MAE384, Spring 2018 Homework #7

This assignment replaces a portion of the final exam and is equivalent to a **take-home exam**. Collaboration with anyone, either within or outside the class, IS NOT ALLOWED. The only exception is open discussion with instructor in recitation sessions and lectures.

Hard copy of report is due **at the start of final exam**. Computer codes used to complete the tasks should be included in the report. Uses of Matlab built-in functions for Fourier transform, such as **fft**, are NOT allowed for this homework.

Task 0 (0 point; mandatory to complete for the report to be accepted)
Provide a statement that **no collaboration occurred in the process of completing this assignment**. This implies that the person submitting the report has not helped anyone or received help from anyone on this assignment. **This statement must be placed in the beginning of the first page of the report**.

Task 1 (3 points)
Background: A function, $f(x)$, is periodic over the interval of $0 \le x \le 128$. (In other words, $f(0) = f(128)$. More generally, $f(x) = f(x+128)$ for any x.) The values of $f(x)$ are known at discrete points of $x = 0, 1, 2, 3, \ldots, 127$, and 128. (Note that $f(128) = f(0)$.) The data file, **hw7task1data.mat**, stores two arrays, **x** and **f**, that contain the discrete values of x and the corresponding values of $f(x)$ at those discrete points. More precisely,

$$\mathbf{x} = [0\ \ 1\ \ 2\ \ 3\ \ \ldots\ 127\ \ 128]\ ,\ \ \mathbf{f} = [f(0)\ \ f(1)\ \ f(2)\ \ f(3)\ \ldots\ f(127)\ \ f(128)]\ .$$

After downloading the data file and placing it in your working directory, the data can be read by Matlab using the "load" command. The following code reads the data and produces a quick plot to confirm that the **x** and **f** arrays have been read correctly:

```
clear
load('hw7task1data.mat')
plot(x,f,'k-')
axis([0 128 -6 6]);xlabel('x');ylabel('f(x)')
```

(a) Express $f(x)$ in Fourier series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\,n\,\pi\,x}{L}\right) + b_n \sin\left(\frac{2\,n\,\pi\,x}{L}\right)\ ,$$

where $L = 128$. Your first task is to use the data given in *Background* to evaluate the Fourier coefficients a_0, a_1, b_1, a_2, b_2, ... , etc. List the values of all coefficients for $n \le 20$ (including $n = 0$). You do not need to evaluate the coefficients with $n > 20$.

(b) Truncate the Fourier series representation of $f(x)$ at $n = 4$ (inclusive). In other words, construct a "filtered" version of $f(x)$ defined by

$$F(x) = a_0 + \sum_{n=1}^{4} a_n \cos\left(\frac{2\,n\,\pi\,x}{L}\right) + b_n \sin\left(\frac{2\,n\,\pi\,x}{L}\right)\ ,$$

using the Fourier coefficients from Part (a). The $F(x)$ here should be a "smoothed" version of the original $f(x)$, since the "noisy" components with large n have been removed. Make a plot of $F(x)$ and the original (unfiltered) $f(x)$. Please collect the two curves in a single plot for a clear comparison.

Task 2 (2.5 points)
For $u(x, t)$ defined on the domain of $0 \le x \le 1$ and $t \ge 0$, find the analytic solution of the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u \quad ,$$

with the boundary conditions,

(i) $u(0, t) = 0$, (ii) $u_x(1, t) = 0$, and (iii) $u(x, 0) = \sin(0.5\,\pi x)$.

(Note: We use u_x to denote $\partial u/\partial x$. Be aware that the second boundary condition is imposed on the partial derivative of u.)

The solution should be in closed form without any unevaluated integral. Please include the hand derivation to show the procedure for finding the analytic solution. Plot the solution, $u(x,t)$, as a function of x at $t = 0$, 0.1, and 0.3. Collect all three curves in a single plot.

Task 3 (2.5 points)
Consider the system given in Task 2 but now solve it numerically by the finite difference method. Let Δx and Δt be the grid spacing in x- and t-direction, and "i" and "j" the indices for the grid points in the two respective directions. To discretize the PDE, use the first-order forward difference scheme to approximate the partial derivative with respect to t:

$$\left(\frac{\partial u}{\partial t}\right)_{i,\,j} \approx \frac{u_{i,\,j+1} - u_{i,\,j}}{\Delta t} \quad .$$

Use the 2nd-order centered difference scheme to approximate the 2nd partial derivative with respect to x:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,\,j} \approx \frac{u_{i-1,\,j} - 2\,u_{i,\,j} + u_{i+1,\,j}}{(\Delta x)^2} \quad .$$

For the second boundary condition, use the first-order backward difference scheme to relate the values of u at the last two grid points in x-direction:

$$\left(\frac{\partial u}{\partial x}\right)_{M,\,j} \approx \frac{u_{M,\,j} - u_{M-1,\,j}}{\Delta x} \quad ,$$

where $i = M$ corresponds to $x = 1$, i.e., the grid point at the right boundary, and $i = M-1$ corresponds to $x = 1-\Delta x$.

With the above setup, choose $\Delta x = 0.1$ and $\Delta t = 0.001$ to construct the finite difference formula for the system and solve it to $t = 0.3$. Plot the numerical solution $u(x, t)$ as a function of x at $t = 0$, 0.1, and 0.3. Please collect all three curves in one plot.