x - 1.33$

2: $y = -2.89x^2 + 12.57x - 8.72$

3: $y = 2.51x^2 - 19.83x + 39.9$

Prob 2(b) Direct matrix solution (Thanks to Christina Hays)

b.

x	y
1	2
1.6	4
3	3
5	3.5

$$y_1 = ax_1^3 + bx_1^2 + cx_1 + d$$
$$y_2 = ax_2^3 + bx_2^2 + cx_2 + d$$
$$y_3 = ax_3^3 + bx_3^2 + cx_3 + d$$
$$y_4 = ax_4^3 + bx_4^2 + cx_4 + d$$

$$\begin{bmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ x_2^3 & x_2^2 & x_2 & 1 \\ x_3^3 & x_3^2 & x_3 & 1 \\ x_4^3 & x_4^2 & x_4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4.096 & 2.56 & 1.6 & 1 \\ 27 & 9 & 3 & 1 \\ 125 & 25 & 5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 3.5 \end{bmatrix}$$ inv(A) * b

$a = 0.5769$ $b = -5.2542$ $c = 14.0177$ $d = -7.3403$

$$y = 0.5769x^3 - 5.2542x^2 + 14.0177x - 7.3403$$

Prob 2(b) Solution by Lagrange interpolation polynomials (Thanks to Julia Dyer)

Lagrange interpolation

$$\varphi_1(x) = \frac{(x-1.6)(x-3)(x-5)}{(1-1.6)(1-3)(1-5)} = \frac{x^3-9.6x^2+27.8x-24}{-4.8} =$$

$$\varphi_1(x) = -0.208x^3+2x^2-5.79x+5$$

$$\varphi_2(x) = \frac{(x-1)(x-3)(x-5)}{(1.6-1)(1.6-3)(1.6-5)} = \frac{x^3-9x^2+23x-15}{2.856}$$

$$\varphi_2(x) = 0.35x^3-3.15x^2+8.05x-5.25$$

$$\varphi_3(x) = \frac{(x-1)(x-1.6)(x-5)}{(3-1)(3-1.6)(3-5)} = \frac{x^3-7.6x^2+14.6x-8}{-5.6}$$

$$\varphi_3(x) = -0.179x^3+1.36x^2-2.61x+1.43$$

$$\varphi_4(x) = \frac{(x-1)(x-1.6)(x-3)}{(5-1)(5-1.6)(5-3)} = \frac{x^3-5.6x^2+9.4x-4.8}{27.2}$$

$$\varphi_4(x) = 0.0368x^3-0.206x^2+0.346x-0.176$$

$$P(x) = \varphi_1(x)\cdot y_1 + \varphi_2(x)\cdot y_2 + \varphi_3(x)\cdot y_3 + \varphi_4(x)\cdot y_4$$

3rd	2nd	1st	0
-0.416	4	-11.58	10
1.4	-12.6	32.2	-21
-0.537	4.08	-7.83	4.29
0.1288	-0.721	1.211	-0.616
0.576	-5.25	14.01	-7.34

$$\boxed{P(x) = 0.576x^3-5.25x^2+14.01x-7.34}$$