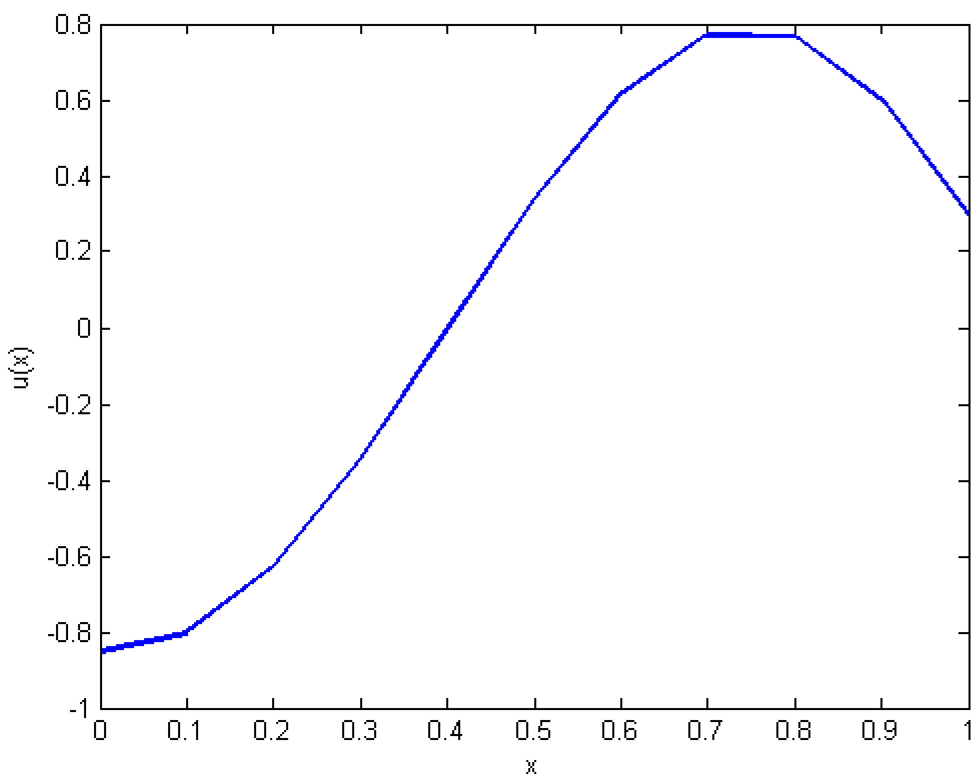
MAE384 2012 HW6 Solutions

Prob 1 Matlab code + plot (prepared by HPH)

```
clear
h = 0.1; N = 1/h;
for i = 1:N-1
    for j = 1:N-1
        if (i == j) && (i == 1)
            A(i,j) = 15*(h^2)+8*(h^3)*((i+1)-1)-2+1;
        elseif (i == j)
            A(i,j) = 15*(h^2)+8*(h^3)*((i+1)-1)-2;
        elseif ((i-j) == 1) || ((j-i) == 1)
            A(i,j) = 1;
        else
            A(i,j) = 0;
        end
    end
end
b(1) = 0.5*h; b(N-1) = -0.3;
for i = 2:N-2
    b(i) = 0;
end
c = inv(A)*(b');
d(1) = c(1)-0.5*h;
for i = 2:N
    d(i) = c(i-1);
end
d(N+1) = 0.3;
xplot = [0:h:1];
plot(xplot,d,'LineWidth',2); xlabel('x');ylabel('u(x)')
```



Prob 1 Some useful derivations by hand (Thanks to Nick Ceraulo)

1. $\dfrac{d^2u}{dx^2} + (15+8x)u = 0 \qquad \begin{matrix} u'(0) = 0.5 \\ u(1) = 0.3 \end{matrix}$

$h = 0.1$ over interval $0 \le x \le 1$

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + (15+8x_i)u_i = 0$$

$$u_{i-1} - 2u_i + u_{i+1} + h^2(15+8x_i)u_i = 0$$

$$u_{i-1} + [h^2(15+8x_i) - 2]u_i + u_{i+1} = 0$$

$x_0 = 0$, $x_{10} = 1$; $h = 0.1$

$$\begin{array}{ll}
i=1 & u_0 + [0.1^2(15+8(0.1))-2]u_1 + u_2 \\
i=2 & u_1 + [0.1^2(15+8(0.2))-2]u_2 + u_3 \\
i=3 & u_2 + [0.1^2(15+8(0.3))-2]u_3 + u_4 \\
i=4 & u_3 + [0.1^2(15+8(0.4))-2]u_4 + u_5 \\
i=5 & u_4 + [0.1^2(15+8(0.5))-2]u_5 + u_6 \\
i=6 & u_5 [0.1^2(15+8(0.6))-2]u_6 + u_7 \\
i=7 & u_6 [0.1^2(15+8(0.7))-2]u_7 + u_8 \\
i=8 & u_7 [0.1^2(15+8(0.8)-2]u_8 + u_9 \\
i=9 & u_8 [0.1^2(15+8(0.9))-2]u_9 + u_{10}
\end{array}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$u'_i = \dfrac{u_{i+1} - u_i}{h} \quad ; \quad \begin{matrix} u'(0) = 0.5 \\ u(1) = 0.3 \end{matrix}$

$\dfrac{u_1 - u_0}{h} = 0.5 \longrightarrow u_1 - u_0 = 0.05$

$u_0 = u_1 - 0.05$

Prob 1, hand derivations continued

$$\begin{bmatrix} -0.842 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1.834 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1.826 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1.818 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1.81 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1.802 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1.794 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1.786 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1.778 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.3 \end{bmatrix}$$

$u_1 = -0.80289$

$u_2 = -0.62603$

$u_3 = -0.34526$

$u_4 = -0.00440$

$u_5 = 0.33725$

$u_6 = 0.61483$

$u_7 = 0.77067$

$u_8 = 0.76775$

$u_9 = 0.60053$

$u_0 = -0.85289$

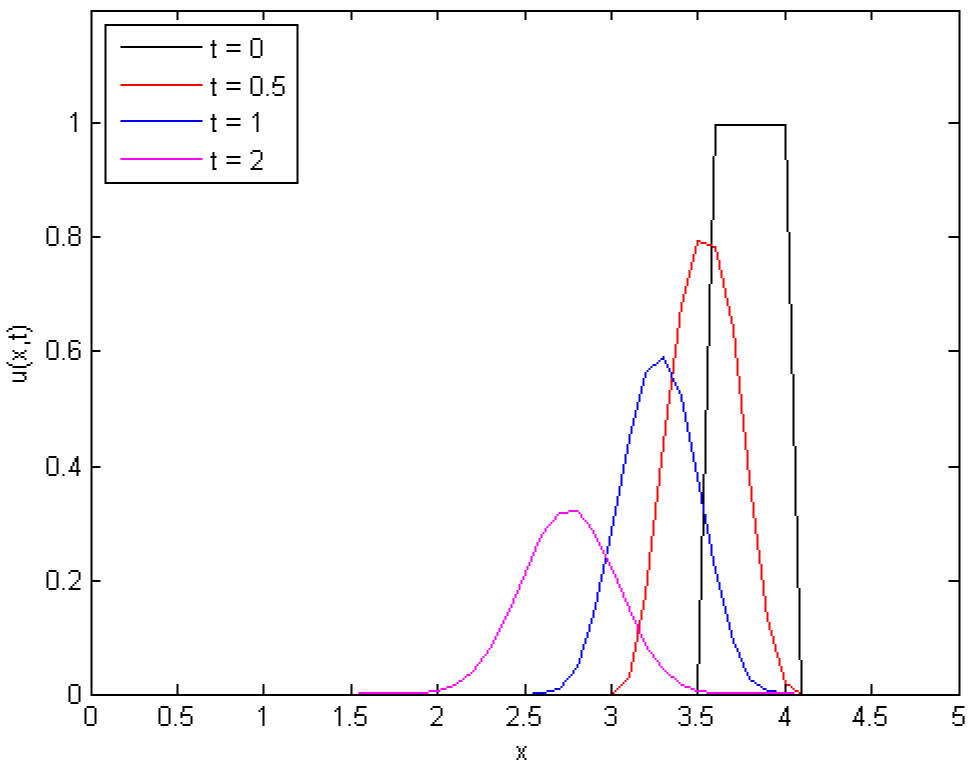
$u_{10} = 0.3$

Prob 2 Matlab code + plot (prepared by HPH)

```
clear
dx = 0.1; dt = 0.1; A = 0.5; B = 0.4;
x = [-0.1:0.1:5.1]; xx = [0:0.1:5];
for k = 1:52
    if (x(k) >= 3.5) && (x(k) <= 4)
        u(k) = 1;
    else
        u(k) = 0;
    end
end
for k = 2:52
    u00(k-1) = u(k);
end
for n = 1:5
    for k = 1:51
        u1(k) = (1-A*dt/dx-B*dt)*u(k)+(A*dt/dx)*u(k+1);
    end
    u1(52) = 0;
    for k = 1:52
        u(k) = u1(k);
    end
end
for k = 2:52
    u05(k-1) = u(k);
end
for n = 1:5
    for k = 1:51
        u1(k) = (1-A*dt/dx-B*dt)*u(k)+(A*dt/dx)*u(k+1);
    end
    u1(52) = 0;
    for k = 1:52
        u(k) = u1(k);
    end
end
for k = 2:52
    u10(k-1) = u(k);
end
for n = 1:10
    for k = 1:51
        u1(k) = (1-A*dt/dx-B*dt)*u(k)+(A*dt/dx)*u(k+1);
    end
    u1(52) = 0;
    for k = 1:52
        u(k) = u1(k);
    end
end
for k = 2:52
    u20(k-1) = u(k);
end
plot(xx,u00,'k-',xx,u05,'r-',xx,u10,'b-',xx,u20,'m-')
axis([0 5 0 1.2])
legend('t = 0','t = 0.5','t = 1','t = 2','Location','NorthWest')
xlabel('x');ylabel('u(x,t)')
```

(plot in next page)

Plot for Prob 2



Prob 2 Some useful derivations by hand (Thanks to Nick Ceraulo)

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t} \; ; \; \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x}$$

$$u_i = u_{i,j}$$

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = 0.5\left[\frac{u_{i+1,j} - u_{i,j}}{\Delta x}\right] - 0.4\left[u_{i,j}\right]$$

$$u_{i,j+1} - u_{i,j} = \frac{0.5\, u_{i+1,j}\,\Delta t}{\Delta x} - \frac{0.5\, u_{i,j}\,\Delta t}{\Delta x} - 0.4\, u\,\Delta t$$

$$u_{i,j+1} = 0.5\, u_{i+1,j} - 0.5\, u_{i,j} - 0.04\, u + u_{i,j}$$

$$\downarrow$$

$$u_{i,j+1} = 0.5\, u_{i+1,j} - 0.5\, u_{i,j} - 0.04\, u_{i,j} + u_{i,j}$$

$$\boxed{u_{i,j+1} = 0.5\, u_{i+1,j} + 0.46\, u_{i,j}}$$

Prob 3 (Thanks to Nick Ceraulo)

3a. $\dfrac{\partial^2 u}{\partial x \partial y} - xyu = 0 \qquad u(x,y) = G(x)H(y)$

$\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) - xyu = 0$

$\dfrac{dG}{dx} \cdot \dfrac{dH}{dy} - xyGH = 0$

$\dfrac{dG}{dx}\dfrac{dH}{dy} = xyGH$

$\dfrac{dG}{dx} \cdot \dfrac{1}{Gx} = Hy \cdot \dfrac{dy}{dH}$

$\dfrac{dG}{dx} \cdot \dfrac{1}{Gx} = c$

$\dfrac{1}{G} dG = cx\,dx$

$\ln G = c\dfrac{x^2}{2}$

$\underline{G = K_1 e^{\frac{1}{2}cx^2}}$

$Hy \cdot \dfrac{dy}{dH} = c$

$Hy \cdot dy = c\,dH$

$y\,dy = c\dfrac{1}{H}dH$

$\dfrac{y^2}{2} = c \ln H$

$\ln H = \dfrac{y^2}{2c}$

$\underline{H = K_2 e^{\frac{y^2}{2c}}}$

$u(x,y) = G(x) \cdot H(y)$

$u(x,y) = K_1 e^{\frac{1}{2}cx^2} \cdot K_2 e^{\frac{y^2}{2c}}$

$u(x,y) = K e^{\frac{1}{2}cx^2 + \frac{y^2}{2c}}$

3b. $$xy\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + yu = 0 \qquad u(x,y) = G(x)H(y)$$

$$xyH\frac{dG}{dx} - G\frac{dH}{dy} + yGH = 0$$

$$xyH\frac{dG}{dx} + yGH = G\frac{dH}{dy}$$

$$y\left(xH\frac{dG}{dx} + GH\right) = G\frac{dH}{dy}$$

$$xH\frac{dG}{dx} + GH = \frac{G}{y}\frac{dH}{dy}$$

$$\frac{x}{G}\frac{dG}{dx} + 1 = \frac{1}{Hy}\frac{dH}{dy}$$

$$\frac{x}{G}\frac{dG}{dx} + 1 = c$$

$$\frac{x}{G}\frac{dG}{dx} = c-1$$

$$\frac{1}{G}dG = (c-1)\frac{1}{x}dx$$

$$\ln G = \ln x(c-1)$$

$$G = k_1 x^{c-1}$$

$$\frac{1}{Hy}\frac{dH}{dy} = c$$

$$\frac{1}{H}dH = cy\,dy$$

$$\ln H = c\frac{y^2}{2}$$

$$H = k_2 e^{\frac{1}{2}cy^2}$$

$$u(x,y) = G(x)\cdot H(y)$$

$$u(x,y) = k_1 x^{c-1} \cdot k_2 e^{\frac{1}{2}cy^2}$$

$$u(x,y) = kx^{c-1}e^{\frac{1}{2}cy^2}$$